Using Copulas in Risk Management

Master’s Thesis by

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Everything should be made as simple as possible,
but not simpler.

Einstein (1879-1955)
Acknowledgements

This thesis was written at ABP Research in the period February 2003 until August 2003. Before I started studying papers about the main topic of this thesis, copulas, I was completely unfamiliar with that subject. Thanks to the help of my supervisors, Roderick Molenaar and Nolke Posthuma at the Research Department of ABP and Bas Werker from Tilburg University, I have been able to complete this ‘test of competence’.

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Naturally I take full responsibility for any errors, which will inevitably be present¹.

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1 Introduction

Traditionally the risk profiles of financial products are estimated under the assumption of (joint) normality. The normal (Gaussian) distribution has all kinds of interesting and, especially from an asymptotical point of view, appealing properties. Using normality assumptions makes life easier, because it often simplifies calculations, it keeps results tractable, it makes sure that limit distributions can be given explicitly, etc. All those attractive properties have caused normality assumptions to form the foundation of many, historically invaluable, asset pricing theories such as Markowitz’ portfolio theory, the Capital Asset Pricing Model, and the Arbitrage Pricing Theory.

Yet it has been known for a long time that in finance, probably more often than not, normality assumptions are not very realistic. One of the earliest papers to report that was Mandelbrot 1963. Heavy tails and skewness are observed frequently for empirical distributions of return series. In order to make a thorough analysis of the risks associated with holding certain assets, derivatives, or portfolios thereof, this observed non-normal behavior of return series must be taken into account.

Financial practitioners regularly observe that in 'bear' markets returns are more correlated than they are in 'bull' markets. This asymmetry in return dependence can have a big influence on the risk profile of a portfolio. If we make a risk profile simply assuming that there is a symmetric dependence relationship between returns, then we could be confronted with large portfolio losses much more often than we had expected. Therefore a basic question that has to be answered is: How to model the joint distribution of different risks?

A joint distribution can be seen as a framework containing two elements:

- All the marginal distributions.
- The dependence structure that exists between the marginal distribution functions.

A copula is a function that completely describes the dependence structure; it contains all the information to link the marginal distributions to their joint distribution. The theory allows one to combine several marginal distribution functions, possibly from different distributional families, with any copula function and by doing so, obtaining a valid multivariate distribution function.

This thesis starts with a mathematical introduction to copula functions, see Section 2. In this section we explore the mathematics behind the statement: "any joint distribution function can be written as a copula function combined with its marginal distributions”. Also some important standard copulas will be treated because they will be important in the definitions of independence and
perfect dependence.

Section 3 deals with all kinds of association concepts, like independence, perfect dependence, and tail dependence. There is also a treatment of the linear correlation coefficient. We explore in which situations it is useful to look at the linear correlation coefficient, and for which purposes it is unsuitable to use the linear correlation coefficient. Some alternative measures of association will be treated.

Section 4 discusses the two most commonly used parametric copula families, the elliptical copulas and the Archimedean copulas. Several characteristics of each of these families will be dealt with. The elliptical copula family includes the Gaussian copula and the $t$-copula.

Sections 5 and 6 describe parameter estimation techniques for copulas. The methods presented there will be based on the Maximum Likelihood Principle. Also we treat the application of these methods to the elliptical copulas, and the Archimedean copulas. Application of this technique to the class of Archimedean copulas is not straightforward. A method will be proposed to simplify matters.

Section 7 treats the topic of simulation, an important tool in finance. Technically speaking, simulating from copulas is not that difficult, if one is able to use the conditional sampling method. This method will be explained, as well as the reason why it is difficult to use this method in practice. For the Gaussian and $t$-copulas we will see a fast and accurate simulation algorithm. For the Archimedean copulas a recursive algorithm will be treated, but this will turn out to be a slow and inaccurate algorithm.

The Applications section, Section 8, shows the implementation of the copula techniques to various different portfolios. A portfolio of currencies, two portfolios with hedge fund indices, and two portfolios with corporate bonds.

In short, this thesis tries to give an answer to the questions: "Why should one use copulas?", and "Which copula is suited for which problem?". Although the theory of copulas dates back to Sklar 1959, the application of copula theory in financial modeling is a much more recent phenomenon. Most papers about the applications of copulas in the field of finance have been written after 1998. Many interesting ideas have only been introduced in the last few months. The Conclusions section will try to answer the two boldfaced questions. The second question however is extremely difficult to answer, and this thesis ‘only’ gives some intuition on how to choose between copulas. There is an enormous amount of very recent literature on that topic, and new ideas appear online almost on a weekly basis.
2 Introducing Copula Functions

2.1 Definition of Copula Functions

Before defining what a copula function is, first a short introduction about the concept of a multivariate distribution function in general will be given. Let $X_1, \ldots, X_n$ be random variables, with marginal distribution functions $F_1, \ldots, F_n$, respectively, and joint distribution function $H$. The dependence structure of the variables is now completely described by the function $H$ in the following way:

$$H(x_1, \ldots, x_n) := P[X_1 \leq x_1, \ldots, X_n \leq x_n]$$

Any of the marginal distribution functions $F_i$ can be obtained from $H$, by letting $x_j \rightarrow \infty$ for all $j \neq i$. The necessary and sufficient properties of a multivariate distribution function are (see Joe 1997):

**Definition 1 (Multivariate or Joint Distribution Function)**

A function $H : \mathbb{R}^n \rightarrow [0, 1]$ is a multivariate distribution function if the following conditions are satisfied:

1. $H$ is right continuous,
2. $\lim_{x_i \rightarrow -\infty} H(x_1, \ldots, x_n) = 0$, for $i = 1, \ldots, n$,
3. $\lim_{x_i \rightarrow \infty} H(x_1, \ldots, x_n) = 1$,
4. For all $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with $a_i \leq b_i$ for $i = 1, \ldots, n$ the following inequality holds:

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1 + \cdots + i_n} H(x_{i_1}, \ldots, x_{i_n}) \geq 0$$

where $x_{j1} = a_j$ and $x_{j2} = b_j$ for all $j$.\footnote{If $H$ has $n$-th order derivatives, then this condition is equivalent to $\frac{\partial^n H}{\partial x_1 \cdots \partial x_n} \geq 0$}

We now try to separate the joint distribution function $H$ into its marginal distribution functions (the margins), and a part that describes the dependence structure. This means that somehow we will have to 'filter out' the information that is contained in all the margins. This can be established by the use of a probability integral transformation (sometimes also called a quantile transformation).

$$H(x_1, \ldots, x_n) = P[X_1 \leq x_1, \ldots, X_n \leq x_n] = P[F_1(X_1) \leq F_1(x_1), \ldots, F_n(X_n) \leq F_n(x_n)] = C(F_1(x_1), \ldots, F_n(x_n))$$
Recall that if the random variable $Y$ has a continuous distribution function $F$, then $F(Y)$ has a Uniform$(0,1)$ distribution. This implies that we have to define the properties of the function $C$, called the copula or the copula function, on a set of standard uniformly distributed random variables.

**Definition 2 (Copula Function)**

An $n$-dimensional copula $C$ is a function from $[0,1]^n$ to $[0,1]$ satisfying the following properties:

1. $C(u_1,\ldots,u_n) = 0$ if $u_i = 0$ for some $i = 1,\ldots,n$.

2. $C(1,\ldots,1,u_i,1,\ldots,1) = u_i$ for all $u_i \in [0,1]$.

3. For all $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ with $a_i \leq b_i$ for $i = 1,\ldots,n$ the following inequality holds:

$$
\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{i_1},\ldots,u_{i_n}) \geq 0
$$

(1)

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j$ and $u_{jk} \in [0,1]$ for all $j$ and $k$.

Property 1 is also sometimes expressed as: "$C$ is grounded", and property 3 as: "$C$ is $n$-increasing". The properties that a function must have to be qualified as a copula function follow from the fact that it has to define a distribution function with standard uniform margins. So how about continuity of the copula? Next theorem ascertains the fact that any copula is uniformly continuous on its domain.

**Theorem 3 (Copula Continuity)**

Let $C$ be any $n$-copula. Then, for all $u, v \in [0,1]^n$,

$$
|C(v) - C(u)| \leq \sum_{i=1}^{n} |v_i - u_i|
$$

**Proof.** See Schweizer and Sklar 1983. ■

---

3The statement: "$C$ is $n$-increasing", is not equivalent with the statement: "$C$ is a non-decreasing function in each argument". If however $C$ is grounded, then "$C$ is $n$-increasing" implies "$C$ is a nondecreasing function in each argument". See Joe 1997.
2 Introducing Copula Functions

2.2 Sklar’s Theorem

In previous section the link between a joint distribution function and a copula has been explained intuitively. In this section it will be formalized mathematically. That is done in the most important theorem regarding copulas, Sklar’s theorem.

**Theorem 4 (Sklar’s Theorem)**

Let $H$ be an $n$-dimensional distribution function with margins $F_1, \ldots, F_n$. Then there exists an $n$-copula $C$ such that for all $x \in \mathbb{R}^n$,

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

(2)

If $F_1, \ldots, F_n$ are all continuous, then $C$ is unique; otherwise $C$ is uniquely determined on $\text{Range}(F_1) \times \cdots \times \text{Range}(F_n)$. Conversely, if $C$ is an $n$-copula and $F_1, \ldots, F_n$ are distribution functions, then the function $H$ defined above is an $n$-dimensional distribution function with margins $F_1, \ldots, F_n$.

**Proof.** See Sklar 1996.

Sklar’s theorem states that for any continuous joint distribution function the univariate margins and the multivariate dependence structure can be separated. That is why a copula is interpreted as the dependence structure: the univariate margins do not contain any information about the dependence structure. We would also like to have an expression for the copula in terms of the margins and the joint distribution function. This can be derived directly as a corollary to Sklar’s theorem. Before doing that we will first need to define the concept of a generalized inverse of a distribution function.

**Definition 5 (Generalized Inverse of a Distribution Function)**

Let $F$ be a univariate distribution function. Then the generalized inverse of $F$ is defined in the following way:

$$F^{-1}(t) := \inf \{x \in \mathbb{R} | F(x) \geq t\} \text{ for all } t \in [0, 1]$$

using the convention $\inf \emptyset := -\infty$.

Now all tools are available to express the copula in terms of the joint distribution function $H$ and the margins $F_1, \ldots, F_n$.

**Corollary 6 (Copula Representation)**

Let $H$ be an $n$-dimensional distribution function with continuous margins $F_1, \ldots, F_n$, and copula $C$ (satisfying Sklar’s theorem). Then for any $u \in [0, 1]^n$,

$$C(u_1, \ldots, u_n) = H(F^{-1}_1(u_1), \ldots, F^{-1}_n(u_n))$$

(3)

The technique of ‘filtering the margins out of the multivariate distribution’ will be illustrated by two examples.
Example 7 (Bivariate Copula Representation)
Let \( X_1 \) and \( X_2 \) be random variables, jointly having a Gumbel bivariate logistic distribution. That means
\[
H(x_1, x_2) = (1 + e^{-x_1} + e^{-x_2})^{-1}
\]
The margins are: 
\[
F_1(x_1) = \lim_{x_2 \to \infty} H(x_1, x_2) = (1 + e^{-x_1})^{-1}
\]
and 
\[
F_2(x_2) = \lim_{x_1 \to \infty} H(x_1, x_2) = (1 + e^{-x_2})^{-1}.
\]
Now we want to specify the copula function \( C \) on the transformed standard uniformly distributed variables \( u_1 \) and \( u_2 \).
From the transformations 
\[
u_1 = F_1(x_1) \quad \text{and} \quad u_2 = F_2(x_2),
\]
we obtain that 
\[
F_1^{-1}(u_1) = -\log (u_1^{-1} - 1) \quad \text{and} \quad F_2^{-1}(u_2) = -\log (u_2^{-1} - 1).
\]
Substitute this into relationship (3) to obtain:
\[
C(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}
\]

Example 8 (Bivariate Gaussian Copula)
Let \( X_1 \) and \( X_2 \) be random variables, jointly having a bivariate standard Gaussian distribution with a linear correlation coefficient \( \rho \), i.e.,
\[
H(x_1, x_2) = \Phi(\rho x_1 x_2)
\]
If we want to find the dependence structure between \( X_1 \) and \( X_2 \), then we can again use relationship (3). The univariate margins of a multivariate standard Gaussian distribution are univariate standard Gaussian. So \( F_1(x_1) = \Phi(x_1) \) and \( F_2(x_2) = \Phi(x_2) \). The transformation of variables therefore becomes:
\[
\begin{align*}
x_1 &= \Phi^{-1}(u_1) \\
x_2 &= \Phi^{-1}(u_2)
\end{align*}
\]
Inserting this into (3) yields us the representation of a bivariate Gaussian copula with linear correlation coefficient \( \rho \):
\[
C_{\rho}^{Gauss}(u_1, u_2) = \Phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2))
\]
In Example 8 we see that a multivariate Gaussian distribution is formed by combining a Gaussian copula with all univariate Gaussian marginals. I.e., if we combine the bivariate Gaussian copula \( C_{\rho}^{Gauss}(u_1, u_2) \) with the marginal distributions \( u_1 = \Phi(\frac{x_1 - \mu_1}{\sigma_1}) \) and \( u_2 = \Phi(\frac{x_2 - \mu_2}{\sigma_2}) \), then we find:
\[
\begin{align*}
H(x_1, x_2) &= \Phi_\rho \left( \Phi^{-1} \left( \Phi \left( \frac{x_1 - \mu_1}{\sigma_1} \right) \right), \Phi^{-1} \left( \Phi \left( \frac{x_2 - \mu_2}{\sigma_2} \right) \right) \right) \\
&= \Phi_\rho \left( \frac{x_1 - \mu_1}{\sigma_1}, \frac{x_2 - \mu_2}{\sigma_2} \right)
\end{align*}
\]
a bivariate Gaussian distribution with mean \( \mu = \left( \mu_1 \quad \mu_2 \right) \), and variance-covariance matrix \( \Sigma = \left( \begin{array}{cc} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{array} \right) \).
Introducing Copula Functions

So far we only described the copula distribution functions. Next, the concept of copula density functions is introduced. This will play a role in the parameter estimation methods for copulas that are based on the Maximum Likelihood Principle.

Corollary 9 (Copula Density)

The relationship between the multivariate density function \( h(x_1, \ldots, x_n) \) and the copula density \( c \), is given by:

\[
 h(x_1, \ldots, x_n) = c(F_1(x_1), \ldots, F_n(x_n)) \prod_{i=1}^{n} f_i(x_i)
\]  

(4)

Proof. Let \( h \) be the \( n \)-dimensional density function that belongs to the distribution function \( H \). Define \( h \) by:

\[
 h(x_1, \ldots, x_n) = \frac{\partial H(x_1, \ldots, x_n)}{\partial x_1 \cdots \partial x_n}
\]

Now substitute (2), to obtain:

\[
 h(x_1, \ldots, x_n) = \frac{\partial C(F_1(x_1), \ldots, F_n(x_n))}{\partial x_1 \cdots \partial x_n}
\]

Using the substitution \( u_i = F_i(x_i) \) for \( i = 1, \ldots, n \) we obtain:

\[
 h(x_1, \ldots, x_n) = \frac{\partial C(u_1, \ldots, u_n)}{\partial u_1 \cdots \partial u_n} \prod_{i=1}^{n} f_i(x_i)
\]

\[
 = c(u_1, \ldots, u_n) \prod_{i=1}^{n} f_i(x_i)
\]

Inserting again \( u_i = F_i(x_i) \) yields the relationship given above. ■
2.3 Copulas and Monotone Transformations

Assume that a researcher is interested in the joint distribution of several variables. He collects data and, somehow, estimates the joint distribution function of the variables of interest. If the research is still in the modeling phase, then it often happens that a decision is taken to do a transformation on one or more variables. E.g., one of the variables is transformed logarithmically. How should the researcher proceed to obtain an estimate for the new joint distribution function?

It turns out that for strictly monotone transformations of random variables, copulas are either invariant, or they change in a simple way. This corresponds with the idea of looking at the copula function as the dependence structure. Typically, the invariance of copulas holds under transformations that are all strictly increasing, as next theorem shows:

**Theorem 10 (Copula Invariance)**

Let \((X_1, \ldots, X_n)\) be a vector of continuous random variables having copula \(C\). If \(\alpha_1, \ldots, \alpha_n\) are strictly increasing functions on \(\text{Range}(X_1), \ldots, \text{Range}(X_n)\), respectively, then \((\alpha_1(X_1), \ldots, \alpha_n(X_n))\) also has copula \(C\).

