Review of Continuous-Time Term-Structure Models

Part II: Arbitrage-Free Models

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1 Introduction

Part II of the lecture notes "Review of Continuous-Time Term-Structure Models" is about arbitrage-free models — defined here as models which fit the initial yield curve exactly, cf. Tuckman (1995, ch. 9). We consider two types of models, namely equilibrium-style models with time-dependent parameters, also called calibrated models, and the Heath, Jarrow and Morton (HJM) modeling framework. In both cases, we focus on one-factor versions of the requisite models. Throughout, the reader is assumed to be familiar with the theory covered in part I of the lecture notes [Lund (1998)].

2 One-factor models and calibration

The wide-spread popularity of one-factor equilibrium models, such as the Vasicek model, stems from their simplicity. At each date, today and in the future, the entire yield curve is a function of a single state variable, the short rate. This feature is very useful when implementing numerical solutions, e.g., when constructing binomial trees to approximate the continuous-time model. However, equilibrium models do not fit the current yield curve exactly, and this tends to limit their effectiveness for pricing fixed-income derivatives. By introducing time-dependent parameters in the model, we can match the current yield curve, while retaining the overall simplicity of the term-structure model. This approach is advocated by, especially, Hull and White (1990) who extend the Vasicek and CIR models with time-dependent parameters. Sometimes, these models are referred to as the *extended* Vasicek and CIR models.

2.1 Vasicek model with time-dependent drift

Under the risk-neutral measure Q, the short rate in the ordinary Vasicek (1977) model evolves according to the SDE

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^Q, \tag{1}$$

where $\theta = \mu - \lambda\sigma/\kappa$ is the risk-neutral mean. Prices of fixed-income derivatives only depend on the distribution of r_t under the Q-measure, so in the following we do not care about the process under the original probability measure (and the true drift).

In general, the model (1) will not fit the current ($t = 0$) yield curve exactly. Therefore, we augment the risk-neutral process with a time-dependent mean $\theta(t)$,

$$dr_t = \kappa(\theta(t) - r_t)dt + \sigma dW_t^Q. \tag{2}$$

The solution to the SDE (2) can be written as

$$r_t = e^{-\kappa t}r_0 + \int_0^t e^{-\kappa(t-s)}\kappa\theta(s)ds + \sigma\int_0^t e^{-\kappa(t-s)}dW_s^Q, \tag{3}$$

see Arnold (1973) or Øksendal (1992). If we define $m(t)$ and x_t by

$$m(t) = e^{-\kappa t} r_0 + \int_0^t e^{-\kappa(t-s)} \kappa\theta(s) ds \tag{4}$$

and

$$dx_t = -\kappa x_t dt + \sigma dW_t^Q, \qquad \text{with } x_0 = 0, \tag{5}$$

respectively, we can write r_t as

$$r_t = m(t) + x_t. \tag{6}$$

Of course, the representations (3) and (6) are equivalent, but the calibration is more straightforward in the latter case, so we use (6) in the sequel.

Absence of arbitrage implies that the price of a zero-coupon bond is given by the risk-neutral expectation:

$$\begin{aligned} P(t,T) &= E_t^Q \left[e^{-\int_t^T r_s ds} \right] \\ &= e^{-\int_t^T m(s)ds} \cdot E_t^Q \left[e^{-\int_t^T x_s ds} \right] \\ &= \exp\left(-\int_t^T m(s)ds \right) \cdot \exp\left[A(T-t) + B(T-t)x_t \right], \end{aligned} \tag{7}$$

where

$$B(\tau) = \frac{e^{-\kappa\tau} - 1}{\kappa} \tag{8}$$

$$A(\tau) = \frac{1}{2}\sigma^2 \int_0^\tau B^2(s)ds = \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2 \left[\frac{1 - e^{-2\kappa\tau} - 4\left(1 - e^{-\kappa\tau}\right)}{2\kappa} + \tau \right]. \tag{9}$$

Note that $P(t,T)$ in (7) is written as the product of a deterministic factor and the bond price in an ordinary Vasicek model with zero mean (under Q).

