Fixed Income Analysis

Term-Structure Models in Continuous Time

Multi-factor equilibrium models (general theory)
The Brennan and Schwartz model
Exponential-affine models

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Outline

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One-Factor Models – 1

- Key features of one-factor (equilibrium) models:
  - All bond prices are a function of a **single** state variable, the short rate.
  - The short rate evolves according to the univariate SDE:

$$dr_t = \mu(r_t)dt + \sigma(r_t)dW_t \,. \qquad (1)$$

  - Using the "absence of arbitrage" assumption and Ito's lemma, we derive a PDE for bond prices:

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2(r) \; + \; \frac{\partial P}{\partial r}\left[\mu(r) - \lambda(r)\sigma(r)\right] \; + \; \frac{\partial P}{\partial t} \; - \; rP = 0, \qquad (2)$$

    with boundary condition $P(T, T) = 1$.

- Advantages of one-factor models:
  - **Simple** model — with a limited number of parameters
  - The state variable (short rate) is **observable**, at least in principle.
  - Numerical solutions (e.g. binomial trees) can be implemented, if necessary.

One-Factor Models – 2

- Problems with one-factor models:
  - Changes in the yield curve are **perfectly correlated** across different maturities.
  - Shape of the yield curve highly restricted (monotonic increasing and decreasing, and hump-shaped, but **not** inversely hump-shaped).
  - Model unable to fit the actual yield curve (when the model parameters are time-invariant, as they are supposed to be). Cause of concern for pricing derivatives (e.g., mortgage-backed securities).

- Solutions:
  - Calibrated one-factor models with time-dependent parameters (advocated by Hull and White (1990) as modifications of the Vasicek and CIR models).
  - Alternatively: HJM models which fit the initial yield curve per construction.
  - Models with multiple factors (but still with time-invariant parameters).

Solution 1: Calibrated one-factor models

- For example, Vasicek with time-dependent drift
$$dr_t = \kappa \{\theta(t) - r_t\} dt + \sigma dW_t^Q, \tag{3}$$
where $\theta(t)$ is chosen to fit the current yield curve exactly.
- Problems solved:
  1. Perfect fit to the current yield curve (including any bond mispricing).
  2. Any shape of the **current** yield curve can be accommodated.
- Problems remaining and new problems:
  1. Still a one-factor model with perfect-correlation assumption. Inadequate for certain derivatives, e.g. options on yield spreads [Canabarro (1995)].
  2. The approach is (inherently) useless for detecting **mispricing of bonds**.
  3. Model will not fit future yield curves, unless parameters are re-calibrated.
  4. Hedging and risk-management applications are problematic — because of the "perfect correlation" assumption.

Solution 2: Multi-Factor Models

- Main assumptions:
  - All bond prices are a function of a m-dimensional state vector X_t.
  - The short rate is a **known** function of X_t, that is $r_t = r(X_t)$.
  - The state variables in X_t evolve according to the multivariate SDE
$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{4}$$
where W_t is an m-dimensional Brownian motion, and $\sigma(X_t)$ diagonal.
- Problems with multi-factor models:
  1. Changes in yield curve are no longer perfectly correlated, but they still lie in an m-dimensional subspace (a great improvement, of course).
  2. We may need "many" factors to fit the entire yield curve.
  3. Factors are, in principle, **unobservable**. What is X_t anyway?
  4. Finding an analytical solution for bond prices may be difficult.
  5. Numerical solutions (for derivatives) can be computationally involved.

Multi-Factor Models — How?

- Without loss of generality, the **short rate** can be taken as one of the m state variables, since $r_t = r(X_t)$.
- Under no-arbitrage assumption all bond prices (still) satisfy:

$$P(t,T) = E_t^Q \left[e^{-\int_t^T r_s ds} \right], \tag{5}$$

  where Q denotes the risk-neutral distribution. **Note:** the risk-neutral process for r_t has yet to be determined.
- Using "traded assets" as additional state variables?
  - Examples: 30Y yield or the consol yield (Brennan-Schwartz model).
  - We must specify how the state variables affect r_t under the Q-measure.
  - Parameter restrictions, since (5) must hold for these assets also.