**Proof.** From Embrechts et al. 2001b:

Let \(F_1, \ldots, F_n\) denote the distribution functions of \(X_1, \ldots, X_n\), and let \(G_1, \ldots, G_n\) denote the distribution functions of \(\alpha_1(X_1), \ldots, \alpha_n(X_n)\). Let \((X_1, \ldots, X_n)\) have copula \(C\) and let \((\alpha_1(X_1), \ldots, \alpha_n(X_n))\) have copula \(C_\alpha\). Then:

\[
C_\alpha (G_1(x_1), \ldots, G_n(x_n)) = \mathbb{P} [\alpha_1(X_1) \leq x_1, \ldots, \alpha_n(X_n) \leq x_n] = \mathbb{P} \left[ X_1 \leq \alpha_1^{-1}(x_1), \ldots, X_n \leq \alpha_n^{-1}(x_n) \right] = C \left( F_1(\alpha_1^{-1}(x_1)), \ldots, F_n(\alpha_n^{-1}(x_n)) \right) = C \left( G_1(x_1), \ldots, G_n(x_n) \right).
\]

Since \(X_1, \ldots, X_n\) are continuous, \(\text{Range}(G_1) = \ldots = \text{Range}(G_n) = [0, 1]\). It follows that \(C_\alpha = C\) on \([0, 1]^n\). ■

The result of this theorem is that a logarithmic or any other strictly increasing transformation of one or more variables means that we only have to change the marginal distributions, not the copula function itself. So what will happen then if there is at least one transformation that is strictly decreasing? It turns out that the copula of the transformed data can be expressed in terms of the ‘old’ copula and its margins.

\[\text{Since } \alpha_1, \ldots, \alpha_n \text{ are all strictly increasing: } G_i(y) = \mathbb{P} (\alpha_i(X_i) \leq y) = \mathbb{P} \left( X_i \leq \alpha_i^{-1}(y) \right) = F_i(\alpha_i^{-1}(y)) \text{ for all } y \in \mathbb{R} \text{ and all } i \in \{1, \ldots, n\}.\]
Theorem 11 Let \((X_1, \ldots, X_n)\) be a vector of continuous random variables having copula \(C_{X_1, \ldots, X_n}\). Let \(\alpha_1, \ldots, \alpha_n\) be strictly monotone functions on \(\text{Range}(X_1), \ldots, \text{Range}(X_n)\) respectively, and let \((\alpha_1(\text{X}_1), \ldots, \alpha_n(\text{X}_n))\) have copula \(C_{\alpha_1(\text{X}_1), \ldots, \alpha_n(\text{X}_n)}\). Furthermore, let \(\alpha_i\) be strictly decreasing for some \(i\). Without loss of generality, let \(i = 1\). Then

\[
C_{\alpha_1(\text{X}_1), \ldots, \alpha_n(\text{X}_n)}(u_1, \ldots, u_n) = C_{\alpha_2(X_2), \ldots, \alpha_n(X_n)}(u_2, \ldots, u_n) - C_{X_1, \alpha_2(X_2), \ldots, \alpha_n(X_n)}(1 - u_1, u_2, \ldots, u_n)
\]

Proof. See Embrechts et al. 2001b.

Using the two above theorems recursively it can be shown that the copula \(C_{\alpha_1(\text{X}_1), \ldots, \alpha_n(\text{X}_n)}\) can be expressed in terms of the copula \(C_{X_1, \ldots, X_n}\) and its margins. This is shown for the bivariate case:

Example 12 Let \(\alpha_1\) be strictly decreasing and let \(\alpha_2\) be strictly increasing:

\[
C_{\alpha_1(\text{X}_1), \alpha_2(X_2)}(u_1, u_2) = u_2 - C_{X_1, \alpha_2(X_2)}(1 - u_1, u_2) = u_2 - C_{X_1, X_2}(1 - u_1, u_2)
\]

Let \(\alpha_1\) and \(\alpha_2\) be strictly decreasing:

\[
C_{\alpha_1(\text{X}_1), \alpha_2(X_2)}(u_1, u_2) = u_2 - C_{X_1, \alpha_2(X_2)}(1 - u_1, u_2) = u_2 - (1 - u_1 - C_{X_1, X_2}(1 - u_1, 1 - u_2)) = u_1 + u_2 - 1 + C_{X_1, X_2}(1 - u_1, 1 - u_2)
\]
2.4 Survival Copulas

Sometimes in financial theory one is interested in the distribution of the remaining lifetime, or survival time, of objects. This is for instance the case in credit risk management, when modeling the times to default of a basket of credits. Assume that we have a portfolio of \( n \) different securities, with stochastic times to default \( X_1, \ldots, X_n \). We want to have information about the joint distribution of these times to default.

\[
\mathcal{H}(x_1, \ldots, x_n) := \mathbb{P}[X_1 > x_1, \ldots, X_n > x_n]
\]

This is called the 'joint survival function'. Assuming that we already found the copula \( C_{X_1, \ldots, X_n} \) for \( \mathcal{H} \), how to obtain then the copula \( \tilde{C} \), the survival copula for \( \mathcal{H} \)? For that we can use the results derived in the previous section. Use Theorem 11, with \( \alpha_i(y) = 1 - y \) for all \( i \), to express \( \tilde{C} \) in terms of \( C_{X_1, \ldots, X_n} \) and its univariate margins. This is shown again for the bivariate case:

**Example 13** (Bivariate Survival Copula)

\[
\tilde{C}(u_1, u_2) = C_{1-X_1, 1-X_2}(1-u_1, 1-u_2) = u_2 - C_{X_1, X_2}(1-u_1, 1-u_2) = u_2 - (1 - u_1 - C_{X_1, X_2}(1-u_1, 1-u_2)) = u_1 + u_2 - 1 + C_{X_1, X_2}(1-u_1, 1-u_2)
\]

A second concept that is often used in the literature is the notion of a joint survival function \( \overline{C} \) for uniform random variables having a joint distribution function given by the copula \( C \).

\[
\overline{C}(u_1, \ldots, u_n) = \mathcal{H}(u_1, \ldots, u_n) = \mathbb{P}[U_1 > u_1, \ldots, U_n > u_n]
\]

The link between the joint survival function \( \overline{C} \) and the survival copula \( \tilde{C} \) is given by:

\[
\overline{C}(u_1, \ldots, u_n) = \tilde{C}(1-u_1, \ldots, 1-u_n)
\]

This link is established for the bivariate case in the next example.

**Example 14** (Bivariate Joint Survival Function)

\[
\overline{C}(u_1, u_2) = \mathbb{P}[U_1 > u_1, U_2 > u_2] = 1 - (\mathbb{P}[U_1 \leq u_1] \cup \mathbb{P}[U_2 \leq u_2]) = 1 - (\mathbb{P}[U_1 \leq u_1] + \mathbb{P}[U_2 \leq u_2] - \mathbb{P}[U_1 \leq u_1, U_2 \leq u_2]) = 1 - u_1 - u_2 + C(u_1, u_2) = \tilde{C}(1-u_1, 1-u_2)
\]
2 Introducing Copula Functions

2.5 Copula Concordance and Fréchet Bounds

In comparing copulas, the concept of concordance ordering is often used:

**Definition 15 (Concordance Ordering of Copulas)**

Copula $C_1$ is said to be smaller than copula $C_2$, if:

$$C_1(u) \leq C_2(u), \text{ for all } u \in [0, 1]^n$$

The partial ordering of the copulas is called a concordance ordering. Not every pair of copulas can be ordered using this concept. However, a lot of parametric copula families are totally ordered. A one-parameter family $\{C_\alpha\}$ is called positively ordered if $C_{\alpha_1} \preceq C_{\alpha_2}$ whenever $\alpha_1 \leq \alpha_2$. Using the concordance ordering, copulas are bounded.

**Definition 16 (Fréchet Bounds)**

Define on $[0, 1]^n$ the functions $C^-$ and $C^+$ as:

$$C^- (u_1, \ldots, u_n) : = \max \left( \sum_{i=1}^n u_i - n + 1, 0 \right)$$

$$C^+ (u_1, \ldots, u_n) : = \min (u_1, \ldots, u_n)$$

The defined functions are called, respectively, the lower and the upper Fréchet bound.

The lower and upper Fréchet bound will play an important role in the theory of dependence structures. They will form the mathematical definitions of the concepts ‘perfectly negatively dependent’ and ‘perfectly positively dependent’.

**Theorem 17 (Fréchet-Hoeffding Bounds Inequality)**

Let $C$ be any $n$-copula, then:

$$C^- \preceq C \preceq C^+$$

**Proof.** The upper bound:

$$C (u_1, \ldots, u_n) = \mathbb{P}(U_1 \leq u_1, \ldots, U_n \leq u_n)$$

$$\leq \min_{i=1, \ldots, n} \mathbb{P}(U_i \leq u_i)$$

$$= \min_{i=1, \ldots, n} u_i$$

$$= C^+ (u_1, \ldots, u_n)$$

---

5The reason for giving this name will become clear in section 3.3.
The lower bound:

\[ C(u_1, \ldots, u_n) = \mathbb{P}(U_1 \leq u_1, \ldots, U_n \leq u_n) \]
\[ = \mathbb{P}\left( \bigcap_{i=1}^{n} \{U_i \leq u_i\} \right) \]
\[ = 1 - \mathbb{P}\left( \bigcup_{i=1}^{n} \{U_i > u_i\} \right) \text{ (DeMorgan’s Law)} \]
\[ \geq 1 - \sum_{i=1}^{n} \mathbb{P}(U_i > u_i) \]
\[ = 1 - \sum_{i=1}^{n} (1 - u_i) \]
\[ = 1 - n + \sum_{i=1}^{n} u_i \]

Because \( C(u_1, \ldots, u_n) \) defines a probability it should be non-negative. Other steps follow from the fact that \( U_i \sim \text{Uniform}(0,1) \). □

\( C^+ \) is an \( n \)-copula in all dimensions, but it can be shown, see Embrechts et al. 2001b, that \( C^- \) is not a copula for \( n \geq 3 \). In spite of that, \( C^- \) is the best possible lower bound, because of the next theorem:

**Theorem 18 (Attainability Lower Fréchet Bound)**

For any \( n \geq 3 \) and any \( u \in [0,1]^n \), there is an \( n \)-copula \( C \) (which depends on \( u \)) such that

\[ C(u) = C^-(u) \]

**Proof.** See Nelsen 1999. □

In appendix A, Figures A1 and A2, the 3-D graphs of the bivariate \( C^+ \) and \( C^- \) are drawn. Another, more common, way to display bivariate copulas is by drawing the contour diagram. This means that we connect all points \( (u_1, u_2) \) having the same value for \( C(u_1, u_2) \). We do this for several fixed values of \( C(u_1, u_2) \). By plotting those lines in the \( (u_1, u_2) \)-plane we get an idea of how the distribution function looks like. For the bivariate upper- and lower Fréchet bound the contour diagrams can also be found in the appendix, Figures A3 and A4.
2.6 The Product Copula

Previous subsection featured the Fréchet bounds, which will serve as the definition in copula terms of perfectly positively dependent and perfectly negatively dependent. We will now see their pendant, the product copula, which will serve as the mathematical formulation for independence.

**Definition 19 (Product Copula)**

Define on $[0, 1]^n$ the function $C^\perp$, called the product copula, as:

$$C^\perp (u) = u_1 \cdots u_n$$

In Figure A5 and A6 in the appendix the 3-D graph and the contour plot for the bivariate product copula are shown.
3 Association Concepts

3.1 Introduction

In practice, if people want to say something about the degree of association between two random variables, they often report the linear correlation coefficient between those variables.

**Definition 20 (Linear Correlation Coefficient)**

Let \((X, Y)\) be a vector of random variables with nonzero and finite variances. The linear correlation coefficient (also called Pearson’s correlation) for \((X, Y)\) is equal to:

\[
\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}
\]

While \(\rho(X_1, X_2)\) is often used in practice as an indicator for dependence, it is not up to that task in a lot of situations. This is mainly caused by the fact that \(\rho(X_1, X_2)\) is a strictly linear concept. An extensive theoretical treatment of the deficiencies of linear correlation can be found in Embrechts et al. 2001b. The same discussion can be found in Embrechts et al. 1999, but this time more focus is placed on practical issues.

A few reasons why linear correlation might not be the right measure of association to use as an indicator for dependence are:

1. Perfect positive dependence between risks does not necessarily imply that their correlation will be 1. Also, perfect negative dependence between risks does not necessarily imply that the correlation will be -1.

2. Zero correlation does not imply independence of risks.

3. Correlation is not invariant under all increasing transformations of the risks. E.g., \(\log(X)\) and \(\log(Y)\) will generally have a different correlation than \(X\) and \(Y\).

Zero correlation only implies independence in case of a (multivariate) normal distribution. In this section we first define the concept of independence and perfect dependence in a copula setting. After that, three different forms of association measures will be treated: concordance measures, dependence measures, and tail dependence measures. The first two are used to describe the joint behavior of two variables in general, while the last one is used to describe the joint behavior of extreme events.

Concordance between two random variables arises if large values of one variable tend to occur simultaneously with large values of the other variable and small values of one variable occur simultaneously with small values of the other.
variable. Concordance measures are able to locate nonlinear associations between the variables that linear correlation might miss completely.

The main advantage of a measure of dependence over a measure of concordance is that it gives us proof of a functional dependence between two variables. A measure of dependence equals 0 if and only if two variables are independent. A measure of dependence equals 1 if and only if each of the two variables is almost surely\(^6\) a strictly monotone function of the other.

A measure of concordance equals -1 or 1 if the two variables are perfectly dependent. Also a measure of concordance equals 0 whenever two variables are independent. However, these relationships do not hold the other way around! So a value of -1, 0, or 1 is no proof of the fact that variables are independent or perfectly dependent.

The main advantage of a measure of concordance over a measure of dependence is that it tells us something about the nature of the association between the variables. A measure of dependence ‘only’ tells us that two variables are functionally dependent on one another. A measure of concordance also tells us if two variables tend to be high or low simultaneously, or that they tend to deviate in opposite directions.

In this section the two most commonly used association measures, Kendall’s tau and Spearman’s rho, will be presented. We will also look at two commonly used measures of dependence, namely Schweizer and Wolff’s sigma and Hoeffding’s dependence index.

---

\(^6\) A condition holds almost surely if it holds with probability 1, or equivalently, if it holds everywhere except for a set of measure 0.
3.2 Independence and Perfect Dependence

The most distinct levels of association between random variables are the situations of independence and of perfect dependence. Formally, $X_1, \ldots, X_n$ are independent if and only if $h(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$ for all $x_1, \ldots, x_n \in \mathbb{R}$, where $h(x_1, \ldots, x_n)$ is the joint density function of $x_1, \ldots, x_n$ and $f_i(x_i)$ is the density function of $x_i$. Next theorem defines independence in copula terms.

**Theorem 21 (Independence and the Product Copula)**

Let $X_1, \ldots, X_n$ be continuous random variables having copula $C$, then $X_1, \ldots, X_n$ are independent if and only if $C = C^\times$.

**Proof.** This follows directly from Sklar’s theorem combined with the definition of independence. ■

Two variables, $X$ and $Y$ are perfectly dependent if each of the variables is a monotone function of the other variable. This definition can be formulated using the copula concepts of comonotonicity and countermonotonicity. The concept of comonotonicity and countermonotonicity are equivalent to "perfect positive dependence" and "perfect negative dependence" respectively.

**Definition 22 (Comonotonicity or Perfect Positive Dependence)**

The random variables $X$ and $Y$ are comonotonic if they have the copula $C^+$. 

**Theorem 23 (Comonotonicity or Perfect Positive Dependence)**

$X$ and $Y$ are comonotonic if and only if

$$(X, Y) \overset{D}{=} (u(Z), v(Z))$$

for some random variable $Z$ and $u$ and $v$ increasing functions. If $X$ and $Y$ are continuous, then comonotonicity is equivalent to $Y = T(X)$ a.s. for some strictly increasing function $T$.

**Proof.** See De Matteis 2001. ■

**Definition 24 (Countermonotonicity or Perfect Negative Dependence)**

The random variables $X$ and $Y$ are countermonotonic if they have the copula $C^-$. 

**Theorem 25 (Countermonotonicity or Perfect Negative Dependence)**

$X$ and $Y$ are countermonotonic if and only if

$$(X, Y) \overset{D}{=} (u(Z), v(Z))$$

for some random variable $Z$ with $u$ an increasing and $v$ a decreasing function. If $X$ and $Y$ are continuous, then countermonotonicity is equivalent to $Y = T(X)$ a.s. for some strictly decreasing function $T$.

**Proof.** See De Matteis 2001. ■
3.3 Measures of Concordance

Let \((x_1, y_1)\) and \((x_2, y_2)\) be two observations from a vector \((X, Y)\) of continuous random variables. Then the the pairs \((x_1, y_1)\) and \((x_2, y_2)\) are called concordant if \((x_1 - x_2)(y_1 - y_2) > 0\), and discordant if \((x_1 - x_2)(y_1 - y_2) < 0\). First a definition of a measure of concordance will be given.

**Definition 26 (Measure of Concordance)**

A numeric measure \(\kappa\) of association between two continuous random variables \(X\) and \(Y\) whose copula is \(C\) is a measure of concordance if it satisfies the following properties:

1. \(\kappa\) is defined for every pair \(X, Y\) of continuous random variables.
2. \(-1 = \kappa_{X,-X} \leq \kappa_{C} \leq \kappa_{X,X} = 1\).
3. \(\kappa_{X,Y} = \kappa_{Y,X}\).
4. If \(X\) and \(Y\) are independent, then \(\kappa_{X,Y} = \kappa_{C} = 0\).
5. \(\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}\).
6. If \(C_1 \prec C_2\), then \(\kappa_{C_1} \leq \kappa_{C_2}\).
7. If \(\{(X_n, Y_n)\}\) is a sequence of continuous random variables with copulas \(C_n\), and if \(\{C_n\}\) converges pointwise to \(C\), then \(\lim_{n \to \infty} \kappa_{C_n} = \kappa_{C}\).

