Fixed Income Analysis

Term-Structure Models in Continuous Time

Fundamental PDE for bond prices (summary)
More on risk-neutral valuation
The Vasicek and CIR one-factor models

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The fundamental PDE for bond prices – 1

- Model building blocks (assumptions):
  1. **Absence of arbitrage** opportunities (in a frictionless market).
  2. **One factor:** the bond price, $P(t,T)$, depends only the short rate, r_t.
  3. **Stochastic process:** r_t follows the SDE $dr_t = \mu(r_t)dt + \sigma(r_t)dW_t$.

- Based on these assumptions, we first derive the APT-restriction

$$\mu_P(t,T) = r_t + \lambda(r_t)\sigma_P(t,T) \quad ; \quad \sigma_P(t,T) = \frac{\partial P}{\partial r}\sigma(r), \tag{1}$$

  where $\mu_P(t,T)$ and $\sigma_P(t,T)$ are the instantaneous **expected return** and **volatility** of the T-maturity bond,

$$d\,P(t,T)\,/\,P(t,T) = \mu_P(t,T)dt + \sigma_P(t,T)dW_t, \tag{2}$$

  and $\lambda(r)$ is the so-called **market price of risk**.

The fundamental PDE for bond prices – 2

- Using Ito's lemma, $\mu_P(t,T)$ may also be written as:

$$\mu_P(t,T)P(t,T) = \frac{\partial P}{\partial r}\mu(r) + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2(r), \tag{3}$$

- From the APT restriction (1) we have

$$\mu_P(t,T)P(t,T) = r_t P(t,T) + \lambda(r_t)\frac{\partial P}{\partial r}\sigma(r_t) \tag{4}$$

- By combining the two equations (3) and (4), we get the **fundamental PDE** which the bond price $P(t,T)$ must satisfy:

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2(r) \; + \; \frac{\partial P}{\partial r}\left[\mu(r) - \lambda(r)\sigma(r)\right] \; + \; \frac{\partial P}{\partial t} \; - \; rP = 0, \tag{5}$$

  with boundary condition $P(T,T) = 1$.

Risk-neutral valuation – basics

- Feynman-Kac representation of the solution to the PDE,

$$P(t,T) = E_t^Q\left[e^{-\int_t^T r_s}\right]. \tag{6}$$

- The expectation is taken under a new probability measure Q corresponding to the drift-adjusted SDE for the short rate

$$dr_t = \{\mu(r_t) - \lambda(r_t)\sigma(r_t)\}\, dt + \sigma(r_t)dW_t^Q, \tag{7}$$

  where W_t^Q is a Brownian motion under the Q-measure.

- We refer to this as **risk-neutral valuation**.

- Risk-neutral valuation in two cases:
  - **SDE:** risk adjustment done by modifying the drift of the short-rate process.
  - **Binomial:** risk adjustment by modifying the probabilities of an up-move.

Risk-neutral valuation – extensions

- Consider a claim with the following **payoff structure**
  - For $t \leq s \leq T$, there is a continuous **annualized** payment of $c(r_s)$. That is, between s and $s + ds$, the payment from the claim is $c(r_s)ds$.
  - At maturity T, there is a final lump-sum payment of $C(r_T)$.
- Using risk-neutral valuation, the price can be expression as:

$$V_t(r) = E_t^Q \left[\int_t^T e^{-\int_t^s r_u du} c(r_s) ds \right] + E_t^Q \left[e^{-\int_t^T r_s ds} C(r_T) \right]. \tag{8}$$

- Note how the future payoffs of $c(r_s)ds$ and $C(r_T)$ are discounted.
- By the Feynman-Kac duality, there is also a PDE representation:

$$\frac{1}{2} \frac{\partial^2 V}{\partial r^2} \sigma^2(r) + \frac{\partial V}{\partial r} [\mu(r) - \lambda(r)\sigma(r)] + \frac{\partial V}{\partial t} + c(r) - rP = 0, \tag{9}$$

subject to the boundary condition $V_T(r) = C(r)$.