2.2 Calibration of the time-dependent parameters

We want to fit the initial (current) yield curve, represented by the discount function $d(T)$. That is,

$$P(0,T) = \exp\left[-\int_0^T m(s)ds + A(T) \right] = d(T), \tag{10}$$

since $x_0 = 0$ by the normalization above. From (10) we get

$$\int_0^T m(s)ds = -\log d(T) + A(T), \tag{11}$$

and after differentiating with respect to T on both sides of the equation we arrive at

$$m(T) = -\frac{d \log d(T)}{dT} + \frac{dA(T)}{dT} = f(0,T) + \frac{1}{2}\sigma^2 B^2(T). \tag{12}$$

This means that $m(T)$ is obtained from the initial forward curve, $f(0,T)$. The time-invariant parameters, κ and σ, must, of course, be specified prior to this calculation. In principle, κ and σ can be backed out from market prices of, e.g., interest-rate caps, but such calculations are outside the scope of the present paper.

In order to determine $\theta(t)$, first note that the derivative of $m(t)$ is given by:

$$\begin{aligned} m'(t) &= -\kappa e^{-\kappa t} r_0 + \kappa\theta(t) - \kappa \int_0^t e^{-\kappa(t-s)} \kappa\theta(s) ds \\ &= \kappa\theta(t) - \kappa m(t). \end{aligned} \tag{13}$$

Using this result, and the definition of $m(t)$ in (12) above, we have

$$\begin{aligned} \kappa\theta(t) &= \kappa m(t) + m'(t) \\ &= \kappa f(0,t) + \frac{1}{2}\kappa\sigma^2 B^2(t) + \frac{\partial f(0,t)}{\partial t} + \sigma^2 B(t) B'(t) \\ &= \kappa f(0,t) + \frac{\partial f(0,t)}{\partial t} + \phi(t), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \phi(t) &= \frac{1}{2}\kappa\sigma^2 B^2(t) + \sigma^2 B(t) B'(t) \\ &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right) \;=\; \mathrm{Var}_0\left[r_t\right] \end{aligned} \tag{15}$$

is the conditional variance of r_t given r_0 (the proof that the first line of (15) simplifies to the second line is somewhat lengthy and tedious, so it is left out here). Finally, we can write the SDE for r_t in the following way:

$$dr_t = \left\{\kappa(f(0,t) - r_t) + \frac{\partial f(0,t)}{\partial t} + \phi(t)\right\} dt + \sigma dW_t^Q, \tag{16}$$

which illustrates how the time-dependent parameters of the SDE are obtained from the initial yield curve, or rather forward curve $f(0,t)$.

2.3 Distribution of future bond prices

The purpose of calibrating the drift $\theta(t)$ to the initial yield curve is, of course, pricing fixed-income derivatives at time $t = 0$. Consider a claim with a single payoff, depending on the term structure at time t, for example a call option on a zero-coupon bond maturing at time T, with exercise (strike) price K. Here, the uncertain payoff is given by

$$C(r_t) = \max\left\{P(r_t,t,T) - K, 0\right\},$$

and the current value (price) of the claim is

$$V_0 = E_0^Q\left[e^{-\int_0^t r_s ds} C(r_t)\right]. \tag{17}$$

Methods for calculating the price (17) in closed form (if possible), or implementing appropriate numerical procedures, will be addressed later. For our present purposes, it suffices to note that we need the time $t = 0$ distribution of the future bond price, $P(t,T)$, which again depends solely on r_t (besides the deterministic parameters) because of the one-factor assumption.

From (7), the bond price at time t is given by

$$P(t,T) = \exp\left[-\int_t^T m(s)ds + A(T-t) + B(T-t)(r_t - m(t))\right], \tag{18}$$

where $m(s)$, $t \leq s \leq T$, is obtained from the calibration to the initial term structure.[1] In the following, we rewrite this expression to something that is (hopefully) easier to interpret. First, note that the forward price of the T-maturity bond is given by:

$$\begin{aligned} \frac{P(0,T)}{P(0,t)} &= \frac{\exp\left[-\int_0^T m(s)ds + A(T)\right]}{\exp\left[-\int_0^t m(s)ds + A(t)\right]} \\ &= \exp\left[-\int_t^T m(s)ds + A(T) - A(t)\right]. \end{aligned} \tag{19}$$

Second, using (19) allows us to write (18) as

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[A^*(t,T) + B(T-t)(r_t - m(t))\right], \tag{20}$$

where $A^*(t,T) = A(T-t) + A(t) - A(T)$. After many lengthy calculations (literally), we obtain the following formula for $A^*(t,T)$:

$$A^*(t,T) = -\frac{1}{2}B^2(T-t)\phi(t) + \frac{1}{2}\sigma^2 B(T-t)B^2(t). \tag{21}$$