Multivariate SDEs

- Multivariate SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{6}$$

  where $X_t = (X_{1t}, \ldots, X_{mt})$.
- The i'th row of (6) is a univariate SDE, whose drift and volatility functions depend on all m state variables:

$$dX_{it} = \mu_i(X_t)dt + \sigma_i(X_t)dW_{it}. \tag{7}$$

  In this setup, the m univariate Brownian motions can be correlated, with Corr$(dW_{it}, dW_{ij}) = \rho_{ij}dt$.
- Consider a scalar function, F(X,t), representing a mapping from $R^m \times R$ to the real line, R. The dynamics of $F(X,t)$ are obtained by applying a multivariate version of **Ito's lemma**.

Ito's lemma (multivariate)

- If X_t evolves according to the vector SDE (6), the function F, given by $F = F(X,t)$ follows the univariate SDE:

$$dF_t = \alpha(X_t)dt + \sum_{i=1}^{m} \beta_i(X_t)dW_{it}, \tag{8}$$

- The drift in (8) is given by:

$$\alpha(X) = \sum_{i=1}^{m} \frac{\partial F}{\partial X_i}\mu_i(X) + \frac{\partial F}{\partial t} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 F}{\partial X_i \partial X_j}\sigma_i(X)\sigma_j(X)\rho_{ij}, \tag{9}$$

where $\rho_{ii} = 1$.

- The i'th volatility coefficient in (8) is given by:

$$\beta_i(X) = \frac{\partial F}{\partial X_i}\sigma_i(X). \tag{10}$$

A General Multi-Factor Model – 1

- As in the one-factor case, we determine (endogenously) the relationship between X_t and bond prices, $P(t,T)$.
- Since $P(t,T)$ is a function of X_t and t,

$$d\,P(t,T) = \mu_P(t,T)P(t,T)dt \;+\; \sum_{i=1}^{m} \sigma_{Pi}(t,T)P(t,T)dW_{it}, \tag{11}$$

and the drift and volatility coefficients are obtained from Ito's lemma.

- Absence of arbitrage implies the APT restriction:

$$\mu_P(t,T) = r_t + \sum_{i=1}^{m} \lambda_i(X_t)\sigma_{Pi}(t,T). \tag{12}$$

- In equation (12), $\lambda_i(X_t)$ is the market price of risk for the i'th factor, and it is independent of T.

A General Multi-Factor Model – 2

- By Ito's lemma, $\mu_P(t,T)$ and $\sigma_{Pi}(t,T)$ can also be written as:

$$\mu_P(t,T)P(t,T) = \sum_{i=1}^{m} \frac{\partial P}{\partial X_i}\mu_i(X) + \frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 P}{\partial X_i \partial X_j}\sigma_i(X)\sigma_j(X)\rho_{ij}$$

$$\sigma_{Pi}(t,T)P(t,T) = \frac{\partial P}{\partial X_i}\sigma_i(X), \quad i = 1, 2, \ldots, m$$

- After substituting these equations into the APT restriction (12), we get the following PDE:

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 P}{\partial X_i \partial X_j}\sigma_i(X)\sigma_j(X)\rho_{ij} + \sum_{i=1}^{m} \frac{\partial P}{\partial X_i}\left[\mu_i(X) - \lambda_i(X)\sigma_i(X)\right] + \frac{\partial P}{\partial t} - r(X)P = 0 \quad (13)$$

with boundary condition $P(T,T) = 1$.

A General Multi-Factor Model – 3

- Feynman-Kac solution:

$$P(t,T) = E_t^Q\left[e^{-\int_t^T r(X_s)ds}\right], \quad (14)$$

- The expectation in (14) is taken under the probability measure corresponding to the risk-neutral (drift-adjusted) process:

$$dX_t = \{\mu(X_t) - \sigma(X_t)\lambda(X_t)\}\,dt + \sigma(X_t)dW_t^Q. \quad (15)$$

- The i'th element of the SDE (15) is

$$dX_{it} = \{\mu_i(X_t) - \lambda_i(X_t)\sigma_i(X_t)\}\,dt + \sigma_i(X_t)dW_{it}^Q. \quad (16)$$

Four examples of multi-factor models

1. Double-Decay (Central-Tendency) model:

$$\begin{aligned} dr_t &= \kappa_1(\mu_t - r_t)dt + \sigma_1 dW_{1t} \quad (17) \\ d\mu_t &= \kappa_2(\theta - \mu_t)dt + \sigma_2 dW_{2t} \quad (18) \end{aligned}$$

2. Fong-Vasicek stochastic volatility model:

$$\begin{aligned} dr_t &= \kappa_1(\mu - r_t)dt + \sqrt{V_t}dW_{1t} \quad (19) \\ dV_t &= \kappa_2(\alpha - V_t)dt + \eta\sqrt{V_t}dW_{2t} \quad (20) \end{aligned}$$

3. Brennan-Schwartz model:

$$\begin{aligned} d\log r_t &= [\alpha(l_t - r_t) - \alpha \log p]\, dt + \sigma_1 dW_{1t} \quad (21) \\ d\, l_t &= \beta_2(r, l)dt + \sigma_2 l_t dW_{2t}, \quad (22) \end{aligned}$$

   where l_t is the consol rate (annuity that never matures).