The Vasicek model – 1

- The first paper about continuous-time term-structure models.
- Vasicek (1977) assumes that the short rate follows the Ornstein-Uhlenbeck process

$$dr_t = \kappa(\mu - r_t)dt + \sigma dW_t. \tag{10}$$

- The market price of risk is assumed to be a constant, $\lambda(r) = \lambda$.
- Main features of the Vasicek model:
  - Mean reversion towards the unconditional mean $\mu = E(r)$.
  - Speed of mean reversion determined by κ (a larger κ means faster mean reversion).
  - The short rate is normally distributed (Gaussian model).
  - Because of the normal distribution, we can obtain closed-form solutions for interest-rate derivatives in many important cases.

The Vasicek model – 2

- PDE for bond prices:

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2 + \frac{\partial P}{\partial r}\left[\kappa(\mu - r) - \lambda\sigma\right] + \frac{\partial P}{\partial t} - rP = 0, \qquad (11)$$

  with boundary condition $P(T,T) = 1$.

- We **guess** that the solution to (11) takes the following form:

$$P(t,T) = \exp\left[A(\tau) + B(\tau)r_t\right], \quad \tau = T - t. \qquad (12)$$

- In order to show that equation (12) is the solution to (11) **and** to determine $A(\tau)$ and $B(\tau)$, we do the following:
  - Calculate the requisite partial derivatives of (12), and substitute these expressions into the PDE (11).
  - If the PDE reduces to two ordinary differential equations (ODEs), we have verified that the solution is of the form (12).
  - Solve the ODEs, subject to the boundary condition $A(0) = 0$ and $B(0) = 0$.

The Vasicek model – 3

- Partial derivatives of (12),

$$\frac{\partial P}{\partial r} = B(\tau)P(t,T), \qquad \frac{\partial^2 P}{\partial r^2} = B(\tau)^2 P(t,T)$$

$$\frac{\partial P}{\partial t} = -\frac{\partial P}{\partial \tau} = -\left[A'(\tau) + B'(\tau)r\right] \cdot P(t,T).$$

- Next, we substitute these expressions into the PDE:

$$\left\{\frac{1}{2}B^2(\tau)\sigma^2 + B(\tau)[\kappa(\mu - r) - \lambda\sigma] - A'(\tau) - B'(\tau)r - r\right\} \cdot P = 0. \qquad (13)$$

- After dividing by P and collecting terms with the factor r, we get

$$\left\{\frac{1}{2}B^2(\tau)\sigma^2 + B(\tau)[\kappa\mu - \lambda\sigma] - A'(\tau)\right\} - \left\{\kappa B(\tau) + B'(\tau) + 1\right\} r = 0. \qquad (14)$$

- Both terms in brackets must be zero (our two ODEs).

The Vasicek model – 4

- System of ODEs for the Vasicek model

$$A'(\tau) = \tfrac{1}{2}\sigma^2(\tau)B^2(\tau) + \{\kappa\mu - \lambda\sigma\}\, B(\tau) \tag{15}$$

$$B'(\tau) = -\kappa B(\tau) - 1 \tag{16}$$

- The PDE boundary condition

$$P(T,T) = \exp\left[A(0) + B(0)r_T\right] = 1 \quad \text{for all } r_T, \tag{17}$$

means that $A(0) = 0$ and $B(0) = 0$ — ODE initial conditions.

- The ODE system has a **recursive** structure — the ODE equation for $B'(\tau)$, i.e. (16), does not involve $A(\tau)$.
- This means that the function $B(\tau)$ only depends on κ and τ.

The Vasicek model – 5

Four steps in finding the solution for $B(\tau)$

1. Multiply all terms by $\exp(\kappa\tau)$ and rearrange,

$$B'(\tau)e^{\kappa\tau} + \kappa B(\tau)e^{\kappa\tau} = -e^{\kappa\tau}\,. \tag{18}$$

2. By the product rule for differentiation the LHS may be written as

$$\frac{d}{d\tau}\left\{e^{\kappa\tau}B(\tau)\right\} = -e^{\kappa\tau}\,. \tag{19}$$

3. Since, for any function, $h(\tau) = h(0) + \int_0^\tau h'(s)ds$,

$$e^{\kappa\tau}B(\tau) = B(0) + \int_0^\tau \frac{d}{ds}\left\{e^{\kappa s}B(s)\right\} ds = -\int_0^\tau e^{\kappa s}ds, \tag{20}$$