Finally, since $m(t) = f(0,t) + \frac{1}{2}\sigma^2 B(t)$, we get

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\frac{1}{2}B^2(T-t)\phi(t) + B(T-t)(r_t - f(0,t))\right], \tag{22}$$

which only involves the current forward curve, $f(0,t)$. As a digression, this expression for $P(t,T)$ will be useful when investigating the relationship between forward and futures prices for zero-coupon bonds.[2]

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[1] It is important to understand that we are looking at the distribution of the future bond price, given the current ($t = 0$) information. Once we observe $P(t,T)$, we can re-calibrate the function $m(s)$, for $s \geq t$, and the new function will generally differ from the one obtained from $f(0,t)$. This is the inherent inconsistency of the calibration approach, see Tuckman (1995, ch. 9) for further discussion. However, we are only interested in prices of contingent claims at time $t = 0$, which leads us to ignore the problem.

[2] It follows from, e.g., Cox et al. (1981) that the price of a futures contract is given by $E_0^Q\left[P(t,T)\right]$. Since r_t is normally distributed with mean $m(t) = f(0,t) + \frac{1}{2}\sigma^2 B^2(t)$ and variance $\phi(t)$, the futures price reduces to

$$E_0^Q\left[P(t,T)\right] = \frac{P(0,T)}{P(0,t)} \exp\left[\frac{1}{2}\sigma^2 B(T-t)B^2(t)\right],$$

which is less than the corresponding forward price since $B(\tau) < 0$ for all τ.

2.4 Calibration in other cases

In the above model, only the drift parameter $\theta(t)$ is time-dependent, whereas σ and κ are still time-invariant (constant) parameters. This suggests the following extension of the model,

$$dr_t = \kappa(t)\left\{\theta(t) - r_t\right\} dt + \sigma(t) dW_t^Q, \tag{23}$$

or an equivalent generalization of the CIR model

$$dr_t = \kappa(t)\left\{\theta(t) - r_t\right\} dt + \sigma(t)\sqrt{r_t} dW_t^Q. \tag{24}$$

One advantage of letting $\sigma(t)$ or $\kappa(t)$ be time-dependent is that the model can match the current volatility structure, in addition to the current yield curve.[3] Not surprisingly, this additional generality comes at a cost, namely that the calibration of the time-dependent parameters becomes much more complex. In particular, there is no longer a simple relationship between $\theta(t)$ and the initial forward curve $f(0,t)$, as in equation (14) above. We refer the interested reader to Hull and White (1990) who analyze the models (23) and (24) and discuss different approaches for calibration of the time-dependent parameters.

In practice, however, the calibration is most often done within a binomial (or trinomial) approximation to the continuous-time model (SDE), using the principle of forward induction, see, e.g., Jakobsen (1992) and Hull and White (1993).

3 The Heath, Jarrow and Morton model

The Heath, Jarrow and Morton (HJM) (1992) framework is similar to the calibration approach in the sense that we can match an arbitrary initial yield curve exactly. With calibrated models (section 2), the starting point is a SDE for r_t with time-dependent parameters, and the yield curve and volatility structure are determined endogenously from the time-dependent parameters. In HJM models, on the other hand, the initial forward curve and volatility structure are specified directly (exogenously). We use the no-arbitrage assumption to derive restrictions on the future movements of the forward curve, which facilitates using the HJM model for pricing fixed-income derivatives.

3.1 A general one-factor HJM model

Under the true probability measure, the forward curve $f(t,T)$ evolves according to

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \quad \text{for all } T \geq t, \tag{25}$$

where $\sigma(t,T)$ is the forward-rate volatility (volatility structure), and $\alpha(t,T)$ the drift of the forward curve. Note that all maturities are affected by the same Brownian

[3] There are several ways to define the volatility structure, or term-structure of volatilities. Generally, we use the volatility of the forward or yield curve, and these volatilities can either be estimated from historical data or backed out from prices of contingent claims, for example interest-rate caps.

motion W_t, that is changes in the forward curve are perfectly correlated (one factor model). The short rate r_t is implicitly defined as the forward rate with $T = t$, that is

$$r_t = f(t,t). \tag{26}$$

From the definition of instantaneous forward rates, the logarithm of bond prices are given by the expression

$$\log P(t,T) = -\int_t^T f(t,u)du. \tag{27}$$

In order to price fixed-income derivatives, we need the distribution of $f(t,T)$, and hence bond prices, under the risk-neutral measure (Q-measure). This involves imposing conditions on the forward-rate dynamics (under Q), so that they are consistent with absence of arbitrage opportunities. We begin by determining the stochastic differential equation for (log) bond prices under the true probability measure

$$\begin{aligned} d\log P(t,T) &= f(t,t)dt - \int_t^T df(t,u)du \\ &= r_t dt - \int_t^T \{\alpha(t,u)dt + \sigma(t,u)dW_t\}\, du \\ &= \left\{r_t - \int_t^T \alpha(t,u)du\right\} dt - \left\{\int_t^T \sigma(t,u)du\right\} dW_t. \end{aligned} \tag{28}$$