Criterion number 6 for measures of concordance is the reason why the relationship "\(\prec\)" defined in Definition 15, is called the concordance ordering. From the properties above we can derive the following link between monotone functional dependence of two variables and the value of their concordance measure:

**Theorem 27** Let \(\kappa\) be a measure of concordance for continuous random variables \(X\) and \(Y\).

1. If \(Y\) is almost surely an increasing function of \(X\), then \(\kappa_{X,Y} = \kappa_{C^+} = 1\).
2. If \(Y\) is almost surely a decreasing function of \(X\), then \(\kappa_{X,Y} = \kappa_{C^-} = -1\).
3. If \(\alpha\) and \(\beta\) are strictly monotone functions on Range\((X)\) and Range\((Y)\), respectively, then \(\kappa_{\alpha(X),\beta(Y)} = \kappa_{X,Y}\).

**Proof.** Use Theorem 10 combined with definition 26. □
3 Association Concepts

3.3.1 Kendall’s Tau

Definition 28 (Kendall’s Tau)
Let \((X_1, Y_1)\) and \((X_2, Y_2)\) be two independent and identically distributed random vectors. Then Kendall’s tau between the random variables \(X\) and \(Y\) is defined as:

\[
\tau_{X,Y} = \mathbb{P}[(X_1 - X_2) (Y_1 - Y_2) > 0] - \mathbb{P}[(X_1 - X_2) (Y_1 - Y_2) < 0]
\]

As can be seen from the definition, Kendall’s tau is defined as the probability of concordance minus the probability of discordance for a pair of observations \((x_i, y_i)\) and \((x_j, y_j)\). Looking at an observed sample with sample size \(n\), there are \(\binom{n}{2}\) distinct pairs \((x_i, y_i)\) and \((x_j, y_j)\). Let \(c\) denote the number of concordant pairs, and \(d\) the number of discordant pairs in the sample. Then a sample estimator for Kendall’s tau is given by\(^1\):

\[
\hat{\tau} = \frac{c - d}{c + d} = \frac{c - d}{\binom{n}{2}} = \binom{n}{2}^{-1} \sum_{i<j} \text{sign} [(X_i - X_j) (Y_i - Y_j)]
\]

So far we have not seen how Kendall’s tau between two variables can be linked to the copula function that describes their dependence structure. Next theorem establishes the general formula for calculation of Kendall’s tau from the copula function.

Theorem 29 (Kendall’s Tau)
Kendall’s tau for a vector of continuous random variables \((X, Y)\) with copula \(C\) is given by:

\[
\tau_{X,Y} = 4 \int_0^1 \int_0^1 C(u,v) \, dC(u,v) - 1 \quad (5)
\]


Calculation of this integral is not always straightforward, but fortunately for a lot of copulas this relationship can be simplified.

\(^1\)Note that the given relationship holds only if we ignore the possibility of ‘ties’ occurring. A tie would mean that either the \(X\)’s or the \(Y\)’s being under comparison are exactly equal. An unbiased estimator for \(\tau\) in the case of ties occurring will be given in Section 6.2.
3 Association Concepts

3.3.2 Spearman’s Rho

**Definition 30 (Spearman’s Rho)**

Let \((X_1, Y_1), (X_2, Y_2)\) and \((X_3, Y_3)\) be three independent and identically distributed random vectors. Then Spearman’s rho between the variables \(X\) and \(Y\) is defined as:

\[
\varrho_{X,Y} = 3 \left( \mathbb{P}[(X_1 - X_2) (Y_1 - Y_3) > 0] - \mathbb{P}[(X_1 - X_2) (Y_1 - Y_3) < 0] \right)
\]

A sample estimator for Spearman’s rho is given by:

\[
\hat{\varrho} = \frac{12}{n(n^2 - 1)} \sum_{i=1}^{n} \left( \text{rank}(X_i) - \frac{n+1}{2} \right) \left( \text{rank}(Y_i) - \frac{n+1}{2} \right)
\]

where the rank of an observation denotes its position in an ordered sample:

\[
\text{rank}(X_i) := 1 + \# \{ j | X_j < X_i \} + \frac{1}{2} \# \{ j | j \neq i \text{ and } X_j = X_i \}
\]

Once again this can be expressed in terms of copulas.

**Theorem 31 (Spearman’s Rho)**

Spearman’s rho for a vector of continuous random variables \((X, Y)\) with copula \(C\) is given by:

\[
\varrho_{X,Y} = 12 \int \int_{[0,1]^2} uv dC(u,v) - 3
\]

The definition of Spearman’s rho can be interpreted as the probability of concordance minus the probability of discordance for two vectors \((X_1, Y_1)\) and \((X_2, Y_3)\). I.e., a pair of vectors with equal margins, but one vector has joint distribution function \(H\), and the other vector has independent components. Consequently Spearman’s rho intuitively measures the distance between the copula \(C\) and the copula of independence \(C^\perp\). In other words, Spearman’s rho measures ‘how close to independence the relationship between the two variables is’. This is formalized mathematically (See De Matteis 2001):

\[
\varrho_{X,Y} = 12 \int \int_{[0,1]^2} [C(u,v) - uv] dudv
\]
3.4 Measures of Dependence

Definition 32 (Measure of Dependence)
A numeric measure \( \delta \) of association between two continuous random variables \( X \) and \( Y \) whose copula is \( C \) is a measure of dependence if it satisfies the following properties:

1. \( \delta \) is defined for every pair \( X, Y \) of continuous random variables.
2. \( \delta_{X,Y} = \delta_{Y,X} \).
3. \( 0 \leq \delta_{X,Y} \leq 1 \).
4. \( \delta_{X,Y} = 0 \) if and only if \( X \) and \( Y \) are independent.
5. \( \delta_{X,Y} = 1 \) if and only if each of \( X \) and \( Y \) is almost surely a strictly monotone function of the other.
6. If \( \alpha \) and \( \beta \) are almost surely strictly monotone functions on \( \text{Range}(X) \) and \( \text{Range}(Y) \), respectively, then \( \delta_{\alpha(X), \beta(Y)} = \delta_{X,Y} \).
7. If \( \{(X_n, Y_n)\} \) is a sequence of continuous random variables with copulas \( C_n \), and if \( \{C_n\} \) converges pointwise to \( C \), then \( \lim_{n \to \infty} \delta_{C_n} = \delta_C \).

Intuitively it is clear that measures of dependence are based on a 'distance' between the copula of \( (X, Y) \) and the product copula \( C^\perp \).

3.4.1 Schweizer and Wolff’s Sigma

Definition 33 (Schweizer and Wolff’s Sigma)
Schweizer and Wolff’s Sigma for a vector of continuous random variables \( (X, Y) \) with copula \( C \) is given by:

\[
\sigma_{X,Y} = 12 \iint_{[0,1]^2} |C(u, v) - uv| \, du \, dv
\]

Notice from this definition the similarity with Spearman’s rho. The difference between the two is that this measure reports the absolute distance between the copula under consideration and the product copula whereas Spearman’s rho reports the ‘signed’ distance.

3.4.2 Hoeffding’s Dependence Index

Definition 34 (Hoeffding’s Dependence Index)
Hoeffding’s Dependence Index for a vector of continuous random variables \( (X, Y) \) with copula \( C \) is given by:

\[
\Phi^2_{X,Y} = 90 \iint_{[0,1]^2} |C(u, v) - uv|^2 \, du \, dv
\]
3.5 Tail Dependence

The treated measures of concordance and dependence tell us something about the joint behavior of two variables over their complete outcome space. But let’s assume that we are especially interested in the behavior of the joint distribution of two variables in the tails of the distribution. This is the case, for instance, when we calculate the Value at Risk of a portfolio.

Generally speaking, if we are interested in extreme portfolio returns, then we should focus on the joint extreme behavior of the univariate returns. The 'general joint behavior' of two variables can be described by Kendall’s tau or Spearman’s rho. Joint tail events are described by coefficients of tail dependence. There are two cases of tail dependence, upper tail dependence, the joint occurrence of high return values, and lower tail dependence, the joint occurrence of low (typically negative) return values.

**Definition 35 (Upper Tail Dependence)**

Let $X$ and $Y$ be random variables with distribution functions $F$ and $G$. The coefficient of upper tail dependence between $X$ and $Y$ is:

$$
\lim_{s \to 1} \frac{P(X > F^{-1}(s) \mid Y > G^{-1}(s))}{P(Y > G^{-1}(s))} = \lambda_u
$$

provided a limit $\lambda_u \in [0, 1]$ exists. If $\lambda_u \in (0, 1]$, then $X$ and $Y$ are said to be asymptotically dependent in the upper tail. If $\lambda_u = 0$ they are asymptotically independent in the upper tail.

Next we will see how upper tail dependence can be represented in a copula terms. Assuming the limits exist, a copula definition for the upper tail dependence is derived in the following way:

$$
\lambda_u = \lim_{s \to 1} \frac{P(Y > G^{-1}(s) \mid X > F^{-1}(s))}{P(Y > G^{-1}(s))}
$$

$$
= \lim_{s \to 1} \frac{\frac{P[X > F^{-1}(s), Y > G^{-1}(s)]}{P[X > F^{-1}(s)]}}{P(Y > G^{-1}(s))}
$$

$$
= \lim_{s \to 1} \frac{C(s, s)}{1 - s}
$$

To see how we perform these calculations in practice an example is treated for the Gumbel-Hougaard copula\(^8\).

\(^8\)This copula will be treated in more detail later on.
Example 36 Suppose we have a bivariate Gumbel-Hougaard copula,

\[ C_{\alpha}^{GH}(u, v) = \exp \left[ - \left[ (- \log u)^\alpha + (- \log v)^\alpha \right]^{\frac{1}{\alpha}} \right] \]

for \( \alpha \in [1, \infty) \)

then the coefficient of upper tail dependence is equal to:

\[
\lambda_u = \lim_{s \to 1} \frac{C(s, s)}{1 - s} = \lim_{s \to 1} \frac{1 - 2s + C(s, s)}{1 - s} = \lim_{s \to 1} \frac{1 - 2s + \exp \left[ - \left[ (- \log s)^\alpha + (- \log s)^\alpha \right]^{\frac{1}{\alpha}} \right]}{1 - s} = \lim_{s \to 1} \frac{1 - 2s + 2^{1/\alpha} \log s}{1 - s} = \lim_{s \to 1} \frac{-2 + 2^{1/\alpha} s^{2^{1/\alpha} - 1}}{-1} = 2 - 2^{1/\alpha}
\]

Thus for \( \alpha > 1 \), \( C_{\alpha}^{GH} \) has upper tail dependence. Because \( \alpha \in [1, \infty) \), it follows that \( \lambda_u^{GH} \in [0, 1] \).

Lower tail dependence is used more often in practice, since it gives a lot of insight in the financial risks that we take by holding a certain portfolio of assets.

Definition 37 (Lower Tail Dependence)

Let \( X \) and \( Y \) be random variables with distribution functions \( F \) and \( G \). The coefficient of lower tail dependence between \( X \) and \( Y \) is:

\[
\lim_{s \to 0} \mathbb{P} \left[ Y \leq G^{-1}(s) \mid X \leq F^{-1}(s) \right] = \lambda_l
\]

provided a limit \( \lambda_l \in [0, 1] \) exists. If \( \lambda_l \in (0, 1] \), then \( X \) and \( Y \) are said to be asymptotically dependent in the lower tail. If \( \lambda_l = 0 \) they are asymptotically independent in the lower tail.

Also lower tail dependence can be defined using copulas:

\[
\lambda_l = \lim_{s \to 0} \frac{\mathbb{P} \left[ Y \leq G^{-1}(s) \mid X \leq F^{-1}(s) \right]}{\mathbb{P} \left[ X \leq F^{-1}(s) \right]} = \lim_{s \to 0} \frac{C(s, s)}{s}
\]

\(^9\)This relationship follows from L'Hôpital's Rule: \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \)
4 Copula Families

4.1 Introduction

This section will deal mainly with parametric copulas. The specific inclusion of the word 'parametric' suggests that there is also something like non-parametric copulas, and indeed this is the case. Despite the fact that this thesis will not use such copulas in the practical section, it is useful to understand some of their basics. Three different forms of non-parametric copulas are distinguished in the current literature. The so-called Deheuvels or empirical copula, the kernel approximations to copulas, and copula approximations.

Deheuvels copula functions were introduced by Deheuvels 1979 and are sometimes also called empirical copulas. The empirical copula function is defined similar to an empirical distribution function, by counting the number of outcomes below certain values and dividing that number by the total number of outcomes. Empirical copulas are used in practice, but have the disadvantage that they are discontinuous functions.

Kernel approximations to copula functions were introduced very recently in Fermanian and Scaillet 2003. No other papers had appeared about this approach when this thesis was written.

Approximations to copulas were introduced in Li et al. 1997. In this approach an approximating functional form for the copula is specified. This approximation formula includes the empirical copula. Two important approximating forms are called the Bernstein polynomial, and the checkerboard copula. A drawback is that, although these approximations do converge to the true copula when the number of observations increases, the tail dependence will always equal zero for the approximations, as proven in Durrleman et al. 2000b. This paper suggests using a perturbation method to modify the copula approximation in such a way that the desired tail dependence can be introduced.

This thesis will only look at copula families that are parametric. That is the most commonly used approach in the literature. Because of the specific nature of this research, multivariate distributions of order higher than two, only two specific classes of parametric copulas will be treated. The extension of two-dimensional copulas to copulas of higher dimension is certainly not a trivial one, but for the two classes treated here it is feasible. Unfortunately there is still not so much literature to be found about copulas of order higher than two. Other classes of copulas can be found in the literature, such as the Fréchet Family (See Hürlimann 2001) and Extreme Value copulas (See Joe 1997).

The first class of copulas that will be treated is the elliptical copulas, specifically the Gaussian and $t$-copulas. Elliptical copulas are copulas derived from multivariate elliptical distributions (definition see Embrechts et al. 1999)
using Sklar’s theorem. Deriving Gaussian and $t$-copulas with more than two dimensions is done equivalently to the bivariate case (See example 8). However, the class of elliptical copulas has an unfavorable property when talking about application in the field of finance. The dependence structure is such that some 'stylized facts' in financial data cannot be represented correctly. For instance, as already mentioned in the Introduction, the asymmetry of the lower and upper tail of a distribution cannot be described properly by an elliptical copula. This is because elliptical copulas exhibit 'radial symmetry'. A radial symmetric bivariate copula is a copula which has the property that $C(u, v) = C(v, u)$ and $C(u, v) = \tilde{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$. Another fact that is sometimes mentioned as a drawback of elliptical copulas is that they do not have closed form expressions.

Therefore another class of copulas might be worth looking at, the so-called Archimedean copulas. This class of copulas allows for a great variety of different dependence structures. Also, all commonly used Archimedean copulas have closed form expressions. Archimedean copulas can be characterized by a single parameter function each, which is enough to specify their complete multivariate form. The main drawback of Archimedean copulas (as they are presented here) however, is that they only have one single parameter to describe the complete dependence structure. The Gaussian and $t$-copula on the other hand have one parameter for each bivariate margin.
4.2 Elliptical Copulas

4.2.1 Gaussian Copula

The Gaussian copula is obtained by 'filtering out' all univariate Gaussian distributions from the multivariate Gaussian distribution. This is done by applying Sklar’s theorem, see Example 8. Here we will look at the general multivariate Gaussian copula of dimension $n$.

**Definition 38 (Multivariate Gaussian Copula)**

Let $\Phi$ denote the standard univariate Gaussian distribution function and let $\Phi^\nu_{\Sigma}$ denote the standard multivariate $n$-dimensional Gaussian distribution function with linear correlation matrix $\Sigma$. The $n$-dimensional Gaussian copula is then defined as:

$$C^{Ga}_\Sigma (u_1, \ldots, u_n) = \Phi^\nu_{\Sigma} (\Phi^{-1} (u_1), \ldots, \Phi^{-1} (u_n))$$

Here we observe a drawback of the elliptical copulas in general. They have no closed form expression. The definition above uses the Gaussian distribution function and inverse distribution function. Equivalently, the following definition is often used:

**Definition 39 (Multivariate Gaussian Copula)**

The $n$-dimensional Gaussian copula with linear correlation matrix $\Sigma$ is defined as:

$$C^{Ga}_\Sigma (u_1, \ldots, u_n) = \int_{-\infty}^{\Phi^{-1} (u_1)} \cdots \int_{-\infty}^{\Phi^{-1} (u_n)} \exp \left( -\frac{\omega^\top \Sigma^{-1} \omega}{2} \right) \frac{1}{|\Sigma| (2\pi)^{n/2}} d\omega_n \cdots d\omega_1$$

where $|\Sigma|$ stands for the determinant of $\Sigma$ and $\omega = (\omega_1, \ldots, \omega_n)$.

Parameter estimation for this copula will be done using methods based on maximum likelihood. Therefore we need an expression for the Gaussian copula density, which is provided by next theorem.

**Theorem 40 (Gaussian Copula Density)**

The copula density for a multivariate Gaussian copula is given by:

$$c^{Ga}_\Sigma (u_1, \ldots, u_n) = \frac{1}{|\Sigma|^{1/2}} \exp \left( -\frac{1}{2} \omega^\top (\Sigma^{-1} - I) \omega \right)$$

where $\omega = (\omega_1, \ldots, \omega_n) = (\Phi^{-1} (u_1), \ldots, \Phi^{-1} (u_n))$.