4. Multi-factor CIR model:

$$\begin{aligned} r_t &= \textstyle\sum_{i=1}^{m} y_{it} \quad (23) \\ dy_{it} &= \kappa_i(\mu_i - y_{it})dt + \sigma_i\sqrt{y_{it}}dW_{it}, \quad i = 1, 2, \ldots, m \quad (24) \end{aligned}$$

   where the m Brownian motions are independent.

The Brennan and Schwartz (1979) model

- State variables in the model:

  r_t the short rate (instantaneous interest rate).

  l_t the yield-to-maturity on a consol bond with a "continuous coupon".

- General stochastic process:

$$\begin{aligned} dr_t &= \beta_1(r_t, l_t)dt + \eta_1(r_t, l_t)dW_{1t} \quad (25) \\ d\, l_t &= \beta_2(r_t, l_t)dt + \eta_2(r_t, l_t)dW_{2t}. \quad (26) \end{aligned}$$

- The particular process used in the paper:

$$\begin{aligned} d\log r_t &= [\alpha(l_t - r_t) - \alpha \log p]\, dt + \sigma_1 dW_{1t} \quad (27) \\ d\, l_t &= \beta_2(r, l)dt + \sigma_2 l_t dW_{2t}. \quad (28) \end{aligned}$$

- For pricing purposes, we do not need to specify $\beta_2(l, r)$ as the second state variable is a traded asset.

BS consol price dynamics

- A consol is an annuity that never matures. If V_t denotes the price of the consol, we have the following relation:

$$V_t = \int_0^\infty e^{-l_t s} ds = \left[-\frac{1}{l_t} e^{-l_t s} \right]_0^\infty = \frac{1}{l_t} \tag{29}$$

- Note: the relationship between $X_t = (r_t, l_t)$ and V_t is known.
- Consol price dynamics:

$$\frac{dV_t}{V_t} = \mu_V(r_t, l_t)dt \ + \ 0 \cdot dW_{1t} \ + \ \sigma_V(r_t, l_t)dW_{2t} \tag{30}$$

where

$$V \cdot \mu_V(r,l) = -l^{-2}\beta_2(l,r) + l^{-3}\eta_2^2(l,r) = l^{-1}\left[-l^{-1}\beta_2(l,r) + l^{-2}\eta_2^2(l,r)\right] \tag{31}$$

$$V \cdot \sigma_V(r,l) = -l^{-2}\eta_2(l,r) = l^{-1}\left[-l^{-1}\eta_2(l,r)\right] \tag{32}$$

since $V_t = l_t^{-1}$ does not depend on r_t.

BS fundamental PDE – 1

- The bond price, $P(t,T)$, satisfies the PDE:

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\eta_1^2(r,l) \ + \ \frac{1}{2}\frac{\partial^2 P}{\partial l^2}\eta_2^2(r,l) \ + \ \frac{\partial^2 P}{\partial r \partial l}\rho\eta_1(r,l)\eta_2(r,l) \ + \\ \frac{\partial P}{\partial r}\{\beta_1(r,l) - \lambda_1(r,l)\eta_1(r,l)\} \ + \\ \frac{\partial P}{\partial l}\{\beta_2(r,l) - \lambda_2(r,l)\eta_2(r,l)\} \ + \ \frac{\partial P}{\partial t} \ - \ rP = 0. \tag{33}$$

- Because the l_t is a known function of a traded asset, we can eliminate $\beta_2(r,l)$ and $\lambda_2(r,l)$ from the above PDE.
- First, we substitute the SDE for the consol price dynamics (30) into the APT relationship used to derive the PDE:

$$\mu_V(r,l) + l \ = \ r + \lambda_2(r,l)\sigma_V(r,l) \tag{34}$$

Why do we add l on the LHS?

