Fixed Income Analysis

Mortgage-Backed Securities — part II

Risk-Management Issues

Modeling burnout and borrower heterogeneity
The price-rate function
Duration and convexity
Cost of convexity

Jesper Lund

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Modeling burnout – 1

- Consider two scenarios:
  1. Interest rates decrease by 2% after the first period, and increase by 1% between first and second period.
  2. Interest rates increase by 1% after the first period, and decrease by 2% between first and second period.
- In both cases, interest rates have dropped by 1% at time $n = 2$, compared to time $n = 0$.
- Should we expect the same level of prepayments then?
- **Probably not** — in the first scenario there has been some prepayment at time $n = 1$, and the borrowers with the lowest transaction costs are the first to prepay their loans.
- Thus, in scenario 1, the remaining borrowers face higher transaction costs (on average) at $n = 2$.
- This problem is known as **burnout**.

Modeling burnout – 2

- If the prepayment function, $\lambda(n, s)$, depends on past interest rates, the pricing problem is no longer Markovian (but path-dependent).
- We want to use binomial models (and not Monte Carlo simulation), so we cannot model burnout in this way.
- Alternative approach [Jakobsen (1994)]
  - Assume we have N mortgage (sub)pools with different prepayment functions, $\lambda_i(n, s)$, $i = 1, \ldots, N$.
  - Within each (sub)pool, there are **no path-dependencies**.
  - The different pools **could** be determined by loan size (this information is now available in Denmark).
  - In Jakobsen (1994), $N = 2$ and the two pools consists of households and firms (corporations). At that time (1994), the loan size information was not available.

Modeling burnout – 3

- **Within each pool**, we use the binomial model and the MBS backward equation to calculate the price of the MBS, $V_i(0, 0)$.
- With relative weights of each (sub)pool, $w_i(0, 0)$, the price of the MBS today is given by

$$V(0,0) = \sum_{i=1}^{N} w_i(0,0) V_i(0,0) \tag{1}$$

- For pools with above-average prepayment rates (typically corporate borrowers), the relative weight will be reduced over time.
- This means that aggregate prepayment will be reduced (since the low-prepayment pools get a greater weight) if there has been prepayment in the past.
- Thus, even though there are no path-dependencies within each pool, we incorporate the burnout feature in the model.

Introduction to risk management

- The importance of this topic cannot be understated . . .
- Before we can talk about risk **management**, we must talk about risk **measurement**. Today, we concentrate on the latter.
- Risk measurement is also important for hedging (reducing risk) or selective risk exposure (hedge funds). For example, buying a MBS and hedging the general interest-rate risk by shorting T-bonds.
- For securities with fixed payments (non-callable bonds), **duration** is the most widely used measure of risk.
- However, for many fixed-income securities the cash flows are stochastic (depend on the evolution of interest rates).
- In general, we need a term-structure model to compute "something like duration" for these securities.

The price-rate function $P(y)$ – 1

- Basic assumption: the term-structure is governed by a one-factor model with state variable y.
- The price of a given fixed-income security is a function of y, denoted $P(y)$.
- We call $P(y)$ the **price-rate function** — since y is an interest rate in most cases.
- What happens to the price if y changes to, say, $y + \Delta$?
- First order Taylor-series approximation:

$$P(y + \Delta) \approx P(y) + P'(y)\Delta \qquad (2)$$

- Computation of $P(y)$ and its derivative:
  - Valuation techniques discussed earlier (forward-risk adjusted measure, binomial and trinomial trees, Monte Carlo simulation, etc).
  - Empirical approaches (curve-fitting using historical data).

The price-rate function $P(y)$ – 2

- We can also express (2) in relative terms

$$\frac{P(y+\Delta) - P(y)}{P(y)} \approx \frac{P'(y)}{P(y)}\Delta = -D(y)\Delta \tag{3}$$

  where $D(y) = -P'(y)/P(y)$ is the new **duration** measure.

- Simple example: coupon-bearing bond with fixed payments, $\{c_i\}$, and y is the yield-to-maturity on the bond

$$P(y) = \sum_{i=1}^{n} c_i \exp[-y t_i] \tag{4}$$

- Here, duration is given by the well-know formula

$$D(y) = \sum_{i=1}^{n} c_i t_i \exp[-y t_i]/P(y) \tag{5}$$

Hedging with duration

- Duration is a relative measure, but $P'(y)$ (sometimes called dollar duration when the sign is changed) is more useful for calculating **hedge ratios**.
- Suppose that we have a portfolio of two assets (the number of each asset is w_1 and w_2) and that both prices depend on y,

$$V(y) = w_1 P_1(y) + w_2 P(y) \tag{6}$$

- First-order approximation to the change in value

$$V(y+\Delta) - V(y) = V'(y)\Delta = \left\{ w_1 P_1'(y) + w_2 P_2'(y) \right\} \Delta \tag{7}$$

- The portfolio is riskless (approximately) if

$$w_2/w_1 = -P_1'(y)/P_2'(y) \equiv H_{12}(y). \tag{8}$$

- $H_{12}(y)$ is called the hedge ratio between securities 1 and 2.