**Proof.** Use relationship (4) and insert the multivariate Gaussian distribution function and the univariate Gaussian density functions to obtain:

$$\frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp \left( -\frac{1}{2} \Sigma^{-1} x \right) = c (\Phi (x_1), \ldots, \Phi (x_n)) \left( \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x_i^2 \right) \right)$$
Now we write out the product and transform the variables: \( u_i = \Phi(x_i), \ i = 1, \ldots, n, \) finding:

\[
C_{\Sigma}^{\alpha}(u_1, \ldots, u_n) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp \left( -\frac{1}{2} \omega^\top \Sigma^{-1} \omega \right)
\]

where \( \omega = (\omega_1, \ldots, \omega_n) = \left( \Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n) \right). \) After getting rid of the redundant terms and some rearranging we are left with the above expression.

In the appendix, Figures A7a, A7b, and A7c, the contour plots for the density of the bivariate Gaussian copula can be found. For \( \rho = 0.2, 0.5 \) and 0.8 a few values of the density are plotted. When \( \rho \) tends to 1, the density will be very close to the line \( u_1 = u_2. \) Note that if the correlation coefficients would be negative, then the plots would all rotate over 90°.

We have already seen in example 8 that a multivariate Gaussian distribution is formed by combining a Gaussian copula with all univariate Gaussian marginals.

### 4.2.2 \( t \)-Copula

The \( t \)-copula is obtained by 'filtering out' all univariate \( t_\nu \)-distributions from the multivariate \( t_{\Sigma, \nu} \)-distribution. Again, this follows directly from Sklar’s theorem.

**Definition 41 (Multivariate \( t \)-Copula)**

Let \( t_\nu \) denote the univariate \( t \)-distribution function with \( \nu \) degrees of freedom, and let \( t_{\Sigma, \nu} \) denote the standard multivariate \( n \)-dimensional \( t \)-distribution function with linear correlation matrix \( \Sigma \) and degrees of freedom \( \nu \). The \( n \)-dimensional \( t \)-copula is then defined as:

\[
C_{\Sigma, \nu}^t(u_1, \ldots, u_n) = t_{\Sigma, \nu} \left( t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n) \right)
\]

Equivalently, the following definition is often used:

**Definition 42 (Multivariate \( t \) Copula)**

The \( n \)-dimensional \( t \)-copula with linear correlation matrix \( \Sigma \) and degrees of freedom \( \nu \), is defined as:

\[
C_{\Sigma, \nu}^t(u_1, \ldots, u_n) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_n)} \frac{\Gamma((\nu + n)/2) \left( 1 + \omega^\top \Sigma^{-1} \omega / \nu \right)^{-(\nu+n)/2}}{|\Sigma|^{1/2} \Gamma(\nu/2)(\nu \pi)^{n/2}} d\omega_n \cdots d\omega_1
\]

where \( |\Sigma| \) stands for the determinant of \( \Sigma \), \( \omega = (\omega_1, \ldots, \omega_n) \) and \( \Gamma(\cdot) \) is the Gamma function.

\(^{10}\)Where \( \rho \) is the linear correlation. Here \( \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \)
Using similar steps as for the Gaussian copula, we can derive the corresponding density for the $t$-copula:

**Theorem 43 (t Copula Density)**

The copula density for a multivariate $t$-copula is given by:

$$c_{\Sigma,\nu}(u_1, \ldots, u_n) = \frac{\Gamma \left( \frac{(\nu + n)/2}{2} \right) [\Gamma \left( \nu/2 \right)]^{n-1} \left( 1 + \omega^T \Sigma^{-1} \omega / \nu \right)^{-\frac{(\nu + n)/2}{2}}}{|\Sigma|^{1/2} \Gamma \left( \frac{(\nu + 1)/2}{2} \right) \prod_{i=1}^{n} \left( 1 + \omega_i^2 / \nu \right)^{-\frac{(\nu + 1)/2}{2}}}
$$

where $\omega = (\omega_1, \ldots, \omega_n) = \left( t_{\nu}^{-1}(u_1), \ldots, t_{\nu}^{-1}(u_n) \right)$.

In the appendix, Figures A8a, A8b, and A8c, several contour plots can be found for a bivariate $t_8$ copula. For $\rho = 0.2, 0.5$ and $0.8$ a few values of the density are plotted. Note that if the number of degrees of freedom for the copula goes to infinity, the $t$-copula will converge to a Gaussian copula (with equal $\Sigma$). If the correlation coefficients would be negative, then the plots would all rotate over $90^\circ$.

A multivariate $t$-distribution is formed by combining a $t$-copula with all univariate $t$ distributed marginals. It is not difficult to see that if we take the limit of a $t$-copula for $\nu \to \infty$, that we will end up having a Gaussian copula.

### 4.2.3 Kendall’s Tau and Spearman’s Rho

The most often-used measures of concordance are Kendall’s tau and Spearman’s rho. In an ideal situation we prefer to have a copula family for which those two measures can stretch over the complete interval $[-1, 1]$. For the Gaussian and $t$-copulas the following result concerning Kendall’s tau can be derived:

**Theorem 44 (Kendall’s Tau for Gaussian and $t$ Copulas)**

Let $(X_1, \ldots, X_n)$ have copula $C$, where $C$ is either a Gaussian or a $t$-copula. Then for all $i$ and $j \in \{1, \ldots, n\}$

$$\tau_{X_i, X_j} = \frac{2}{\pi} \arcsin \rho_{ij} \quad (9)$$

where $\rho_{ij}$ denotes the linear correlation coefficient between $X_i$ and $X_j$.

**Proof.** For a proof of an extended version of this theorem see LINDSKOG ET AL. 2001.

This result shows us that for elliptical copulas, Kendall’s tau is a transformation of the linear correlation coefficient. Also the theorem implies that
for the Gaussian and \( t \)-copula, Kendall’s tau can cover the complete interval \([-1, 1]\). For Spearman’s rho the relationship is more complicated. In Hult and Lindskog 2001 it is proven that Spearman’s rho is not invariant in the class of elliptical distributions with continuous univariate margins and a fixed covariance matrix. Therefore, when measuring the degree of concordance in the data and in the theoretical copulas, people in general only use Kendall’s tau. For the sake of completeness a result for the Gaussian case is included.

**Theorem 45 (Spearman’s Rho for Gaussian Copulas)**

Let \((X_1, \ldots, X_n)\) be non-degenerate random variables, having a Gaussian copula \( C \). Then for all \( i \) and \( j \in \{1, \ldots, n\} \)

\[
\rho_{X_i, X_j} = \frac{6}{\pi} \arcsin \left( \frac{\rho_{ij}}{2} \right)
\]

where \( \rho_{ij} \) denotes the linear correlation coefficient between \( X_i \) and \( X_j \).

**Proof.** For a proof of an extended version of this theorem see Hult and Lindskog 2001. ■

So for variables with a dependence relationship that is perfectly described by a Gaussian copula, Spearman’s rho is a transformation of the linear correlation coefficient and it can cover the complete interval \([-1, 1]\).

### 4.2.4 Tail Dependence

If one is particularly interested in the joint occurrence of extreme values, then the already defined notion of tail dependence is important. The Gaussian copula is assumed to describe the dependence structure of returns in a lot of portfolio theories. This often happens implicitly, by postulating a multivariate Gaussian distribution for the asset returns. When trying to understand why the assumption of joint normality might not be very realistic, copula theory can provide an answer.

First notice that in case of elliptical distributions the upper and lower tail dependence is exactly equal. Therefore, for elliptical distributions the notation \( \lambda \) will be used, being upper-, as well as the lower tail dependence coefficient.

Secondly, if we want to calculate the degree of tail dependence between two variables having copula \( C \), we see that it is not straightforward if we don’t have an explicit formula for the copula function\(^\text{12}\). Consider a pair of random variables \((X, Y)\) having a bivariate copula \( C \). \( X \) has distribution function \( F \), and \( Y \) has distribution function \( G \). To proceed we can first use l’Hôpital’s Rule:

\(^{12}\)This is the case for the Gaussian and \( t \)-copula. They are given in terms of either distribution functions and inverses thereof or as integral formulas.
\[
\lambda = \lim_{s \to 1} \mathbb{P} \left[ Y > G^{-1} (s) \mid X > F^{-1} (s) \right]
\]
\[
= \lim_{s \to 1} \frac{C(s, s)}{1 - s}
\]
\[
= -\lim_{s \to 1} \frac{dC(s, s)}{ds}
\]
\[
= -\lim_{s \to 1} \left[ -2 + \frac{\partial}{\partial a} C(a, b) \big|_{a=b=s} + \frac{\partial}{\partial b} C(a, b) \big|_{a=b=s} \right]
\]
\[
= -\lim_{s \to 1} \left[ -2 + \mathbb{P}[V \leq s \mid U = s] + \mathbb{P}[U \leq s \mid V = s] \right]
\]
\[
= \lim_{s \to 1} \left( \mathbb{P}[V > s \mid U = s] + \mathbb{P}[U > s \mid V = s] \right)
\]
\[
= 2 \lim_{s \to 1} \mathbb{P}[V > s \mid U = s]
\]

Now we use the fact that \( F = G \) (because \( C \) is a bivariate Gaussian or \( t \)-copula) for a quantile transformation and we find:

\[
\lambda = 2 \lim_{s \to 1} \mathbb{P} \left[ F^{-1} (V) > F^{-1} (s) \mid F^{-1} (U) = F^{-1} (s) \right]
\]
\[
= 2 \lim_{s \to \infty} \mathbb{P} \left[ F^{-1} (V) = s \right]
\]
\[
= 2 \lim_{s \to \infty} \mathbb{P} \left[ Y > s \mid X = s \right]
\]

Now all tools are available to obtain a result for the degree of tail dependence for a bivariate Gaussian copula.

**Theorem 46 (Tail Dependence for the Bivariate Gaussian Copula)**

For a bivariate Gaussian copula \( C_{\Phi}^{G^a} (u, v) \) the degree of tail dependence is equal to:

\[
\lambda = 2 \lim_{s \to \infty} \left[ 1 - \Phi \left( s \sqrt{1 - \rho} \right) \right]
\]

where \( \rho \) is the degree of linear correlation between \( x = F^{-1} (u) \) and \( y = G^{-1} (v) \).

Thus, unless \( \rho \) is equal to 1, the Gaussian copula has asymptotic tail independence.

**Proof.** For the bivariate Gaussian copula \( F = G = \Phi \) (because the margins of a bivariate Gaussian distribution are both univariate Gaussian). A second property that is used is: \( Y \mid X = s \sim N (\rho s, 1 - \rho^2) \). This result yields us:

\[
\lambda = 2 \lim_{s \to \infty} \mathbb{P} \left[ Y > s \mid X = s \right]
\]
\[
= 2 \lim_{s \to \infty} \left[ 1 - \mathbb{P} [Y \leq s \mid X = s] \right]
\]
\[
= 2 \lim_{s \to \infty} \left[ 1 - \Phi \left( \frac{s - \rho s}{\sqrt{1 - \rho^2}} \right) \right]
\]

\[13\] Because if \( s / 1 \) then \( F^{-1} (s) \to \infty \). Now rename \( F^{-1} (s) \) into \( s \).
Simplifying this gives us the requested formula. ■

For a bivariate $t$-copula another result can be obtained.

**Theorem 47 (Tail Dependence for the Bivariate $t$ Copula)**

For a bivariate $t$-copula $C_{\nu,\rho}(u, v)$ the degree of tail dependence is equal to:

$$
\lambda = 2 \left( 1 - t_{\nu+1} \left( \frac{\sqrt{\nu + 1} \sqrt{1 - \rho}}{\sqrt{1 + \rho}} \right) \right)
$$

where $\rho$ is the degree of linear correlation between $x = F^{-1}(u)$ and $y = G^{-1}(v)$. Thus, unless $\rho$ is equal to -1, the $t$-copula has asymptotic tail dependence.

**Proof.** For the bivariate $t_\nu$ copula $F = G = t_\nu$ (because the margins of a bivariate $t_{\nu,\rho}$-distribution are both univariate $t_\nu$-distributions). The second fact that is to be used is that:

$$
\sqrt{\frac{\nu + 1}{\nu + s^2}} \cdot \frac{Y - \rho s}{\sqrt{1 - \rho^2}} \sim t_{\nu+1}
$$

From the result for the Gaussian copula we can conclude that even if the degree of linear correlation is very close to 1, very extreme events occur independently in each margin. In practice this could be unrealistic. Let’s look at a portfolio of assets for instance. If we know that the value of one of the assets drops with a very significant amount, does that affect our beliefs about the other assets in our portfolio? If we insist on using a Gaussian copula to model the joint distribution of the asset values, then we implicitly give the answer: ”no”, to the previous question. By using a $t$-copula instead of a Gaussian one, we have control over the degree of tail dependence by changing the degrees of freedom. Furthermore, we have the Gaussian copula as the limiting case of the $t$-copula if the degrees of freedom $\nu$ goes to infinity.

In Table 1 the degree of tail dependence for a bivariate $t$-copula is tabulated for several degrees of freedom and degrees of linear correlation. As can be seen from the table, the $t$-copula has tail dependence for all values of $\rho \neq -1$. The strength of the dependence is increasing in $\rho$ and decreasing in $\nu$. For $\nu$ equals infinity we have back the case of a Gaussian copula.

<table>
<thead>
<tr>
<th>$\nu$ \ $\rho$</th>
<th>-0.5</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.0577</td>
<td>0.1817</td>
<td>0.2722</td>
<td>0.3910</td>
<td>0.5594</td>
<td>0.7177</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0.0117</td>
<td>0.0756</td>
<td>0.1438</td>
<td>0.2532</td>
<td>0.4366</td>
<td>0.6298</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1.3-10^{-1}</td>
<td>0.0069</td>
<td>0.0261</td>
<td>0.0819</td>
<td>0.2360</td>
<td>0.4627</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>9.3-10^{-3}</td>
<td>1.6-10^{-1}</td>
<td>0.0019</td>
<td>0.0151</td>
<td>0.0979</td>
<td>0.3054</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>7.6-10^{-4}</td>
<td>4.2-10^{-1}</td>
<td>1.5-10^{-1}</td>
<td>0.0030</td>
<td>0.0435</td>
<td>0.2110</td>
<td>1</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Tail dependence for a bivariate $t$-copula for several values of $\rho$ and $\nu$. 
4 Copula Families

4.3 Archimedean Copulas

4.3.1 Definition and Properties

Archimedean copulas are copulas ‘generated’ or characterized by a single parameter function $\varphi$ that satisfies certain properties. Using this function $\varphi$ we can specify the complete multivariate copula function. Unfortunately, bivariate Archimedean copulas cannot always be extended straightforwardly. In a lot of cases technical conditions are needed to ensure the fact that the extension is a valid $n$-copula. An advantage however of Archimedean copulas, is that they do have closed form expressions and they can be classified easily due to the generating function $\varphi$.

First we look at the two-dimensional case. Here is an overview of how the properties of the function $\varphi$, that forms the basic definition for any Archimedean copula.

**Definition 48 (Pseudo-Inverse of a Function)**

Let $\varphi$ be a continuous, strictly decreasing function from $[0, 1]$ to $[0, \infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of $\varphi$ is the function $\varphi^{-1} : [0, \infty] \to [0, 1]$ given by:

$$
\varphi^{-1}(t) = \begin{cases} 
\varphi^{-1}(t), & 0 \leq t \leq \varphi(0), \\
0, & \varphi(0) \leq t \leq \infty.
\end{cases}
$$

It follows from the definition that $\varphi^{-1}$ is continuous and non-increasing on $[0, \infty]$, and strictly decreasing on $[0, \varphi(0)]$. Furthermore, $\varphi^{-1}(\varphi(u)) = u$ on $[0, 1]$ and

$$
\varphi(\varphi^{-1}(t)) = \begin{cases} 
t, & 0 \leq t \leq \varphi(0), \\
\varphi(0), & \varphi(0) \leq t \leq \infty.
\end{cases}
$$

Finally, if $\varphi(0) = \infty$, then $\varphi^{-1} = \varphi^{-1}$. Now we define the Archimedean copula function, using the function $\varphi$.

**Theorem 49 (Two Dimensional Archimedean Copula)**

Let $\varphi$ and $\varphi^{-1}$ be defined as above. Let $C$ be the function from $[0, 1]^2$ to $[0, 1]$ given by

$$
C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)).
$$

Then $C$ is a copula if and only if $\varphi$ is convex.

**Proof.** See Nelsen 1999.

Proving that (10) indeed defines a proper copula is not so easy. Crucially is proving that it satisfies (1). But as a consequence of this theorem we only have to verify the properties of $\varphi$ in practice. This is shown for a bivariate Gumbel-Hougaard copula.
Example 50 (Bivariate Gumbel-Hougaard Family)
If we choose \( \varphi(t) = (-\ln t)^{\alpha} \), with \( \alpha \in [1, \infty) \), we call \( C_\alpha(u, v) \) the Gumbel-Hougaard Family\(^{14}\). The expression for the copula family is given by:

\[
C_{\alpha}^{GH}(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)) = \exp\left[-((\ln u)^{\alpha} + (\ln v)^{\alpha})^{\frac{1}{\alpha}}\right]
\]

Let’s check if the required properties for \( \varphi \) are satisfied.