4. Finally, multiply by $\exp(-\kappa\tau)$, and calculate the integral

$$B(\tau) = -e^{-\kappa\tau}\int_0^\tau e^{\kappa s}ds = \frac{e^{-\kappa\tau} - 1}{\kappa}. \tag{21}$$

The Vasicek model – 6

Finding the solution for $A(\tau)$

- The ODE is $A'(\tau) = \frac{1}{2}\sigma^2(\tau)B^2(\tau) + \{\kappa\mu - \lambda\sigma\}\, B(\tau)$.
- No special "tricks" are needed here since $A(\tau)$ is not on the RHS.
- We calculate $A(\tau)$ by straightforward integration of the RHS

$$\begin{aligned} A(\tau) &= A(0) + \int_0^\tau A'(s)ds \\ &= \frac{1}{2}\sigma^2 \int_0^\tau B^2(s)ds + [\kappa\mu - \lambda\sigma] \int_0^\tau B(s)ds. \end{aligned} \tag{22}$$

- We know $B(\tau)$, and after **a lot** of calculations we get

$$\begin{aligned} A(\tau) &= -R(\infty)\,(\tau + B(\tau)) \;-\; \frac{\sigma^2}{4\kappa}B^2(\tau), \qquad \text{where} \\ R(\infty) &= \mu - \frac{\lambda\sigma}{\kappa} - \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2. \end{aligned} \tag{23}$$

The CIR model – 1

- Similar to the Vasicek model — except that the short rate is restricted to be positive (non-negative).
- Stochastic process for the short rate:

$$dr_t = \kappa(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{24}$$

- This process has a **reflecting barrier** at 0, hence $r_t \geq 0$.
- Market price of risk: $\lambda(r) = (\lambda/\sigma)\sqrt{r}$.
- PDE for bond prices:

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2 r + \frac{\partial P}{\partial r}\left[\kappa(\mu - r) - \lambda r\right] + \frac{\partial P}{\partial t} - rP = 0, \tag{25}$$

with boundary condition $P(T,T) = 1$.

The CIR model – 2

- Again we guess that $P(t, t+\tau) = \exp\left[A(\tau) + B(\tau) r_t\right]$.
- We substitute the partial derivatives into the PDE (25),

$$\left\{\frac{1}{2}B^2(\tau)\sigma^2 r \;+\; B(\tau)[\kappa(\mu - r) - \lambda r] \;-\; A'(\tau) - B'(\tau) r \;-\; r\right\} \cdot P = 0. \quad (26)$$

- After dividing by P and collecting terms with the factor r, we get

$$\left\{\frac{1}{2}B^2(\tau)\sigma^2 - B(\tau)(\kappa + \lambda) - B'(\tau) - 1\right\} r \;+\; \left\{B(\tau)\kappa\mu - A'(\tau)\right\} = 0. \quad (27)$$

- From the two brackets, we get the ODE system:

$$A'(\tau) = \kappa\mu B(\tau) \quad (28)$$

$$B'(\tau) = \tfrac{1}{2}\sigma^2 B^2(\tau) - (\kappa + \lambda) B(\tau) - 1, \quad (29)$$

with initial conditions $A(0) = 0$ and $B(0) = 0$.

Does equation (12) always work?

- Q: Do we always get $P(t, t+\tau) = \exp\left[A(\tau) + B(\tau) r_t\right]$?
- A: **No** — as counter-example let $\lambda(r) = 0$ and

$$dr_t = \kappa(\mu - r_t)dt + \sigma r_t^{\gamma} dW_t. \quad (30)$$

- If the above guess is correct, the PDE becomes

$$\left\{\frac{1}{2}B^2(\tau)\sigma^2 r^{2\gamma} \;+\; B(\tau)\kappa(\mu - r) \;-\; A'(\tau) - B'(\tau) r \;-\; r\right\} \cdot P = 0. \quad (31)$$

- After dividing by P and collecting terms we have,

$$\left\{\frac{1}{2}B^2(\tau)\sigma^2\right\} r^{2\gamma} \;-\; \left\{B(\tau)\kappa + B'(\tau) + 1\right\} r \;+\; \left\{B(\tau)\kappa\mu - A'(\tau)\right\} = 0. \quad (32)$$

- The three expressions in brackets cannot be zero at the same time (unless $\gamma = 0$ or $\gamma = 1/2$), so our guess is **wrong**.