Finance II

Jan 19, 2005

10.00-15.00

All notation should be clearly defined.
Arguments should be complete and careful.

1. Consider the following boundary value problem in the domain $[0,T]\times R$ for an unknown function $F(t,x)$.

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) + F(t,x)k(t,x) &= 0, \\ F(T,x) &= \Phi(x). \end{aligned}$$

   Here μ, σ, k and Φ are assumed to be known functions.

   Derive a Feynman-Kač representation for this problem. In this formula it must be quite clear exactly at which points the various functions should be evaluated. In other words - if you suppress variables you must explain **very** clearly exactly what you mean. ....(20p)

   If you think this problem is hard you may (with loss of 7 points) assume that $k = 0$.

2. Consider the standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= rB_t dt. \end{aligned}$$

   Prove that every contract of the form $\Phi(S_T)$ can be replicted. (20p)

3. Consider a two-dimensional Black-Scholes market i.e. a market consisting of a risk-free asset, B, with P-dynamics given by

$$\begin{cases} dB_t = rB_t dt, \\ B_0 = 1, \end{cases}$$

and two stocks, X and Y, with P-dynamics given by

$$\begin{cases} dX_t &= \alpha X_t dt + \sigma X_t dW_t, \\ dY_t &= \beta Y_t dt + \delta Y_t dW_t. \end{cases}$$

Here W is a one-dimensional P-Wiener processes and r, α, σ, β and δ are assumed to be constants such that

$$r \neq \frac{\delta\alpha - \sigma\beta}{\delta - \sigma}.$$

Assume that the filtration is the natural filtration generated by the Wiener process W. Show that this model is **not** free of arbitrage.
.... (20p)

4. Consider an short rate model described by

$$dr = \mu(t, r)dt + \sigma(t, r)dW, \tag{1}$$

under a risk-neutral martingale measure Q. Here W denotes a Q-Wiener process. Definite, as usual the, yield curve process $Y(t, T)$ implicitly by

$$p(t, T) = e^{-Y(t,T)(T-t)},$$

where $p(t, T)$ dentoes the zero coupon price. We say that the short rate model possesses a **flat term structure** if, at every time t, the yield curve is horizontal, i.e. if

$$Y(t, T) = Y(t, t),$$

for all t and T such that $t \leq T$.

(a) Prove that if the term structure is flat then it must hold that

$$Y(t, T) = r_t.$$

.... (5p)

(b) Assume that the term structure is flat for the short rate model above. What can you then say about μ and σ?

.... (15p)

Hint: The term structure equation can be used without proof.

5. Suppose that the stock price (in US dollars) of the US company $ACME$ has the following dynamics under the objective measure P:

$$dS_t = \alpha_S S_t dt + \sigma_S S_t dW_t^1.$$

The exchange rate (SEK per US dollar) is denoted by X. It is assumed to have dynamics of the form

$$dX_t = \alpha_X X_t dt + X_t \sigma_X dW_t^2$$

The entities α_S, α_X, σ_S and σ_X are assumed to be known constants. The P-Wiener processes W^1 and W^2 are assumed to be independent. The US short rate is denoted by r_f and the Swedish short rate is denoted by r_d. They are both assumed to be constant and deterministic.

Your task is to give the price, in Swedish kronor (SEK), of a European call option on $ACME$, with exercise date T and exercise price K SEK. Please note that the stock price is in US dollars whereas the strike price is in SEK. You are allowed to use, without proof, the standard Black-Scholes formula which is given below. ....(20p)

Black-Scholes formula:

$$F(t,s) = sN\left[d_1(t,s)\right] - e^{-r(T-t)} KN\left[d_2(t,s)\right].$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$