BS fundamental PDE – 2

- Second, we substitute (31) and (32) into (34):

$$-l^{-1}\beta_2(r,l) + l^{-2}\eta_2^2(r,l) + l \;=\; r - \lambda_2(r,l)l^{-1}\eta_2(r,l). \qquad (35)$$

- Third, after multiplying by l on both sides of (35), we get

$$\beta_2(r,l) - \lambda_2(r,l)\eta_2(r,l) \;=\; l^{-1}\eta_2^2(r,l) + l^2 - rl \qquad (36)$$

- Finally, we substitute (36) into (33):

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\eta_1^2(r,l) \;+\; \frac{1}{2}\frac{\partial^2 P}{\partial l^2}\eta_2^2(r,l) \;+\; \frac{\partial^2 P}{\partial r \partial l}\rho\eta_1(r,l)\eta_2(r,l) \;+$$
$$\frac{\partial P}{\partial r}\{\beta_1(r,l) - \lambda_1(r,l)\eta_1(r,l)\} \;+$$
$$\frac{\partial P}{\partial l}\left\{l^{-1}\eta_2^2(r,l) + l^2 - rl\right\} \;+\; \frac{\partial P}{\partial t} \;-\; rP = 0, \qquad (37)$$

which is the BS PDE.

Assessment of the BS model

- Advantages of the Brennan-Schwartz model:
  - State variables are observable (in principle), and they can be interpreted as short and long-run factors.
  - Only one market price of risk (preference) parameter in the model.

- Problems with the Brennan-Schwartz model:
  - No analytical solution for bond prices. The PDE can only be solved with **numerical methods** — either by finite-difference PDE solutions or Monte Carlo evaluation of the Feynman-Kac formula.
  - In most bond markets, there are no actively traded consol bonds.
  - Technical problems with the BS model: by the definition of l_t,

$$V_t \;=\; l_t^{-1} \;=\; \int_t^{\infty} P(t,s)ds \;=\; F(r_t, l_t), \qquad (38)$$

    but the requisite parameter constraint(s) are not imposed in the BS model.
  - This problem is, in fact, an argument **against** using traded assets (yields) as state variables (not just in the BS model).

Exponential-affine models – 1

- Fundamental PDE for a general multi-factor model:

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 P}{\partial X_i \partial X_j}\sigma_i(X)\sigma_j(X)\rho_{ij} + \sum_{i=1}^{m}\frac{\partial P}{\partial X_i}\left[\mu_i(X) - \lambda_i(X)\sigma_i(X)\right] + \frac{\partial P}{\partial t} - r(X)P = 0 \qquad (39)$$

- The Brennan and Schwartz model with $X_t = (r_t, l_t)$ does **not** lead to an analytical solution of (39) for bond prices.
- There are several term-structure models with an analytical solution for $P(t,T)$, and for **most** of these models we get

$$P(t, t+\tau) = \exp\left[A(\tau) + B(\tau)' X_t\right]. \qquad (40)$$

- Models with bond prices of the form (40) are called **exponential-affine** models [Duffie and Kan (1996)].

Exponential-affine models – 2

- What are the sufficient conditions for obtaining (40) as the solution to (39)?
- All bond prices, solutions to (39), depend on:
  1. The mapping from X_t to r_t, given by $r_t = r(X_t)$.
  2. m risk-neutral drifts:

  $$\mu_i^*(X) = \mu_i(X) - \lambda_i(X)\sigma_i(X) \qquad (41)$$

  3. $m(m+1)/2$ variance-covariance terms: $\sigma_i(X)\sigma_j(X)\rho_{ij}$.
- Sufficient conditions for exponential-affine models:

$$\begin{aligned} r(X) &= w_0 + w_1' X && (42) \\ \mu_i^*(X) &= a_i + b_i' X, \quad i = 1, \ldots, m && (43) \\ \sigma_i(X)\sigma_j(X)\rho_{ij} &= c_{ij} + d_{ij}' X, \quad i,j = 1, \ldots, m && (44) \end{aligned}$$

- That is, all "coefficients" in the PDE are **linear** in X.

Exponential-affine models – 3

- The function $A(\tau)$ and the $m\times 1$ vector (of functions) $B(\tau)$ depend on the specific model.
- $A(\tau)$ and $B(\tau)$ are obtained as the solution to an ODE system with dimension $(m+1)$.
- Same procedure as with one-factor models:
  - First, we **guess** that the solution is of the form (40).
  - Second, we substitute the requisite partial derivatives in to the PDE.
  - Finally, we collect terms with the factor X_i $(i = 1, 2, \ldots, m)$ and a constant (remaining terms).
  - This provides the $m+1$ ODEs which must be solved somehow (perhaps numerically, using Runge-Kutta integration)
  - Boundary conditions for the ODE: $A(0) = 0$ and $B(0) = 0_{m\times 1}$.