An application of Ito's lemma provides the SDE for $P(t,T)$,

$$dP(t,T) = \mu_P(t,T)P(t,T)dt + \sigma_P(t,T)P(t,T)dW_t, \tag{29}$$

where

$$\sigma_P(t,T) = -\int_t^T \sigma(t,u)du \tag{30}$$

$$\mu_P(t,T) = r_t - \int_t^T \alpha(t,u)du + \frac{1}{2}\sigma_P(t,T)^2. \tag{31}$$

Since all bond prices are driven by the same Brownian motion (perfect correlation), absence of arbitrage implies the APT restriction

$$\mu_P(t,T) = r_t + \lambda(t)\sigma_P(t,T), \quad \text{for all } T, \tag{32}$$

where $\lambda(t)$ is the market price of risk at time t. Using equations (30) and (31), the no-arbitrage condition can be written as

$$\int_t^T \alpha(t,u)du = \lambda(t)\int_t^T \sigma(t,u)du + \frac{1}{2}\left(\int_t^T \sigma(t,u)du\right)^2. \tag{33}$$

If we differentiate with respect to T on both sides of the equation, we get

$$\alpha(t,T) = \lambda(t)\sigma(t,T) + \sigma(t,T)\int_t^T \sigma(t,u)du, \tag{34}$$

which means that the forward-rate drift must be a function of the volatility structure and the market price of risk, $\lambda(t)$. Thus, under the true probability measure, the SDE for forward rates is given by

$$df(t,T) = \left[\lambda(t)\sigma(t,T) + \sigma(t,T)\int_t^T \sigma(t,u)du\right]dt + \sigma(t,T)dW_t. \tag{35}$$

At this stage, two things are worth emphasizing. First, the SDE is still specified under the true (original) probability measure. Second, the drift of $f(t,T)$ depends on $\lambda(t)$, that is investor preferences. If W_t^Q is a Brownian motion under the Q-measure, we have the following relationship to the true probability measure,

$$dW_t^Q = dW_t + \lambda(t)dt, \tag{36}$$

and after substituting this into (35) we obtain the SDE under the Q-measure (risk-neutral distribution),

$$\begin{aligned} df(t,T) &= \left[\sigma(t,T)\int_t^T \sigma(t,u)du\right]dt + \sigma(t,T)dW_t^Q \\ &= -\sigma(t,T)\sigma_P(t,T)dt + \sigma(t,T)dW_t^Q, \end{aligned} \tag{37}$$

where the second line follows from (30). Note that the drift of this SDE is independent of $\lambda(t)$, so valuation of fixed-income derivatives is truly preference-free.[4] The reason for this independence is that we are pricing interest-rate derivatives relative to the current yield curve (forward curve), and the yield curve reflects all relevant investor preferences.

There are two inputs to a HJM model: the initial forward curve, $f(0,T)$, and the volatility structure, $\sigma(t,T)$. The first is simply the current ($t=0$) forward curve, whereas the latter must be specified somehow — either from historical estimates or backed out from prices of interest-rate derivatives (like "implied volatility" in the Black-Scholes model).

In any case, the purpose of using the HJM model is pricing (new) derivatives whose payoff depend on the future term structure. Therefore, it is convenient to have an expression like (22) for the HJM model. Starting from the initial forward curve, $f(0,s)$, and using the SDE under the Q-measure (37), we have that

$$\begin{aligned} \log P(t,T) &= -\int_t^T f(t,s)ds \\ &= \int_t^T \left\{-f(0,s) - \int_0^t df(u,s)\right\}ds \\ &= \int_t^T \left\{-f(0,s) + \int_0^t \sigma(u,s)\sigma_P(u,s)du - \int_0^t \sigma(u,s)dW_u^Q\right\}ds. \end{aligned} \tag{38}$$

Second, note that

$$-\int_t^T f(0,s)ds = -\int_0^T f(0,s)ds + \int_0^t f(0,s)ds = \log\left[\frac{P(0,T)}{P(0,t)}\right]. \tag{39}$$

[4] We can compare this to the Black-Scholes model where the price of a call option is independent of the drift of the stock price (expected return).

