Finance II

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Answers and partial solutions

1. Defining the process X by
$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$
the process Z by
$$Z_t = e^{\int_0^t k(s,X_s)ds} F(t, X_t),$$
and doing the usual calculations we obtain
$$F(t, x) = E_{t,x}\left[e^{\int_t^T k(s,X_s)ds}\Phi(X_T)\right]$$

2. See the textbook, Chapter 6.

3. Se the textbook, chapter 12.

4. To start, we have
$$dp(t, T) = \mu(t, T)p(t, T)dt + p(t, T)\gamma(t, T)dW_t,$$
under P, so under a martingale measure we must have
$$dp(t, T) = r_t p(t, T)dt + p(t, T)\gamma(t, T)dW_t.$$
where r is the short rate. The price of the consol is given by
$$C_t = \int_t^\infty p(t, s)ds.$$
Using arguments like in HJM we obtain the C-dynamics
$$dC_t = -p(t, t)dt + \int_t^\infty r_t p(t, s)dtds + \int_t^\infty p(t, s)\gamma(t, s)dW_t ds$$

We thus obtain

$$dC_t = -p(t,t)dt + \left(\int_t^\infty r_t p(t,s) ds \right) dt + \left(\int_t^\infty p(t,s)\gamma(t,s) ds \right) dW_t$$

so

$$dC_t = \{C_t r_t - 1\} dt + \sigma(t) dW_t$$

where

$$\sigma(t) = \int_t^\infty p(t,s)\gamma(t,s) ds.$$

5. By definition, since we are only holding the stock, we have

$$V_t = h_t S_t.$$

We thus have

$$h_t = \frac{V_t}{S_t}$$

Supressing t we obtain the h-dynamics as

$$\begin{aligned} dh &= \frac{1}{S} dV - \frac{V}{S^2} dS + \frac{V}{S^3}(dS)^2 - \frac{1}{S^2} dV dS \\ &= \frac{1}{S} dV - \frac{hS}{S^2} dS + \frac{hS}{S^3}(dS)^2 - \frac{1}{S^2} dV dS \end{aligned}$$

The self financing condition is

$$dV_t = h_t dS_t + h_t dD_t = hS \{(\alpha+\beta)dt + (\sigma+\gamma)dW\}$$

We thus obtain

$$\begin{aligned} dh &= h(\alpha+\beta)dt + h(\sigma+\gamma)dW \\ &- h(\alpha dt + \sigma dW) \\ &+ h\sigma^2 dt \\ &- h(\sigma^2 + \sigma\gamma)dt \end{aligned}$$

so, after simplification,

$$dh_t = h_t(\beta - \sigma\gamma)dt + h_t \gamma dW_t.$$

This is GBM, with solution

$$h_t = e^{(\beta - \sigma\gamma - \frac{1}{2}\gamma^2)t + \gamma W_t}.$$