- \( \varphi(t) \) is continuous.
- \( \varphi(1) = 0 \).
- \( \varphi'(t) = -\frac{\alpha}{t} (-\ln t)^{\alpha-1} \), and so \( \varphi \) is a strictly decreasing function from \([0, 1]\) to \([0, \infty]\).
- \( \varphi''(t) = \frac{\alpha(\alpha-1)}{t^{2}} (-\ln t)^{\alpha-2} \geq 0 \) for \( \alpha \geq 1 \), and so \( \varphi \) is convex.

In Example 50 we can verify that for the bivariate Gumbel-Hougaard copula \( \varphi(0) = \infty \). Any generating function with this property is called a strict generator. The corresponding copulas are called strict Archimedean copulas. In the literature people usually work with strict Archimedean copulas if the number of dimensions is larger than two. Because all of the Archimedean copulas that are used in the Applications section indeed have a strict generator we will from now on assume that we are dealing with strict Archimedean copulas only. Next theorem establishes two nice properties that hold for Archimedean copulas.

**Theorem 51** Let \( C \) be an Archimedean copula with generator \( \varphi \). Then:

1. \( C \) is symmetric, \( C(u, v) = C(v, u) \) for all \( u, v \in [0, 1] \).
2. \( C \) is associative, \( C(C(u, v), w) = C(u, C(v, w)) \) for all \( u, v, w \in [0, 1] \).

**Proof.** Assume that \( u, v, w \in [0, 1] \), then

1. \( C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)) = \varphi^{-1}(\varphi(v) + \varphi(u)) = C(v, u) \).
2. 

\[
C(C(u, v), w) = \varphi^{-1}\left(\varphi\left(\varphi^{-1}(\varphi(u) + \varphi(v))\right) + \varphi(w)\right)
= \varphi^{-1}(\varphi(u) + \varphi(v) + \varphi(w))
= \varphi^{-1}(\varphi(u) + \varphi(\varphi^{-1}(\varphi(v) + \varphi(w))))
= C(u, C(v, w))
\]

\(^{14}\)Some authors simply refer to this family as the Gumbel Family.
Looking at the above proof of the associativity property for Archimedean copulas one could get an idea for constructing multivariate extensions of bivariate Archimedean copulas.

\[
C(C(u, v), w) = \varphi^{-1}(\varphi(u) + \varphi(v) + \varphi(w)) = C(u, v, w)
\]

In general, if we take any bivariate copula \(C(u, v)\) and try to extend it by constructing \(C(C(u, v), w)\), then this will usually not work. For Archimedean copulas however this extension technique does work. Now a definition for \(n\)-dimensional Archimedean copulas can be given. Continuously using the associativity property suggests that a general \(n\)-dimensional Archimedean copula takes the form:

\[
C(u_1, \ldots, u_n) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_n)) = \varphi^{-1}\left(\sum_{i=1}^{n} \varphi(u_i)\right)
\]  

(11)

But we need an extra technical condition to make sure that this defines a proper \(n\) copula for all \(n \geq 2\). Once again the difficulty is that a copula has to satisfy formula (1). For strict Archimedean copulas we can derive a technical requirement.

**Theorem 52** Let \(\varphi\) be a continuous, strictly decreasing, function from \([0, 1]\) to \([0, \infty]\) such that \(\varphi(0) = \infty\) and \(\varphi(1) = 0\), and let \(\varphi^{-1}\) denote the inverse of \(\varphi\). If \(C\) is the function from \([0, 1]^{n}\) to \([0, 1]\) given by (11), then \(C\) is an \(n\)-copula for all \(n \geq 2\) if and only if \(\varphi^{-1}\) is completely monotone\(^{15}\) on \((0, \infty)\).

**Proof.** See Schweizer and Sklar 1983. ■

Note that for non strict Archimedean copula a similar rule can be derived. If \(\varphi^{-1}\) is \(m\)-monotone on \([0, \infty)\) for some \(m \geq 2\), that is, the derivatives of \(\varphi^{-1}\) alter in sign up to and including the \(m\)-th order on \([0, \infty)\), then the function given in (11) defines a proper \(n\)-copula for \(2 \leq n \leq m\). In practice people only look at multivariate copulas with a strict generator. It can be derived that any strict Archimedean copula can only model positive dependence.

**Theorem 53** If the inverse \(\varphi^{-1}\) of a strict generator \(\varphi\) of an Archimedean copula \(C\) is completely monotone, then \(C \succ C^+\).

**Proof.** See Nelsen 1999. ■

In appendix B there is a table containing the Archimedean copulas that will be used in the remainder of this thesis. Also some properties of these copulas

\(^{15}\)A function \(g(t)\) is completely monotone on the interval \(I\) if it has derivatives of all orders which alternate in sign, i.e.

\[
(-1)^k \frac{d^k g(t)}{dt^k} \geq 0
\]

for all \(t\) in the interior of \(I\) and \(k = 0, 1, 2, \ldots\).
can be found there. All of the Archimedean copulas listed are proper \( n \)-copulas for all \( n \geq 2 \). The copulas in the table are from Nelsen 1999. Since a lot of Archimedean copulas have not been given a name, the ones that have no name (yet) have been numbered in Roman numerals.

4.3.2 Kendall’s Tau
Kendall’s tau can be found in general by calculating a double integral of the copula function \( C \), see formula (5). In a lot of cases this integral is not straightforward to solve, and one could try to use numerical integration methods. For the Archimedean copulas however these calculations can be simplified. That is because Kendall’s tau can be written as a single integral over the ratio of the generator and its derivative.

Theorem 54 (Kendall’s Tau for Archimedean Copulas)
Let \( X \) and \( Y \) be random variables with an Archimedean copula \( C \), generated by \( \varphi \). Then Kendall’s tau of \( X \) and \( Y \) is given by

\[
\tau_{X,Y} = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} \, dt
\]


In appendix B a simple formula (if available) is shown for Kendall’s tau for each of the Archimedean copulas. This formula is simply the result of previous theorem. Also in the appendix is the complete interval that Kendall’s tau can cover depending on the copula parameter.

4.3.3 Tail Dependence
Because of the specific form of Archimedean copulas, also tail dependence can be expressed in terms of the copula-generating function. Elliptical copulas are radial symmetric, which typically means that \( C = \hat{C} \), and consequently they have the same coefficient of upper and lower tail dependence. This is generally not the case for Archimedean copulas. We derive for an Archimedean copula the expressions for the coefficients of upper and lower tail dependence in terms of their generating function.

Theorem 55 (Upper Tail Dependence for Strict Archimedean Copulas)
Let \( \varphi \) be a strict generator of the Archimedean copula \( C(u,v) \). The coefficient of upper tail dependence is given by:

\[
\lambda_u = 2 - \lim_{s \to 0} \frac{\varphi^{-1}(2s)}{\varphi^{-1}(s)}
\]

Moreover, it can be shown that the Frank copula is the only Archimedean copula that exhibits the property of radial symmetry (see Nelsen 1999).
Proof. From the definition of upper tail dependence, \( \lambda_u = \lim_{s \to 1} \frac{\mathcal{C}(s,s)}{1-s} \), we derive (again using L'Hôpital's Rule):

\[
\begin{align*}
\lambda_u &= \lim_{s \to 1} \frac{1-2s + C(s,s)}{1-s} \\
&= \lim_{s \to 1} \frac{1-2s + \varphi^{-1}(2\varphi(s))}{1-s} \\
&= \lim_{s \to 1} \frac{-2 + \varphi^{-1}(2\varphi(s)) \cdot 2\varphi'(s)}{-1} \\
&= 2 - 2\lim_{s \to 1} \frac{\varphi^{-1}(2\varphi(s))}{\varphi^{-1}(\varphi(s))} \\
&= 2 - 2\lim_{s \to 0} \frac{\varphi^{-1}(2s)}{\varphi^{-1}(s)} \\
&= 2 - 2\lim_{s \to 0} \frac{\varphi^{-1}(2s)}{\varphi^{-1}(s)}
\end{align*}
\]

Theorem 56 (Lower Tail Dependence for Strict Archimedean Copulas)
Let \( \varphi \) be a strict generator of the Archimedean copula \( C(u,v) \). The coefficient of lower tail dependence is given by:

\[
\lambda_l = 2 \lim_{s \to 0} \frac{\varphi^{-1}(2s)}{\varphi^{-1}(s)}
\]

Proof. Use the definition of lower tail dependence, \( \lambda_l = \lim_{s \to 0} \frac{\mathcal{C}(s,s)}{s} \). Now proceed in similar fashion as the derivation above.

The coefficients of upper and lower tail dependence for the Archimedean copulas can be found in appendix B. The numbers there are the direct results of the previous two theorems.

4.3.4 Distribution Level Curves
The theory in this paragraph will be the basic ingredient for simulation from Archimedean copulas. The usual way of thinking about copulas, or joint distribution functions in general, is: "What is the probability that the variables of interest are jointly below certain values?". In this subsection we will apply the reverse way of reasoning: "Assume that we choose a certain probability level, which values for the variables correspond to that probability?"

The level set of a bivariate copula \( C(u,v) \) for some \( t \in [0,1] \) is given by \( \{ (u,v) \in [0,1]^2 | C(u,v) = t \} \). For a bivariate Archimedean copula that means

\[
\varphi^{-1}(\varphi(u) + \varphi(v)) = t
\]

\[\text{Use the fact that } f'(x) = \frac{1}{f^{-1}(f(x))}.\]

\[\text{Because if } s \to 1 \Rightarrow \varphi(s) \to 0.\]
or equivalently
\[ \varphi(u) + \varphi(v) = \varphi(t) \]

It can be proven that the level curves for Archimedean copulas are convex (See Nelsen 1999).

Now we first define the \( H \)-measure:

**Definition 57 (H-Measure)**

Assume that \( X \) and \( Y \) are random variables in \( \mathbb{R} \) with bivariate joint distribution function \( H \). Let \( A \) be a subset of \( \mathbb{R}^2 \). Then the \( H \)-measure of \( A \subset \mathbb{R}^2 \) is defined by

\[
P((X, Y) \in A)
\]

where

\[(X, Y) \sim H\]

Denote the level curves of a bivariate Archimedean copula by

\[ A_t(C) := \left\{ (u, v) \in [0, 1]^2 | C(u, v) = t \right\} \]

Another interesting quantity that we could look at is the region in \( [0, 1]^2 \) that lies on or below a certain level curve. That is, the region defined by the set \( \left\{ (u, v) \in [0, 1]^2 | C(u, v) \leq t \right\} \). For a bivariate Archimedean copula this is the set \( \left\{ (u, v) \in [0, 1]^2 | \varphi(u) + \varphi(v) \geq \varphi(t) \right\} \). Next theorem gives an explicit form for the \( C \)-measure of the region that lies on or below a level curve.

**Theorem 58 (C-Measure of Region On or Below a Level Curve)**

Let \( C(u, v) \) be a bivariate Archimedean copula with generator \( \varphi \). Let \( K_C(t) \) denote the \( C \)-measure of the set \( \left\{ (u, v) \in [0, 1]^2 | C(u, v) \leq t \right\} \), or equivalently, the \( C \)-measure of the set \( \left\{ (u, v) \in [0, 1]^2 | \varphi(u) + \varphi(v) \geq \varphi(t) \right\} \). Then for any \( t \in [0, 1] \)

\[
K_C(t) = t - \frac{\varphi(t)}{\varphi'(t^+)}
\]

where \( \varphi'(t^+) \) is equal to \( \lim_{s \rightarrow t^+} \varphi'(s) \).

**Proof.** See Nelsen 1999. □

But what does this \( C \)-measure tell us, how should we read it? The interpretation of this measure, and consequently its importance for simulation purposes, follows from next corollary.
Corollary 59 (C-Measure of Region On or Below a Level Curve)

If \((U, V)\) has distribution function \(C\), where \(C\) is an Archimedean copula generated by \(\varphi\), then the function \(K_C\) given by (13) is the distribution function of the random variable \(C(U, V)\).

The function \(K_C\) as defined above is used in Genest et al. 1995 to find an estimate for the parameter in a bivariate Archimedean copula. Their approach can be extended to the multivariate case\(^{19}\). In this paper the function \(K_C\) will be used for purposes of simulating random variates from Archimedean copulas only (also for the multivariate case).

\(^{19}\) The reason that it is not used for estimation purposes in this thesis, is because of the fact that \(n\)-th derivatives have to be taken. With \(n\) the dimension of the copula. This could not be done analytically by the software.
5 Copula Estimation

5.1 Introduction

When estimating parameters in a parametric framework it is common to use the method of Maximum Likelihood. That is because this method has several appealing properties. The statistical properties of a Maximum Likelihood Estimator (MLE) include (providing certain regularity conditions are satisfied):

1. Consistency,
2. Asymptotic normality,
3. Asymptotic efficiency (it attains the Cramér-Rao Lower Bound), and
4. Invariance with respect to reparametrization.

These properties are the main reason why people search for an MLE in the first place. But in practice sometimes things can be less clear. This is especially because of two reasons.

1. The method of Maximum Likelihood depends upon explicit distributional assumptions.
2. Finite sample properties can be quite different from the appealing asymptotical properties.

In this thesis the three most commonly used ML based estimation methods for copula estimation will be presented. First there is the method of Exact Maximum Likelihood (EML). This method is the well-known 'classical' ML estimation procedure. This means that we will jointly estimate the parameters for the margins and the parameter(s) describing the dependence structure, the copula parameter(s). Notice however, that the whole idea behind the use of copulas is not used in this method. When applying EML no separation of margins from the dependence structure is needed. A disadvantage of using the EML method in practice, is that for higher dimensional distributions this method will become computationally intensive (Bouyé et al. 2000). Another disadvantage is that the copula parameters will be estimated under distributional assumptions for the margins.

A second approach that can be used is called the method of Inference Functions for Margins (IFM). In applying this method we do need the copula theory of separation of margins and dependence structure. We write down the Likelihood Function using the copula representation. This means that we split the parameters into specific parameters for the marginal distributions and common parameters for the dependence structure (the copula parameters). Now we perform a two-step estimation technique. First we estimate the parameters of the univariate marginal distributions. Using the resulting parameter estimates we
maximize the likelihood function to find the copula parameter estimates in the second step. In general we find that the parameters as estimated by the EML and the IFM method are different. This estimation technique, just like the EML method, has the important disadvantage that distributional assumptions for the margins must be made. Consequently the quality of the resulting copula parameter estimates depend upon the quality of the assumptions that were made about the margins.

The third approach, called Canonical Maximum Likelihood (CML), also uses the copula idea of a separation of margins and dependence structure. The drawback of the EML and IFM method is that we estimated the parameters of the copula under distributional assumptions for the margins. Our estimates for the copula parameters will be margin-dependent. It would be better to use the observed margins in the estimation procedure instead of an assumption about the distribution. We can do that by transforming the data into uniform variates by using the empirical distributions for all margins. This procedure was first suggested by Genest et al. 1995. A misspecification of the margins can yield biased estimates for the EML and the IFM method. By using the CML method that problem is avoided.

After treating the three ML based estimation methods we will show how to apply the CML method to find parameter estimates for the Gaussian, the $t$-, and the Archimedean copulas. When applying this technique to the Archimedean copulas we will encounter a practical problem.

Deriving the copula density requires that the derivative of the copula function is taken $n$ times. The computer program that we use (OX), is unable to perform this differentiation analytically/symbolically, just as most other software packages. Therefore an approach is proposed in which one tries to 'fit' all bivariate copula margins instead of trying to 'fit' the complete density. The idea behind this is: "If the true Data Generating Process (DGP) has the dependence structure of an Archimedean copula, then all its bivariate margins are identical".
5.2 Exact Maximum Likelihood (EML)

Let’s assume that we have a sample \( \{x_1^t, \ldots, x_n^t\}_{t=1}^T \) containing the values of \( n \) different variables over \( T \) periods. The joint distribution of these variables is described by the copula \( C(F_1(x_1^t), \ldots, F_n(x_n^t)) \), with \( F_1, \ldots, F_n \) all the marginal distributions. The expression for the log-likelihood function then becomes:

\[
\ell(\theta) = \sum_{t=1}^T \ln c(F_1(x_1^t), \ldots, F_n(x_n^t)) + \sum_{t=1}^T \sum_{i=1}^n \ln f_i(x_i^t) \tag{14}
\]

with \( \theta \) a \( k \times 1 \) vector of parameters, including both the margin and the copula parameters. By maximizing the log-likelihood with respect to \( \theta \), we find the ML estimator, \( \hat{\theta}_{ML} \). For this estimator it holds that:

\[
\sqrt{T} \left( \hat{\theta}_{ML} - \theta_0 \right) \to N(0, J^{-1}(\theta_0)) \tag{15}
\]

with \( J(\theta_0) \) the Fisher information matrix.