Finally, by combining (38) and (39), we get

$$
\begin{aligned}
P(t,T) &= \frac{P(0,T)}{P(0,t)} \times \\
&\exp\left\{\int_t^T \int_0^t \sigma(u,s)\sigma_P(u,s)du\,ds - \int_t^T \left[\int_0^t \sigma(u,s)dW_u^Q\right] ds\right\}. \qquad (40)
\end{aligned}
$$

As in (22), we can express $P(t,T)$ as the forward price multiplied by a random factor. However, contrary to the Vasicek model with time-dependent drift, the random factor involves the entire path of the Brownian motion, W_s^Q, between 0 and t, not just a single random variable, like the short rate at time t. This path-dependency can be problematic, especially when implementing numerical approximations to the HJM model. For example, binomial trees cannot be recombining (so the dimension grows exponentially with the number of time steps), and when using Monte Carlo methods, we must simulate the movements of the entire forward curve which is more time-consuming that simulating the path of a single state variable.

3.2 Examples of HJM models

Our discussion of the HJM model thus far has been cast in general terms, and we have not presented any specific models, that is parameterizations of the volatility structure, $\sigma(t,T)$. A thorough discussion of the pros and cons of different specifications is outside the scope of the present paper, so we just give a few examples. First, the Vasicek model corresponds to

$$\sigma(t,T) = \sigma e^{-\kappa(T-t)}, \qquad (41)$$

where the volatility structure is monotonically decaying unless $\kappa = 0$ (which is the Ho and Lee (1986) model in continuous time). Some empirical studies have found that the volatility structure is humped (first increasing, then decreasing from some point). This can be accommodated with the following specification:

$$\sigma(t,T) = [\sigma_0 + \sigma_1(T-t)]\, e^{-\kappa(T-t)} f(t,T)^{\gamma}. \qquad (42)$$

Moreover, if $\gamma \neq 0$, the forward-rate volatilities depend on the level of forward rates. This is a form of conditional heteroskedasticity, similar to the CIR model where the short-rate volatility is proportional to the square root of the short rate.

3.3 Markovian HJM models

Having noted the potential difficulties with path-dependencies in the general HJM model in section 3.1, we turn to a special case of the HJM model where the term structure can always be expressed as a function of a finite number of state variables. We obtain this case, known as *Markovian* HJM models, by imposing certain restrictions on the volatility structure. In any model, bond prices are given by

$$P(t,T) = E_t^Q\left[e^{-\int_t^T r_s ds}\right], \qquad (43)$$

so a Markovian HJM model (as defined above) corresponds to a Markovian stochastic process for the short rate (under Q). The bond price in (43) is given by a conditional expectation, and if the conditional distribution of r_s only depends on a finite number of state variables X_t (the Markov property), the bond price will be a function of these state variables.

We start by writing the short rate of the general HJM model in differential form. The short rate is given by

$$r_t = f(t,t) = f(0,t) - \int_0^t \sigma(s,t)\sigma_P(s,t)ds + \int_0^t \sigma(s,t)dW_s^Q, \tag{44}$$

and the differential form with respect to t is

$$\begin{aligned} dr_t &= \left[\frac{\partial f(0,t)}{\partial t} - \sigma(t,t)\sigma_P(t,t) - \int_0^t \frac{\partial \sigma(s,t)}{\partial t}\sigma_P(s,t)ds \right. \\ &\quad \left. - \int_0^t \sigma(s,t)\frac{\partial \sigma_P(s,t)}{\partial t}ds + \int_0^t \frac{\partial \sigma(s,t)}{\partial t}dW_s^Q\right] dt + \sigma(t,t)dW_t^Q. \end{aligned} \tag{45}$$

Since $\sigma_P(t,t) = 0$ and $\partial\sigma_P(s,t)/\partial t = -\sigma(s,t)$, this reduces to

$$\begin{aligned} dr_t &= \left[\frac{\partial f(0,t)}{\partial t} - \int_0^t \frac{\partial \sigma(s,t)}{\partial t}\sigma_P(s,t)ds + \int_0^t \frac{\partial \sigma(s,t)}{\partial t}dW_s^Q \right. \\ &\quad \left. + \int_0^t \sigma^2(s,t)ds\right] dt + \sigma(t,t)dW_t^Q. \end{aligned} \tag{46}$$