Convexity

- In general, a second-order approximation is more accurate

$$\frac{P(y+\Delta)-P(y)}{P(y)} \approx -D(y)\Delta + \frac{1}{2}C(y)\Delta^2 \tag{9}$$

- In equation (9), $C(y)$ is **convexity**,

$$C(y) = \frac{d^2P(y)/dy^2}{P(y)} \tag{10}$$

- If $C(y) > 0$, the second term on the RHS of (9) is always positive — no matter the sign of Δ.
- This means that positive convexity is desirable — other things equal.
- You don't get anything for free — we need to look at the **cost of convexity**.

Cost of convexity – 1

- Assumptions:
  - The current term structure is flat, and $r = 0.10$.
  - The term-structure is governed by the Ho-Lee model with $\sigma = 0.02$.
- Prices of zero-coupon bonds today,

$$P(0,T) = \exp[-rT] = \exp[-0.1 \cdot T] \tag{11}$$

- Duration and convexity of a zero:

$$D(0,T) = \frac{-dP(0,T)}{dr}/P(0,T) = T \tag{12}$$

$$C(0,T) = \frac{-d^2P(0,T)}{dr^2}/P(0,T) = T^2 \tag{13}$$

- Consider two portfolios: **Portfolio A** has 100% in the 15Y zero, and **Portfolio B** has 50% each in the 5Y and 25Y zeros.

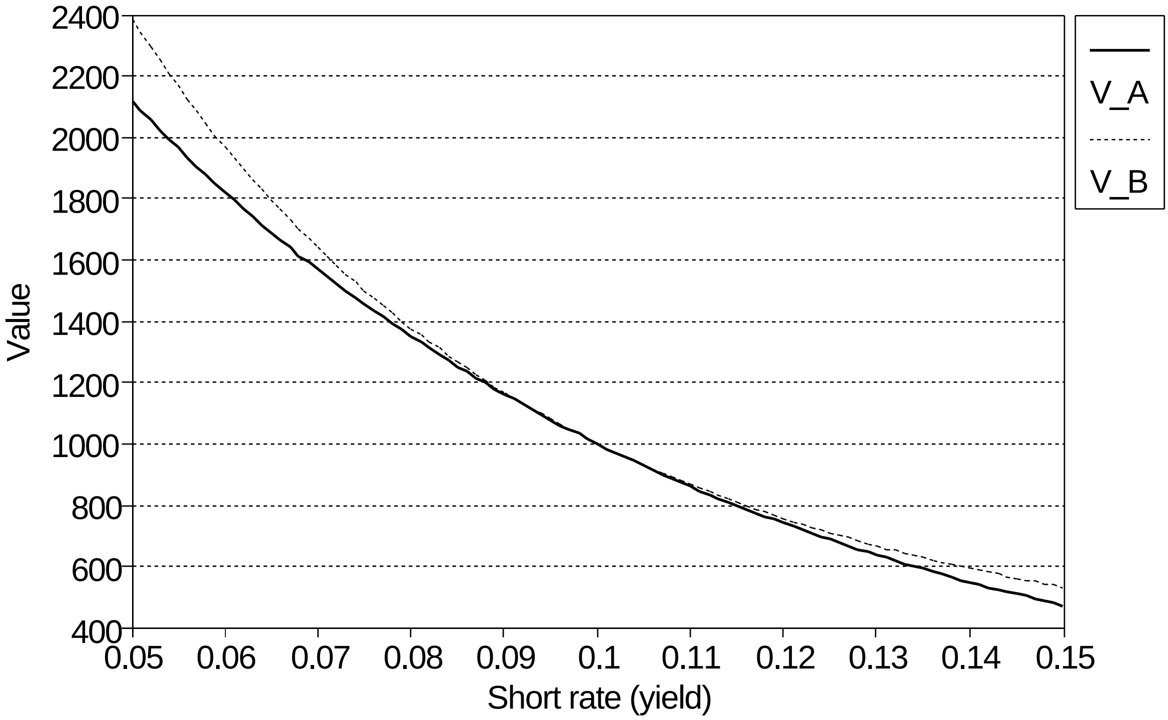
Cost of convexity – 2

- Duration and convexity for the two portfolios:

Portfolio	5Y	15Y	25Y	Duration	Convexity
A	0	100	0	15	225
B	50	0	50	15	325

- The duration and convexity on portfolio B are calculated as follows: $D = 0.5 \cdot (5 + 25) = 15$ and $C = 0.5 \cdot (5^2 + 25^2) = 325$.
- Portfolio B has the same duration as A, but higher convexity.
- According to (9) — and Figure 1 — this means that the return on portfolio B is greater than on A — no matter whether interest rates go up or down (positive or negative Δ).

Figure 1: value of A and B at t=0 as a function of yields (static analysis)



Cost of convexity – 3

- **Question:** why don't we all buy portfolio B and short A? According to Figure 1, the worst than can happen is that we don't gain anything (if interest rates don't move).
- **Answer:** we are misinterpreting a **static** analysis.
- We really need a **dynamic** analysis — which will show that the yield curve cannot continue to be flat (as we assume in Figure 1).
- Absence of arbitrage requires that for all bonds (all T)

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\frac{\sigma^2}{2}(T-t)^2 t - (T-t)\left(r_t - f(0,t)\right)\right]. \qquad (14)$$

- In Figure 2 we use this formula to compute the value (at $t = 1$) of portfolios B and A for different future short rates, r_t.
- The risk of B is clear now — if rates **don't change enough**, we lose money compared to A.

Figure 2: difference btw. V_B and V_A
at time t=1 as a function of r(1)