---

\(^{20}\) Use formula (4) for the likelihood function for 1 period.
5.3 Inference Functions for Margins (IFM)

As stated previously, the EML method does not use the copula idea of a separation of margins from the dependence structure. The IFM method on the other hand is based on that principle. As a consequence of Sklar's theorem we can write the log-likelihood as:

\[
\ell (\theta) = \sum_{t=1}^{T} \ln c \left( F_1 (x^t_1; \theta_1), \ldots, F_n (x^t_n; \theta_n); \alpha \right) + \sum_{t=1}^{T} \sum_{i=1}^{n} \ln f_i \left( x^t_i; \theta_i \right)
\]

with \( \theta = (\theta_1, \ldots, \theta_n, \alpha) \). Now we split the estimation of \( \theta \) in two steps:

- Estimation of the parameters in the univariate margins:
  \[
  \hat{\theta}_i = \arg \max_{\theta_i} \ell_i (\theta_i) := \arg \max_{\theta_i} \sum_{t=1}^{T} \ln f_i \left( x^t_i; \theta_i \right) \quad \text{for } i = 1, \ldots, n.
  \]

- Estimation of the copula parameter, using the estimates obtained for the margins:
  \[
  \hat{\alpha} = \arg \max_{\alpha} \ell^C (\theta) := \arg \max_{\alpha} \sum_{t=1}^{T} \ln c \left( F_1 (x_1^t; \hat{\theta}_1), \ldots, F_n (x_n^t; \hat{\theta}_n); \alpha \right)
  \]

Now the IFM estimator is defined as:

\[
\hat{\theta}_{IFM} = (\hat{\theta}_1, \ldots, \hat{\theta}_n, \hat{\alpha})
\]

It can be shown that the IFM estimator verifies the property of asymptotic normality

\[
\sqrt{T} \left( \hat{\theta}_{IFM} - \theta_0 \right) \rightarrow \mathcal{N} \left( 0, \mathcal{V}^{-1} (\theta_0) \right)
\]

(16)

with \( \mathcal{V} (\theta_0) \) the information matrix of Godambe. Define the score function as

\[
g (\theta) = \left( \frac{\partial}{\partial \theta_1} \ell^1, \ldots, \frac{\partial}{\partial \theta_n} \ell^n, \frac{\partial}{\partial \alpha} \ell^C \right)
\]

then the Godambe information matrix can be written as (see JOE 1997):

\[
\mathcal{V} (\theta_0) = \left( \mathbb{E} \left[ \frac{\partial}{\partial \theta} g (\theta)^\top \right] \right)^{-1} \left( \mathbb{E} [g (\theta)^\top g (\theta)] \right) \left( \mathbb{E} \left[ \frac{\partial}{\partial \theta} g (\theta)^\top \right] \right)^{-1}\top
\]
5.4 Canonical Maximum Likelihood (CML)

When we use the IFM method we will end up with an estimate for $\alpha$ that depends on the distributional assumptions that we made about the margins. How can we change our approach and get an estimate for $\alpha$ that is margin independent? First transform the data $(x_1, \ldots, x_n)$ into uniform variates $(\tilde{u}_1, \ldots, \tilde{u}_n)$ using the empirical distribution functions of the univariate margins. After that we estimate the copula parameters $\alpha$ in the following way:

$$\hat{\alpha} = \arg \max \sum_{i=1}^{T} \ln c(\tilde{u}_1, \ldots, \tilde{u}_n; \alpha)$$

In practice transforming the data into uniform variates is performed using a transformation of the empirical distribution function. Define $F^T_1, \ldots, F^T_n$, the empirical distribution functions, in the following way:

$$F^T_j(x) := \frac{1}{T} \sum_{t=1}^{T} I(X^t_j \leq x)$$

where $I(\cdot)$ is the indicator function. In the estimation procedure we will use $\frac{T}{T+1} F^T_j$ to avoid possible problems with unboundedness of the copula density if some of the $u_i$’s are equal to 1. This means that the transformation of the data into standard uniform random variates is performed by:

$$\tilde{u}_j^t = \frac{T}{T+1} F^T_j(x^t_j) \quad \text{for} \ j = 1, \ldots, n.$$ 

The resulting estimator for $\alpha$ is also called a Pseudo-Likelihood Estimator or Omnibus Estimator. More information about this procedure and the properties of the estimator can be found in Genest et al. 1995.

Note that the difference between IFM and CML is that IFM first estimates the marginal distributions under distributional assumptions and then transforms the data into uniform variates while the CML method directly transforms the data into uniform variates using the empirical distribution of the data.

Another point worth noticing is that the empirical distribution function can also be described in terms of rank numbers. For the definition of the rank, see formula (6). Write $R^t_j$ to denote the rank of observation $t$ for variable $j$. Now the data transformation can also be written in the following way:

$$\tilde{u}_j^t = \frac{R^t_j}{T+1}$$

(17)

In Genest et al. 1995 it is proven that (under certain regularity conditions):

$$\sqrt{T} (\hat{\alpha}_{CML} - \alpha_0) \rightarrow N(0, \nu)$$
where \( \alpha_0 \) denotes the true copula parameter (under the assumption that the data generating process is driven by this specific copula) and \( V \) is not dependent upon \( T \). In Genest and Werker 2002 conditions are given under which CML estimators are semiparametrically efficient in large samples, and it is argued that for most copulas these requirements are not satisfied.
5.5 Applying CML to the Gaussian and $t$-copulas

Assume from now on that we have transformed our data into uniform random variates using the rank statistics of our data. So now we are left with a sample \( \{\tilde{u}_1, \ldots, \tilde{u}_n\} \). Next the pseudo log-likelihood can be written down,

\[
\ell(\alpha) = \sum_{t=1}^{T} \ln c(\tilde{u}_1^t, \ldots, \tilde{u}_n^t; \alpha)
\]

Maximizing the pseudo log-likelihood with respect to \( \alpha \) gives us the CML estimator.

Using the Gaussian copula density, as stated in (8), the log-likelihood is given by

\[
\ell^{Ga}(\Sigma) = -\frac{T}{2} \ln |\Sigma| - \frac{1}{2} \sum_{t=1}^{T} \omega_t^T \left( \Sigma^{-1} - I \right) \omega_t
\]

with \( \omega = (\Phi^{-1}(\tilde{u}_1^t), \ldots, \Phi^{-1}(\tilde{u}_n^t)) \). The CML estimate for \( \Sigma \), denoted by \( \widehat{\Sigma} \), is now found by maximizing the log-likelihood for \( \Sigma \). In this case we could even write down the expression for \( \widehat{\Sigma} \) explicitly (see Bouye et al. 2000), but the explicit solution for \( \widehat{\Sigma} \) will not yield us a proper correlation matrix due to the fact that we did not use the exact empirical distribution functions to transform our data into uniform variables. By applying the transformation in (17), the explicit ML solution is slightly distorted. Therefore we will maximize the likelihood numerically.

But how can we maximize the log-likelihood over all valid correlation matrices \( \Sigma \)? We start by taking any matrix \( A \ (n \times n) \), and compute the matrix \( B = AA^T \). Now the matrix \( B \) is a symmetric positive semi definite matrix, a covariance matrix. Now \( \Sigma \) is formed by:

\[
\Sigma = \begin{bmatrix}
\frac{1}{\sqrt{\rho_{11}}} & 0 & \cdots & 0 \\
0 & \frac{1}{\sqrt{\rho_{22}}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{\sqrt{\rho_{nn}}}
\end{bmatrix} \cdot B \cdot \begin{bmatrix}
\frac{1}{\sqrt{\rho_{11}}} & 0 & \cdots & 0 \\
0 & \frac{1}{\sqrt{\rho_{22}}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{\sqrt{\rho_{nn}}}
\end{bmatrix}
\]

Now we are left with a valid correlation matrix. We perform a numerical search over the matrix \( A \) to find the maximum likelihood estimator \( \Sigma \).

For the $t$-copula one can find that the expression for the log-likelihood is equivalent to (equal to, except for constant multiplication or addition factors):

\[
\ell^t(\Sigma, \nu) \propto -\frac{T}{2} \ln |\Sigma| - \left( \frac{\nu + n}{2} \right) \sum_{t=1}^{T} \ln \left( 1 + \frac{1}{\nu} \omega_t^T \Sigma^{-1} \omega_t \right) + \left( \frac{\nu + 1}{2} \right) \sum_{t=1}^{T} \sum_{i=1}^{n} \ln \left( 1 + \frac{1}{\nu} \omega_{t,i}^2 \right)
\]
with $\omega = (t_{\nu}^{-1}(\nu_1'), \ldots, t_{\nu}^{-1}(\nu_n'))$; see Bouyé et al. 2000. Maximizing this with respect to $\Sigma$ and $\nu$ gives us the CML estimates for the $t$-copula. Once again, use the indirect numerical maximization over $\Sigma$ as described above, to make sure that the resulting estimate for $\Sigma$ is indeed a valid correlation matrix.

If we use the CML technique, how will the estimator 'behave' if we know that the data is from an elliptical copula? That is, how accurate can we expect our estimate to be? Note that for a two-dimensional elliptical copula there is only one unknown parameter in $\Sigma$, namely the linear correlation coefficient $\rho$ between the variables.

Shown in Table 2 are the results of a simulation study to find the answers on the above questions. First a bivariate dataset of length $T$ was simulated from a Gaussian copula. Then $\rho$ was estimated under the assumption that the data are from a Gaussian copula. Next, $\rho$ is estimated assuming that the data were generated by a $t$-copula. After that a bivariate dataset of length $T$ from a $t_{10}$ copula\(^{21}\) was simulated. Then $\rho$ was estimated, first assuming that the data were from a $t$-copula, and then assuming that they were from a Gaussian copula.

These simulations were performed 1000 times (the number of replications), and for different values of $T$, namely $T = 50, 100, 200$ and $500$. In appendix A, Figures A9a until A9d, the density plots of the estimate for $\rho$ can be found\(^{22}\). The real value for $\rho$ in the simulations was 0.5. Reported in Table 2 is the average of the 1000 estimates for $\rho$ and a 95% confidence interval for the estimate for $\rho$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>Data: Gaussian</th>
<th>Data: Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Est. Assumption: Gaussian</td>
<td>Est. Assumption: $t$</td>
</tr>
<tr>
<td>50</td>
<td>$\rho$,95% c.i.</td>
<td>0.531, [0.288 , 0.708]</td>
</tr>
<tr>
<td>100</td>
<td>$\rho$,95% c.i.</td>
<td>0.516, [0.366 , 0.651]</td>
</tr>
<tr>
<td>200</td>
<td>$\rho$,95% c.i.</td>
<td>0.510, [0.409 , 0.608]</td>
</tr>
<tr>
<td>500</td>
<td>$\rho$,95% c.i.</td>
<td>0.507, [0.440 , 0.575]</td>
</tr>
<tr>
<td></td>
<td>Data: $t_{10}$</td>
<td>Data: $t_{10}$</td>
</tr>
<tr>
<td></td>
<td>Est. Assumption: $t$</td>
<td>Est. Assumption: Gaussian</td>
</tr>
<tr>
<td>50</td>
<td>$\rho$,95% c.i.</td>
<td>0.513, [0.235 , 0.723]</td>
</tr>
<tr>
<td>100</td>
<td>$\rho$,95% c.i.</td>
<td>0.511, [0.330 , 0.658]</td>
</tr>
<tr>
<td>200</td>
<td>$\rho$,95% c.i.</td>
<td>0.507, [0.390 , 0.613]</td>
</tr>
<tr>
<td>500</td>
<td>$\rho$,95% c.i.</td>
<td>0.503, [0.430 , 0.577]</td>
</tr>
</tbody>
</table>

Table 2: Estimation results of a simulation study for estimating $\rho$ using CML.

The first thing that catches the eye is the fact, that especially for smaller values of $T$, there seems to be a bias in estimating $\rho$. This is a consequence of

\(^{21}\)A $t$-copula with 10 degrees of freedom was chosen because empirical evidence seems to suggest that the number of degrees of freedom in a $t$-copula would be between 8 and 12 for e.g. asset and equity returns. See Mashal et al. 2003.

\(^{22}\)Performing the simulations and drawing the Figures 10a - d took, respectively, about 15, 30, 60 and 160 minutes.
the fact that we do not exactly use the EDF, but a slightly adjusted version of it, see formula (17).

The second thing that should be noted is that the estimation for $\rho$ seems to be almost as accurate under the wrong assumption as it is under the right one. *Does this mean that a $t$-copula has no benefits over a Gaussian one?* Clearly the answer on that question must be: ‘No’. The fact that the shown estimation results for $\rho$ practically do not change if we assume a Gaussian copula instead of a $t$-copula for the Data Generating Process does not mean that all the characteristics of the two copulas are equal. Most importantly the strict positive coefficients of upper and lower tail dependence can make a big impact if one is interested in tail events. If we use copulas to model a joint distribution and after that, we use the copula to simulate a Value at Risk, then it will definitely make a difference whether we assumed a Gaussian or a $t$-copula.

But is it really necessary to use this time-consuming method? Why is the correlation between the variables not simply estimated by using Pearson’s product-moment correlation estimator? This is because that estimator “... being suitable for data from uncontaminated multivariate normal distributions has a very bad performance for heavier tailed or contaminated data”, according to LINDSKOG 2000b. More about this issue can be found in Section 6.
5.6 Applying CML to the Archimedean Copulas

We want to estimate the parameter $\alpha$ of a bivariate strict Archimedean copula $C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$ using CML. First we derive the copula density $c(u, v)$:

\[
\begin{align*}
\frac{\partial c(u, v)}{\partial u} & = \frac{\partial}{\partial u} \left[ \frac{\partial}{\partial v} C(u, v) \right] \\
& = \frac{\partial}{\partial u} \left[ \varphi'(v) \varphi^{-1}(\varphi(u) + \varphi(v)) \right] \\
& = \frac{\partial}{\partial u} \left[ \frac{\varphi'(v)}{\varphi'[\varphi^{-1}(\varphi(u) + \varphi(v))]} \right] \\
& = \frac{\varphi'(v)}{\varphi'(C(u, v))} \frac{1}{\varphi'(C(u, v))} = \frac{\varphi''(C(u, v))}{\varphi'(C(u, v))} \frac{\varphi'(u)}{\varphi'(C(u, v))} \\
& = \frac{\varphi''(C(u, v)) \varphi'(u) \varphi'(v)}{[\varphi'(C(u, v))]^3} \\
& = \varphi'(v) \left[ \frac{1}{\varphi'[\varphi^{-1}(\varphi(u) + \varphi(v))]} \right] \frac{1}{\varphi'(C(u, v))} \\
& = \frac{\varphi''(C(u, v)) \varphi'(u) \varphi'(v)}{[\varphi'(C(u, v))]^3}.
\end{align*}
\]

Now we are left with a compact expression for the density function of an Archimedean copula. If the number of dimensions is increased, however, one has to make more and more use of the chain rule for differentiating functions, and the resulting formula for the copula density will become more and more complex. So how to proceed when one wants to use CML to estimate the Archimedean copula parameter? If the software package were able to calculate derivatives analytically, then we would have no problem. Unfortunately this is not the case for most software. How to proceed now?

Intuitively one can say: If all the bivariate copula margins are fitted appropriately, then the resulting complete multivariate copula density will be close to the joint distribution of the data as well. For Archimedean copulas we can also say: "If the true Data Generating Process (DGP) has the dependence structure of an Archimedean copula, then all its bivariate margins are identical".

This we will prove now. Assume that we are dealing with a (strict) $n$-dimensional Archimedean copula $C(u_1, \ldots, u_n)$, how do the bivariate margins look like?

---

23 It is not necessary for applying this technique that the Archimedean copula is strict. However, as stated before, in practice people only use strict multivariate Archimedean copulas. To simplify notation we will therefore assume that we are dealing with a strict generating function $\varphi$.

24 $f^{-1}(f^{-1}(t)) = t$, now taking the derivative on both sides we get: $f'(f^{-1}(t)) f^{-1'}(t) = 1$. And so: $f^{-1'}(t) = \frac{1}{f'(f^{-1}(t))}$.
For all $i, k \in 1, \ldots, n$ the bivariate margin (without loss of generality assume that $i < k$) looks like:

$$C_{i,k}(u_i, u_k) = C(1, \ldots, 1, u_i, 1, \ldots, 1, u_k, 1, \ldots, 1)$$

$$= \varphi^{-1}(\varphi(u_i) + \varphi(u_k) + (n - 2) \varphi(1))$$

$$= \varphi^{-1}(\varphi(u_i) + \varphi(u_k))$$

So all bivariate margins of Archimedean copulas $C(u_1, \ldots, u_n)$ with generator $\varphi(\cdot)$ are bivariate Archimedean with the same generating function $\varphi$. This means that all bivariate margins, are completely identical, since there is only one dependence parameter in this copula$^{25}$.

All bivariate margins contain the copula parameter $\alpha$, which can be estimated for each margin separately by applying CML. Using the above results we know that the expression for the pseudo log-likelihood for the bivariate $i, k$-margin is given by

$$\ell(\alpha) = \sum_{t=1}^{T} \ln c(\tilde{u}_i^t, \tilde{u}_k^t; \alpha)$$

$$= \sum_{t=1}^{T} \ln \left[ -\frac{\varphi''(C(\tilde{u}_i^t, \tilde{u}_k^t; \alpha)) \varphi'(\tilde{u}_i^t) \varphi'(\tilde{u}_k^t)}{[\varphi'(C(\tilde{u}_i^t, \tilde{u}_k^t; \alpha))]^3} \right]$$

Maximizing this with respect to $\alpha$ for all bivariate margin separately yields $\frac{1}{2} n (n - 1)$ estimates. Because all bivariate margins in the Archimedean copula $C(u_1, \ldots, u_n)$ are identical, we will take as an estimate for $\alpha$ the average of the $\frac{1}{2} n (n - 1)$ estimates.