Note how the drift is path-dependent since it involves the entire path of the Brownian motion W_s^Q between 0 and t. However, if the volatility structure satisfies the restriction

$$\frac{\partial \sigma(s,t)}{\partial t} = -\kappa(t)\sigma(s,t) \tag{47}$$

for some $\kappa(t)$, the differential form (46) simplifies to

$$\begin{aligned} dr_t &= \left[\frac{\partial f(0,t)}{\partial t} + \kappa(t)\int_0^t \sigma(s,t)\sigma_P(s,t)ds - \kappa(t)\int_0^t \sigma(s,t)dW_s^Q \right. \\ &\quad \left. + \int_0^t \sigma^2(s,t)ds\right] dt + \sigma(t,t)dW_t^Q, \end{aligned} \tag{48}$$

which in light of the identity

$$r_t - f(0,t) = -\int_0^t \sigma(s,t)\sigma_P(s,t)ds + \int_0^t \sigma(s,t)dW_s^Q,$$

can be written as

$$dr_t = \left\{\frac{\partial f(0,t)}{\partial t} + \kappa(t)(f(0,t) - r_t) + \phi_t\right\} dt + \sigma(t,t)dW_t^Q, \tag{49}$$

with

$$\phi_t = \int_0^t \sigma^2(s,t)ds. \tag{50}$$

The differential form of ϕ_t is

$$
\begin{aligned}
d\phi_t &= \left(\sigma^2(t,t) + 2\int_0^t \sigma(s,t)\frac{\partial\sigma(s,t)}{\partial t}ds\right)dt \\
&= \left(\sigma^2(t,t) - 2\kappa(t)\phi_t\right)dt, \qquad (51)
\end{aligned}
$$

and there are no path-dependencies in the drift of either (49) and (51).

In summary, if the volatility structure $\sigma(s,t)$ obeys the restriction (47), the short rate is governed by a bivariate Markovian SDE consisting of equations (49) and (51), respectively. The mean reversion parameter $\kappa(t)$ in (49) is time-varying, and the coefficient is obtained from the volatility structure. Moreover, if $\sigma(s,t)$ is non-stochastic, ϕ_t in (50) reduces to a time-dependent function, and the short-rate dynamics become equivalent to the extended Vasicek process (23), cf. Hull and White (1993).

The Markov restriction on the volatility structure (47) has the form of an ordinary differential equation. Specifically,

$$\frac{\partial\sigma(s,u)/\partial u}{\sigma(s,u)} = \frac{\partial \log\sigma(s,u)}{\partial u} = -\kappa(u), \quad \text{for } u \geq s. \qquad (52)$$

Integrating from $u = s$ to $u = t$ in (52) yields

$$\log\sigma(s,t) - \log\sigma(s,s) = -\int_s^t \kappa(u)du,$$

which can be written as

$$\sigma(s,t) = \sigma(s,s)e^{-\int_s^t \kappa(u)du}, \qquad (53)$$

where $\sigma(s,s)$ is the short-rate volatility at time s, cf. (49). Thus, the volatility structure has to be of the form (53) in order to obtain a Markovian HJM model. Note that $\sigma(s,s)$ can depend on the short rate at time s, so it does not have to be a deterministic function. For example, the following specification can be used

$$\sigma(s,t) = \sigma r_s^\gamma e^{-\kappa(t-s)}, \qquad (54)$$

which reduces to the Vasicek volatility structure if $\gamma = 0$. On the other hand, a specification like (42), where the forward-rate volatility depend on the level of the forward rate itself, is not consistent with a Markovian HJM model.

To complete this section, we turn to the distribution of future (time t) bond prices. If the volatility structure is of the form (53), it can be shown that (40) simplifies to

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left\{-\frac{1}{2}\beta^2(t,T)\phi_t + \beta(t,T)\left[f(0,t) - r_t\right]\right\}, \qquad (55)$$

where

$$\beta(t,T) = -\frac{\sigma_P(t,T)}{\sigma(t,t)} = \int_t^T e^{-\int_t^s \kappa(u)du}ds, \qquad (56)$$

and ϕ_t is defined above. See Ritchken and Sankarasubramanian (1995) for a proof. If $\kappa(u)$ is constant, we get a closed-form expression for (56),

$$\beta(t,T) = \int_t^T e^{-\kappa(s-t)}ds = \frac{1 - e^{-\kappa(T-t)}}{\kappa}, \qquad (57)$$

which, apart from the sign, is the factor loading in the Vasicek model.

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