Gaussian central-tendency model – 1

- Stochastic process for the short rate:

$$\begin{aligned} dr_t &= \kappa_1(\mu_t - r_t)dt + \sigma_1 dW_{1t} && (45) \\ d\mu_t &= \kappa_2(\theta - \mu_t)dt + \sigma_2 dW_{2t}, && (46) \end{aligned}$$

- The Brownian motions are dependent, $\text{Corr}(dW_{1t}, dW_{2t}) = \rho dt$, and the market prices of risk are constants, λ_1 and λ_2.
- PDE:

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma_1^2 + \frac{1}{2}\frac{\partial^2 P}{\partial \mu^2}\sigma_2^2 + \frac{\partial^2 P}{\partial r \partial \mu}\rho\sigma_1\sigma_2 + \frac{\partial P}{\partial r}\left[\kappa_1(\mu - r) - \lambda_1\sigma_1\right] + \frac{\partial P}{\partial \mu}\left[\kappa_2(\theta - \mu) - \lambda_2\sigma_2\right] - \frac{\partial P}{\partial \tau} - rP = 0, \quad (47)$$

- We guess that

$$P(t, t+\tau) = \exp\left[A(\tau) + B_1(\tau) r_t + B_2(\tau)\mu_t\right]. \quad (48)$$

Gaussian central-tendency model – 2

- Substitution of the partial derivatives of the function (48) into the PDE (47) gives

$$\left\{\frac{1}{2}B_1^2(\tau)\sigma_1^2 + \frac{1}{2}B_2^2(\tau)\sigma_2^2 + B_1(\tau)B_2(\tau)\rho\sigma_1\sigma_2 + B_1(\tau)\left[\kappa_1(\mu - r) - \lambda_1\sigma_1\right]\right.$$
$$\left.+B_2(\tau)\left[\kappa_2(\theta - \mu) - \lambda_2\sigma_2\right] - A'(\tau) - B_1'(\tau)r - B_2'(\tau)\mu - r\right\} P = 0 \quad (49)$$

- After dividing by P and collecting terms we get

$$\left\{\frac{1}{2}\sigma_1^2 B_1^2(\tau) + \frac{1}{2}\sigma_2^2 B_2^2(\tau) + \rho\sigma_1\sigma_2 B_1(\tau)B_2(\tau)\right.$$
$$\left.-\lambda_1\sigma_1 B_1(\tau) + (\kappa_2\theta_2 - \lambda_2\sigma_2)B_2(\tau) - A'\tau)\right\}$$
$$-\left\{\kappa_1 B_1(\tau) + B_1'(\tau) + 1\right\} r$$
$$+\left\{\kappa_1 B_1(\tau) - \kappa_2 B_2(\tau) - B_2'(\tau)\right\}\mu = 0 \quad (50)$$

Gaussian central-tendency model – 3

- Since (50) must hold for all values of r and μ, we have

$$B_1'(\tau) = -\kappa_1 B_1(\tau) - 1 \quad (51)$$

$$B_2'(\tau) = \kappa_1 B_1(\tau) - \kappa_2 B_2(\tau) \quad (52)$$

$$A_1'(\tau) = \frac{1}{2}\sigma_1^2 B_1^2(\tau) + \frac{1}{2}\sigma_2^2 B_2^2(\tau) + \rho\sigma_1\sigma_2 B_1(\tau)B_2(\tau)$$
$$-\lambda_1\sigma_1 B_1(\tau) + (\kappa_2\theta - \lambda_2\sigma_2)B_2(\tau). \quad (53)$$

- ODE solutions:

$$B_1(\tau) = \frac{e^{-\kappa_1\tau} - 1}{\kappa_1} \quad (54)$$

$$B_2(\tau) = \frac{e^{-\kappa_2\tau} - 1}{\kappa_2} - \frac{e^{-\kappa_1\tau} - e^{-\kappa_2\tau}}{\kappa_1 - \kappa_2} \quad (55)$$

$$A(\tau) = \int_0^\tau A'(s)ds, \qquad \text{where } A'(s) \text{ is the RHS of (53).} \quad (56)$$