---

$^{25}$Note that all bivariate margins of a multivariate Gaussian (or $t$-) copula are also bivariate Gaussian (or $t$). But because they have a parameter for each bivariate margin, the margins are not completely identical.
6 Implementation Issues

6.1 Introduction

The CML technique, explained previous section, is the most commonly used technique for estimating the correlation matrix $\Sigma$ in a Gaussian or $t$-copula. Unfortunately it is also a time-consuming method. If we want to test the copula technique in a rolling windows framework\textsuperscript{26}, a lot of CML estimates will have to be made, and consequently this will take a lot of time. In this section a different technique will be shown, which is faster for relatively small datasets (number of observations up to 1000). Note that for large datasets the CML technique is faster. If we have a sample size of $n$ dimensions with $T$ data points, the CML method consumes time of order $O(n^2 \times T)$ while the alternative technique, the transformed Kendall’s tau estimator, has a processing time of order $O(n^2 \times T^2)$.

The basic idea will be, that instead of looking at a sample as a total, we look at all combinations of two variables that exist in our sample. That means, if we have $n$ different variables, we will look at $\binom{n}{2} = \frac{1}{2}n(n-1)$ combinations of two variables. The technique that is proposed is based on relationship (9) between the linear correlation coefficient and Kendall’s tau for elliptical copulas. First we estimate Kendall’s tau, using a sample estimate. This estimate for Kendall’s tau is transformed into a correlation coefficient by using formula (9). Now we have found one of the coefficients $\rho$ from the correlation matrix $\Sigma$, parameter in the Gaussian and $t$-copula. This we do for all combinations of two variables in our sample, and so we ‘complete’ the whole matrix. If we now want to estimate the degrees of freedom parameter in the $t$-copula, then we maximize the log-likelihood with respect to $\nu$ numerically, using the fixed estimate for the matrix $\Sigma$ from the first step.

The reason that this method is advocated is because it saves a lot of time. Note that several authors always use this technique when estimating $\Sigma$ in elliptical copulas. Amongst others this is done in Meneguzzo and Vecchiato 2002, Mashal and Zeevi 2002 and Mashal and Naldi 2002.

In Bouyé et al. 2000 this technique is slightly adjusted, using an iterative procedure. This approach however is much more computationally intensive and, as Mashal and Zeevi 2002 remark: ”the difference between the iterative procedure (...) and the plug-in estimator (...) is negligible.” In Section 6.2 a simulation study is performed to see how the ‘transformed Kendall’s tau estimator’ performs. It turns out that the estimator for the linear correlation is as accurate as the estimation using CML. Once again note that this ‘transformed Kendall’s tau estimator’ is a much more robust estimator for linear correlation than Pearson’s product-moment correlation estimator.

\textsuperscript{26}This will be explained in the Applications section.
One practical problem that has to be taken care of, is that the resulting estimate for $\Sigma$ is not always positive semi-definite. This is of course a problem with all estimators for $\Sigma$ that use the plug-in technique. We will show how to transform the initial estimate for the matrix $\Sigma$ into a valid correlation matrix.

The method presented in this section to estimate the matrix $\Sigma$ in an elliptical copula will be used in the Applications section.
6.2 The Transformed Kendall’s Tau Estimator for \( \rho \)

The technique of estimating one component \( \rho_{ij} \) from \( \Sigma \) is based on the (earlier mentioned, see formula (9)) relationship that holds for elliptical copulas:

\[
\tau_{X_i, X_j} = \frac{2}{\pi} \arcsin \rho_{ij}
\]

Solving this equation for \( \rho_{ij} \) gives

\[
\rho_{ij} = \sin \left( \frac{\pi}{2} \tau_{X_i, X_j} \right)
\]

If we have a good estimate for \( \tau \), then we could plug this in and obtain an estimate for \( \rho \). In theory most of the variables that we will look at are continuously distributed. The sample version of Kendall’s tau that was given in subsection 3.3.1 was not corrected for the occurrence of ties. A tie would mean that either \( x_{t_1}^i \) and \( x_{t_2}^j \) or \( x_{t_1}^i \) and \( x_{t_2}^j \) are exactly equal. An unbiased estimator for \( \tau \) also in the case of ties occurring is given by

\[
\hat{\tau}_{X_i, X_j} = \frac{c - d}{\sqrt{c + d + e_{x_i}} \sqrt{c + d + e_{x_j}}}
\]

where \( c \) denotes the number of concordant pairs, \( d \) denotes the number of discordant pairs, \( e_{x_i} \) denotes the number of pairs for which \( x_{t_1}^i = x_{t_2}^i \), \( t_1 \neq t_2 \) and \( e_{x_j} \) denotes the number of pairs for which \( x_{t_1}^j = x_{t_2}^j \), \( t_1 \neq t_2 \). Remember that for a concordant pair \( (x_{t_1}^i - x_{t_2}^i) (x_{t_1}^j - x_{t_2}^j) > 0 \) and for a discordant pair \( (x_{t_1}^i - x_{t_2}^i) (x_{t_1}^j - x_{t_2}^j) < 0 \). The total sample size that is used is \( T \). This \( \hat{\tau}_{X_i, X_j} \) is an unbiased estimator for \( \tau_{X_i, X_j} \). For more information, see Lindskog 2000b.

The plug-in estimator for \( \rho \) that we obtain is however not unbiased27. To find out more about the behavior of this estimator for \( \rho \) another simulation study was performed. First a dataset with length \( T \) from a bivariate Gaussian copula, with a predetermined value for \( \rho \), was generated and \( \rho \) was estimated. This was repeated 1000 times. Also from a bivariate \( t_{10} \) copula \( T \) periods of data were generated, using the same predetermined value for \( \rho \), and again \( \rho \) was estimated. This was repeated 1000 times. In Table 3 the average value of the estimate and a 95% confidence interval are reported for \( T = 50, 100, 200 \) and 500. In appendix A, Figures A10a until A10d, the density of the estimates for the case \( \rho = 0.5 \) are plotted28. The values for \( \rho \) are 0.1, 0.5, and 0.9.

---

27 This follows from Jensen’s inequality combined with the non-linearity of the function \( f(x) = \sin x \).

28 Performing the simulations and drawing the Figures A11a - d took, respectively, about 9, 35, 140 and 900 seconds. Note that the number of simulations per figure is half as much as for Figures A9a-A9d.
Table 3: Estimating $\rho$ using a transformed Kendall’s tau estimator.

Comparing this table to the results in Table 2 shows us that the bias in the estimates is somewhat smaller, especially for smaller values of $T$. The estimates seem to be slightly underestimating the true value for $\rho$. This can be explained by the fact that for a strictly concave function $f(x)$ it holds that

$$\mathbb{E}\{f(x)\} < f(\mathbb{E}\{x\})$$

If we take $f(x) = \sin(x)$ then around $x = \frac{\pi}{20}$ ($= 0.1 * \frac{\pi}{10}$), $x = \frac{\pi}{10}$, and $x = \frac{9\pi}{20}$ the function $f(x)$ is concave. We actually want to estimate $f(\mathbb{E}\{x\})$ but due to the concavity of $f$ our estimate $\mathbb{E}\{f(x)\}$ is somewhat lower on average. In practice it seems that the differences are small.
6.3 The Plug-in Estimator for $\Sigma$

Although for sample sizes up to 1000 time periods $T$ the method using transformed Kendall’s tau estimators is faster than the standard CML estimation method as explained in Section 5.5, for much larger sample sizes it will save time to use the standard CML method. If we have a sample size of $n$ dimensions with $T$ data points, the CML method (based on all pairs of variables) consumes time of order $O(n^2 \times T)$ while the transformed Kendall’s tau estimator has a processing time of order $O(n^2 \times T^2)$ and is consequently slower for large sample sizes.

An advantage of the transformed Kendall’s tau estimator is that it only uses techniques that are stable/robust, while the CML estimation technique might give problems in some cases due to the numerical optimization and the inversion of a matrix that can be near singular.

A disadvantage in general with plug-in estimators for covariance and correlation matrices is that on forehand one usually has no guarantee that the resulting matrix will be positive semi-definite. In Lindskog 2000b this problem is addressed and several solutions are presented to transform an initial plug-in estimate that is not positive semi-definite (psd) into a matrix that is psd. Actually we want to have a matrix that is positive definite, because we need this matrix to find an estimate for $\nu$ in the $t$-copula. To find that, we will maximize the log-likelihood for the $t$-copula using the fixed estimate for $\Sigma$ from the first step. The log-likelihood for the $t$-copula includes taking the inverse of the matrix $\Sigma$, and therefore we actually want our estimate for $\Sigma$ to be positive definite (pd).

Denote by $\tilde{\Sigma}$ the initial plug-in estimate for $\Sigma$ that we found, either using the CML method or the transformed Kendall’s tau method. Now assume that this matrix turns out to be only a pseudo-correlation matrix. This means that the matrix is not positive semi-definite, but:

1. $\tilde{\Sigma}$ is symmetric,
2. all the diagonal elements of $\tilde{\Sigma}$ are equal to 1, and
3. all elements of $\tilde{\Sigma}$ are smaller or equal to 1 in absolute value.

The methods for transforming $\tilde{\Sigma}$ into a psd matrix $\hat{\Sigma}$ are compared in Lindskog 2000b based on the distance measure

$$d^2 (\hat{\Sigma}, \tilde{\Sigma}) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\hat{\Sigma}_{ij} - \tilde{\Sigma}_{ij})^2$$

Lindskog’s conclusion is that the method that is called the ”eigenvalue method” performs best.
Because $\Sigma$ is symmetric we can write

$$\Sigma = \Lambda P'$$

where $\Lambda$ is a diagonal matrix with the eigenvalues of $\Sigma$ on the diagonal and $P$ is a matrix with the corresponding normed eigenvectors of $\Sigma$ as columns. As we assumed that $\Sigma$ was not positive semi-definite, it must be that at least one of the eigenvalues of $\Sigma$ is negative. In practice those negative eigenvalues will be very small in absolute value. We now replace the negative eigenvalues in $\Lambda$ by $\delta$, a very small positive number$^{29}$, and set

$$\Sigma^\ast = P\Lambda^\ast P'$$

where $\Lambda^\ast$ is the matrix $\Lambda$ with the negative values replaced by $\delta$’s. By construction $\Sigma^\ast$ will be symmetric and positive definite. However, the diagonal elements of $\Sigma^\ast$ will not be equal to 1. Therefore we scale this matrix in the following way

$$\hat{\Sigma} = D\Sigma^\ast D$$

where $D$ is a diagonal matrix with diagonal elements $\frac{1}{\sqrt{\Sigma_{ii}}}$, for $i = 1, \ldots, n$.

Now we have obtained our final estimate $\hat{\Sigma}$, a valid correlation matrix. Note that if we have found an initial estimate for $\Sigma$ with all eigenvalues nonnegative, but one or more of them exactly equal to zero, that we will use the exact same technique to transform this matrix.

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$^{29}$ Choose the number $\delta$ in such a way that it is as small as possible without giving numerical problems when inverting the final estimate for $\Sigma$. 
7 Simulation

7.1 General Simulation Algorithm

In solving problems in finance one often uses simulation techniques, for instance to estimate the Value at Risk of a portfolio. This section provides algorithms for simulation of data from copulas. The scatter plots will show what the difference between several copulas is, and how the choice of the margins will influence the resulting joint distribution.

Assume that we want to simulate data \((x_1, \ldots, x_n)\) from a certain copula \(C\) with specific marginals \(F_1, \ldots, F_n\). This can be performed in two steps:

1. Simulate uniform random variates, \((u_1, \ldots, u_n)\) from the desired copula \(C\).

2. Now transform the data using the distribution functions \(F_1, \ldots, F_n\) that we want to have as margins. So \((x_1, \ldots, x_n) = (F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n))\).

This second step can be performed for well-known distributions by most statistical packages. It can also be done using a numerical root finding procedure. This section will focus on the construction of the first step.

A general method to simulate random variates from a chosen copula is established using conditional sampling. Define

\[ C_i(u_1, \ldots, u_i) := C(u_1, \ldots, u_i, 1, \ldots, 1) \]

This is a specific \(i\)-dimensional margin of \(C\). Special cases are \(C_1(u_1) = u_1\) and \(C_n(u_1, \ldots, u_n) = C(u_1, \ldots, u_n)\). Now assume first that we have already simulated the uniform variates \(u_1, \ldots, u_{i-1}\) from the copula \(C\). It follows that the distribution of \(u_i\) conditional on the values of \(u_1, \ldots, u_{i-1}\) is given by

\[
C_i(u_i | u_1, \ldots, u_{i-1}) = \frac{\mathbb{P}(U_i \leq u_i | U_1 = u_1, \ldots, U_{i-1} = u_{i-1})}{\partial u_1 \cdots \partial u_{i-1} C_i(u_1, \ldots, u_i)} / \frac{\partial u_1 \cdots \partial u_{i-1} C_i^{-1}(u_1, \ldots, u_i)}{\partial u_1 \cdots \partial u_{i-1}}
\]

This implies that a complete algorithm for simulation of an \(n\)-variate uniform random variate \((u_1, \ldots, u_n)\) from the copula \(C\) is given by:
Algorithm 60 (Simulation of Random n-Variates from a Copula $C$)

- Simulate a random variate $u_1$ from $U(0,1)$.
- Simulate a random variate $u_2$ from $C_2(u_2 | u_1)$.
- $\ldots$
- Simulate a random variate $u_n$ from $C_n(u_n | u_1, \ldots, u_{n-1})$.

Drawing a random variate $u_i$ from $C_i(u_i | u_1, \ldots, u_{i-1})$ is done by first drawing $v$ randomly from $U(0,1)$, and now solving the equation $v = C_i(u_i | u_1, \ldots, u_{i-1})$ for $u_i$ by numerical root finding.

The above approach for simulation can be computationally intensive. A second (and more serious) problem is that in some cases it is very difficult (or even impossible) to perform this iterative procedure because of the analytical differentiation that is needed. Most computer packages do not perform analytical differentiation, so one would have to find a general formula for $\frac{\partial^{i-1}C_{i-1}(u_1, \ldots, u_{i-1})}{\partial u_1 \ldots \partial u_{i-1}}$.

Fortunately, for the elliptical copulas easier simulation algorithms are known.
7 Simulation

7.2 Simulating from Gaussian and \( t \) Copulas

The commonly used algorithm for simulation of standard uniform variates from a Gaussian copula is this:

**Algorithm 61** *(Simulation of Random \( n \)-Variates from \( C_{\Sigma}^{Ga} \))*

- Find the Cholesky decomposition \( A \) of \( \Sigma \).
- Simulate \( n \) independent random variates \( z_1, \ldots, z_n \) from \( N(0,1) \).
- Set \( y = Az \), where \( z = (z_1, \ldots, z_n) \).
- Set \( u_i = \Phi(y_i), \ i = 1, \ldots, n. \)
- Now \( (u_1, \ldots, u_n) \) is a random \( n \)- variate from \( C_{\Sigma}^{Ga} \).

A similar algorithm also exists for simulating standard uniform variates from a \( t \)-copula.

**Algorithm 62** *(Simulation of Random \( n \)-Variates from \( C_{\Sigma,v}^{t} \))*

- Find the Cholesky decomposition \( A \) of \( \Sigma \).
- Simulate \( n \) independent random variates \( z_1, \ldots, z_n \) from \( N(0,1) \).
- Simulate a random variate \( s \) from \( \chi^2_v \) independent of \( z_1, \ldots, z_n \).
- Set \( w = Az \), where \( z = (z_1, \ldots, z_n) \).
- Set \( y = \frac{s}{\sqrt{v}}w. \)
- Set \( u_i = t_v(y_i), i = 1, \ldots, n. \)
- Now \( (u_1, \ldots, u_n) \) is a random \( n \)- variate from \( C_{\Sigma,v}^{t} \).

After obtaining the random \( n \)-variates \( (u_1, \ldots, u_n) \) from copula \( C \), we transform them to get the margins \( F_1, \ldots, F_n \) that we want the data to have. So we obtain \((x_1, \ldots, x_n) = (F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n))\). In appendix A, Figures A11a until A11e, scatter plots of Gaussian and \( t \)-copulas can be found. All simulations are such that Kendall’s tau is equal to 0.5 (So that means that the linear correlation coefficient is equal to \( \frac{1}{2} \sqrt{2} \approx 0.71 \)). The scale of the \( x \), and \( y \)-axis is equal for each plot individually, so that one can clearly see the difference in dispersion between \( x \) and \( y \)-data. Notice however that when comparing the plots they can have different scales.

The first one (Figure A11a) comes from a Gaussian copula and has two standard normal margins. This is equivalent to simulating two standard normal random variables with a correlation coefficient of \( \frac{1}{2} \sqrt{2} \).
The second plot (Figure A11b) shows data from a Gaussian copula, but this time both margins are $t_{10}$. One can clearly see that the variation in the $x$-, as well as the $y$-data has become larger.

The third plot (Figure A11c) shows data from a $t_4$ copula, and both margins are standard normal. Comparing this plot to the first one, it is obvious that the dispersion in the data is not larger (because $x$ and $y$ are still standard normal), but the simultaneous number of occurrences of high $x$ and $y$-values or low $x$ and $y$-values is much larger. This is because the dependence structure between the two variables is a $t_4$ copula which, contrary to the Gaussian copula, does have tail dependence.

The fourth plot (Figure A11d) shows data from a $t_4$ copula, and this time both margins are $t_8$. Now there is not only tail dependence, but also the dispersion in the data is much bigger than for the standard normal margins.

In the fifth plot (Figure A11e) one of the margins is standard normal, the other one is $t_8$, and the dependence is given by a $t_4$ copula.
7.3 Simulating from Archimedean Copulas

Because of the special form that Archimedean copulas have, it is possible to rewrite the general formulas for the conditional copulas.

**Theorem 63** Let $C$ be an Archimedean $n$-copula with generator $\varphi$. Then for $i = 2, \ldots, n$

$$C_i (u_1|u_1, \ldots, u_{i-1}) = \frac{\varphi^{-1}(i-1)(\varphi(u_1) + \cdots + \varphi(u_i))}{\varphi^{-1}(i-1)(\varphi(u_1) + \cdots + \varphi(u_{i-1}))}$$

where $\varphi^{-1}(m)(\cdot)$ denotes the $m$-th derivative of the function $\varphi^{-1}$.

**Proof.** See Meneguzzo and Vecchiato 2002.

Inserting this relationship into Algorithm 60 yields a procedure that is straightforward to implement if one makes use of software that can calculate derivatives of functions analytically. If that is not the case, then one could try to find a general expression for the $n$-th derivative of the function $\varphi^{-1}$. In a lot of cases however, this will be a difficult task.

There is however an easier way to proceed. To understand it, one can first look at the 'standard' way of simulating bivariate data from an Archimedean copula. This is where the theory of section 4.3.4 comes in. Simulation of uniform data from a bivariate Archimedean copula is based on next theorem.

**Theorem 64** The joint distribution function $H(s,t)$ of the random variables $S = \frac{\varphi(U)}{\varphi(U) + \varphi(V)}$ and $T = C(U,V) = \varphi^{-1}(\varphi(u) + \varphi(v))$ is given by

$$H(s,t) = sK_C(t) \quad \text{for all } (s,t) \in [0,1]^2$$

Hence $S$ and $T$ are independent and $S$ is uniformly distributed on $[0,1]$.

**Proof.** Based on Nelsen 1999.

Assume that $C$ is absolutely continuous (the proof for the general case can be found in Genest and Rivest 1993). The joint density $h(s,t)$ of $S$ and $T$ is given by

$$h(s,t) = \frac{\partial^2 C(u,v)}{\partial u \partial v} \frac{\partial (u,v)}{\partial (s,t)}$$

where $\frac{\partial^2 C(u,v)}{\partial u \partial v}$ is equal to $-\frac{\varphi''(C(u,v)) \varphi'(u) \varphi'(v)}{[\varphi'(C(u,v))]^3}$ (derivation see paragraph 5.6). $\frac{\partial (u,v)}{\partial (s,t)}$ is the Jacobian of the transformation $u = \varphi^{-1} (s \varphi(t))$, $v = \varphi^{-1} ((1-s) \varphi(t))$ for which

$$\frac{\partial (u,v)}{\partial (s,t)} = \begin{bmatrix} \frac{\varphi'(t)}{\varphi'(t)} & \frac{s \varphi'(t)}{\varphi'(t)} \\ \frac{\varphi'(t)}{\varphi'(t)} & \frac{(1-s) \varphi'(t)}{\varphi'(t)} \end{bmatrix} = \begin{bmatrix} \frac{\varphi(t)}{\varphi'(t)} & \frac{s \varphi'(t)}{\varphi'(t)} \\ \frac{(1-s) \varphi'(t)}{\varphi'(t)} & \frac{\varphi(t)}{\varphi'(t)} \end{bmatrix}$$
So \( \det \left( \frac{\partial (u,v)}{\partial (s,t)} \right) = \frac{\varphi'(t)}{} \frac{\varphi'(u)}{\varphi'(v)} \) which is negative. Therefore \( \frac{\partial (u,v)}{\partial (s,t)} = -\frac{\varphi(t)}{\varphi'(u)} \frac{\varphi'(t)}{\varphi'(v)} \). Given the transformations \( s = \frac{\varphi(u)}{\varphi(u) + \varphi(v)} \) and \( t = C(u,v) \) the joint density is:

\[
h(s,t) = \left( -\frac{\varphi''(t)}{[\varphi'(t)]^3} \right) \left( \frac{\varphi(t)}{\varphi'(u)} \frac{\varphi'(t)}{\varphi'(v)} \right)
\]

This implies

\[
H(s,t) = \int_0^s \int_0^t \frac{\varphi''(t)}{[\varphi'(t)]^2} dy dx = s \left[ y - \frac{\varphi'(y)}{\varphi''(y)} \right]_0^t = sK_C(t)
\]

and so the conclusion follows. \( \square \)

Based on this theorem is the following algorithm to simulate bivariate data from an Archimedean copula.

**Algorithm 65 (Simulation of Bivariate Data from an Archimedean Copula)**

1. Generate two independent Uniform(0,1) random variates \( s \) and \( q \).
2. Set \( t = K_C^{-1}(q) \), with \( K_C \) defined as above.
3. Set \( u = \varphi^{-1}(s \varphi'(t)) \) and \( v = \varphi^{-1}((1-s) \varphi(t)) \).
4. \((u,v)\) is generated from the Archimedean copula \( C \) with generator \( \varphi \).

In this algorithm the procedure \( t = K_C^{-1}(q) \) will be performed using a numerical root finding technique. We have seen now how to simulate bivariate data from an Archimedean copula, but how can this be extended to the multivariate case?

A method often described in the literature is based on MARSHALL AND OLKIN 1988. First one simulates \( n \) random variates \( u_1^n, ..., u_n^n \) independently from a Uniform(0,1) distribution. Then an extra 'merging' variable has to be simulated with which the variables \( u_1^n, ..., u_n^n \) are transformed into the variates \( u_1, ..., u_n \) which have the desired dependence structure. The problem with simulating this 'merging' variable is that only its Laplace Transform (or equivalently moment generating function or characteristic function) is known. For some copulas it is known how to derive the distribution function of the merging variable (for example for the Cook and Johnson copula), but for most of the Archimedean copulas that are in the appendix, the Inverse Laplace Transform
is unknown.

One could try to find the Inverse Laplace Transform numerically (there is no known functional inverse in a lot of cases). But even then simulating is not an easy task. Also one could try to use an algorithm given in Devroye 1986 to simulate random variates from a distribution for which only a Characteristic Function is given. Unfortunately for the cases that are treated here, the characteristic functions do not have desirable properties.

Another idea is to use the associativity property of the Archimedean copulas, as seen in Theorem 51. By using that property recursively we establish:

\[
C(u_1, \ldots, u_n) = C(C(u_1, \ldots, u_{n-1}), u_n)
\]

This suggests using the following algorithm for simulating random \(n\)-variates from an Archimedean copula (for the proof, see Lindskog 2000a).

**Algorithm 66** (Simulation of \(n\)-Variates from an Archimedean Copula)

- Generate a random variate \(q\) from a Uniform(0,1) distribution.
- Set \(t_1 = K_C^{-1}(q)\).
  Generate a Uniform(0,1) random variate \(s_1\) independent of \(q\).
  Set \(a_1 = \varphi^{-1}(s_1 \varphi(t_1))\) and \(u_n = \varphi^{-1}((1 - s_1) \varphi(t_1))\).
  
  ... 

- Set \(t_i = K_C^{-1}(a_{i-1})\).
  Generate a Uniform(0,1) random variate \(s_i\) independent of \(q\) and \(s_1, \ldots, s_{i-1}\).
  Set \(a_i = \varphi^{-1}(s_i \varphi(t_i))\) and \(u_{n-i+1} = \varphi^{-1}((1 - s_i) \varphi(t_i))\).
  
  ... 

- Set \(t_{n-1} = K_C^{-1}(a_{n-2})\).
  Generate a Uniform(0,1) random variate \(s_{n-1}\) independent of \(q\) and \(s_1, \ldots, s_{n-2}\).
  Set \(u_1 = \varphi^{-1}(s_{n-1} \varphi(t_{n-1}))\) and \(u_2 = \varphi^{-1}((1 - s_{n-1}) \varphi(t_{n-1}))\).

- \((u_1, \ldots, u_n)\) is generated from an Archimedean copula \(C\) with generator \(\varphi\).
After implementation of this algorithm some test were performed to see whether an input of data from a specific Archimedean copula with parameter \( \alpha \) would result in estimates for \( \alpha \) that were close to the true value that was put in. It turns out to be the case that there is an undervaluation of \( \alpha \), except for the estimate for the bivariate margin \( C(u_1, u_2, 1, \ldots, 1) \). If the generated variates have a large distance, measured in steps of the generation, then the dependence between them tends to be lower\(^{30}\). This automatically means that the value for \( \alpha \) in those bivariate margins will be lower, as our Archimedean copulas treated here are all positively ordered. So typically the estimated value of \( \alpha \) from the margin \( C(u_1, 1, \ldots, 1, u_n) \) will be the lowest.

This is a result of the fact that any small inaccuracy (for instance in calculating \( K_{\alpha}^{-1}(\cdot) \) numerically) remains present for the rest of the iterative generating process. The fact that all the Archimedean copulas that are treated here have only one parameter and they are positively ordered with respect to that parameter, results in inaccuracies that cause a weaker dependence relationship.

To illustrate this we look at the parameter estimates from data generated data from a Cook and Johnson copula with \( n = 6 \) and 5000 time periods, see Table 4 middle column. The value for \( \alpha \) that was used is \( \alpha = 2 \). In the table are the estimates for \( \alpha \) for each bivariate margin (using the CML technique). Fortunately for the Cook and Johnson copula\(^{31}\), there is another simulation algorithm available. That algorithm makes use of the earlier mentioned Inverse Laplace Transform technique.

**Algorithm 67 (Random Variate Generation from a Cook and Johnson Copula)**

- Generate \( n \) independent random variates \( y_1, \ldots, y_n \) from an Exponential distribution with parameter \( \lambda = 1 \).
- Generate a random variate \( z \) from a Gamma\( (\frac{1}{\alpha}, 1) \) distribution independent of \( y_1, \ldots, y_n \).
- Set \( u_i = \left(1 + \frac{y_i}{z}\right)^{-1/\alpha} \) for \( i = 1, \ldots, n \).
- Now \( u = (u_1, \ldots, u_n) \) is generated from a Cook and Johnson copula with parameter \( \alpha \).

Now we use this algorithm to generate random variates from a Cook and Johnson copula with \( n = 6 \), and \( \alpha = 2 \). The estimates for \( \alpha \) for each bivariate margin can be found in Table 4, the right column.

---

\(^{30}\)The distance in generating steps between \( u_i \) and \( u_j \) is equal to \( |j - i - 1| \).

\(^{31}\)Some authors call this the Clayton copula.
As can be seen in the table, some of the estimates for the first algorithm have a 40% smaller implied $\alpha$, than the value that was implemented in the simulation. The second algorithm performs well. The problem is, that for the other copulas such algorithms are unknown. If tests are performed for all of the Archimedean copulas that are presented in the appendix and the number of dimensions is restricted to 4, then the largest difference between the true $\alpha$ and the smallest $\alpha$ implied by the simulation is between 10% and 30% (depending on which Archimedean copula we look at). It seems that smaller values for $\alpha$ produce a larger relative difference.

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<th>Algorithm 67</th>
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<td>3-6</td>
<td>1.26</td>
<td>2.00</td>
</tr>
<tr>
<td>4-5</td>
<td>1.68</td>
<td>1.91</td>
</tr>
<tr>
<td>4-6</td>
<td>1.41</td>
<td>2.01</td>
</tr>
<tr>
<td>5-6</td>
<td>1.71</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Table 4: Comparison of the estimates for $\alpha$ (true value is 2) in a Cook and Johnson copula using two different simulation algorithms.

In appendix A, Figures A12a until A12g, plots of simulated data (10,000 points) from Archimedean copulas can be found. All of the margins are taken to be standard Gaussian. The parameter values are chosen in such a way that the simulated data have a Kendall’s tau of $\frac{1}{3}\textsuperscript{32}$. One can clearly see the differences in tail dependence between the copulas. With these specific parameters chosen, all positive tail dependence coefficients have a value of $\frac{1}{7}$. Looking at the characteristics of the copulas in appendix B, we see that the Gumbel-Hougaard, the Type I and the Type III copula should also have upper tail dependence. The reason that we do not observe this in the data is because with these specific parameter values (chosen in such a way that the value for Kendall’s tau is equal to $\frac{1}{3}$), the upper tail dependence becomes 0. In Figures A13a, A13b and A13c plots of simulated data from these three copulas can be found, this time giving them a parameter value such that their upper tail dependence is equal to $\frac{1}{7}$.

In appendix A, Figure 14, the results of simulating 25,000 data from a Type I copula with $\alpha = 1$. This copula has one standard Gaussian, and one $t_6$ margin.

\textsuperscript{32}Because this is the only value for Kendall’s tau that can be obtained (or at least approximated) by all the Archimedean copulas that are treated here.
8 Applications

8.1 Exchange Rates

In this section the described copula techniques will be applied on a dataset consisting of four exchange rate series, the British Pound (GBP), the German Mark (DEM), the Japanese Yen (JPY) and the Swiss Franc (CHF). We will try to estimate the Value at Risk and the Conditional Value at Risk of an equally weighted portfolio of these currencies. All rates are expressed in US Dollars (USD). The dataset is from the January 2nd 1999 up until June 19th 2003 (1121 data points). Source is The Pacific Exchange Rate Service, see http://pacific.commerce.ubc.ca.xr/.

One of the implicit assumptions of non time-dependent copulas is that all series are stationary. Therefore, instead of working with price series, the returns $r_t$ (in %) implied by the exchange rates have been used. To verify the assumption of stationarity one can perform a stationarity test. Suited for this purpose is the Dickey-Fuller test, which postulates using the model

$$r_t - r_{t-1} = \delta + (\theta - 1) r_{t-1} + \varepsilon_t$$

The test now uses as null hypothesis $H_0: \theta = 1$. This means that the null hypothesis is the assumption of the existence of a unit root. For each of the return series the null hypothesis was rejected using a significance level of 1%. In appendix C, Tables C1 until C4, the test results can be found. For all four return series we look at some descriptive statistics to get an idea of the distributions.

<table>
<thead>
<tr>
<th></th>
<th>GBP</th>
<th>DEM</th>
<th>JPY</th>
<th>CHF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.89E-03</td>
<td>0.59E-03</td>
<td>-2.79E-03</td>
<td>4.97E-03</td>
</tr>
<tr>
<td>Median</td>
<td>0</td>
<td>-37.79E-03</td>
<td>-6.56E-03</td>
<td>-22.63E-03</td>
</tr>
<tr>
<td>Maximum</td>
<td>2.09</td>
<td>2.75</td>
<td>2.97</td>
<td>2.76</td>
</tr>
<tr>
<td>Minimum</td>
<td>-2.08</td>
<td>-2.47</td>
<td>-2.88</td>
<td>-2.86</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>0.48</td>
<td>0.64</td>
<td>0.68</td>
<td>0.66</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.08</td>
<td>0.17</td>
<td>0.34</td>
<td>0.17</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.85</td>
<td>4.01</td>
<td>4.74</td>
<td>4.18</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>35.14</td>
<td>53.30</td>
<td>163.48</td>
<td>70.72</td>
</tr>
<tr>
<td>Probability$^{34}$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 5: Descriptive statistics for the returns (in %) on exchange rate series.

Now for each of the Archimedean copulas mentioned in the appendix we estimate the parameter $\alpha$. These results, along with a weighted average of the log-likelihood value for the bivariate margins, can be found in Table 6. The computations took about 80 seconds. Also in the table are Kendall’s tau as

$^{33}$Formally this would require testing $H_0: \theta = 1, \delta = 0$. In practice usually only the first requirement is tested. For more information, see Verbeek 2000.

$^{34}$This is the $p$-value belonging to the Jarque-Bera test statistic.
well as the degree of lower tail dependence corresponding to the copula with this particular value for $\alpha$. In most cases these values could easily be computed using the formulas for Kendall’s tau given in appendix B. For the other cases one can use a numerical approximation of the integral formula (12).

<table>
<thead>
<tr>
<th>Copula</th>
<th>Parameter estimate</th>
<th>Weighted Log-Lik.</th>
<th>Kendall’s tau</th>
<th>Lower Tail Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cook and Johnson</td>
<td>1.23</td>
<td>243.09</td>
<td>0.38</td>
<td>0.57</td>
</tr>
<tr>
<td>Ali-Mikhail-Haq</td>
<td>0.80</td>
<td>170.49</td>
<td>0.23</td>
<td>0</td>
</tr>
<tr>
<td>Gumbel-Hougaard</td>
<td>1.92</td>
<td>319.14</td>
<td>0.48</td>
<td>0</td>
</tr>
<tr>
<td>Frank</td>
<td>5.41</td>
<td>307.83</td>
<td>0.48</td>
<td>0</td>
</tr>
<tr>
<td>Type I</td>
<td>1.40</td>
<td>245.96</td>
<td>0.52</td>
<td>0.61</td>
</tr>
<tr>
<td>Type II</td>
<td>3.98</td>
<td>298.50</td>
<td>0.47</td>
<td>0</td>
</tr>
<tr>
<td>Type III</td>
<td>1.30</td>
<td>131.49</td>
<td>0.44</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 6: Estimation of $\alpha$ for the different Archimedean copula functions.

It is interesting to note that for most Archimedean copulas the bivariate margin for DEM-CHF was easiest to model, while the margin GBP-DEM was the most difficult one to model. Based on the weighted value for the log-likelihood, it seems that the Gumbel-Hougaard, the Frank and the Type II copula give the most accurate fit. These three copula types have in common that neither of them has lower tail dependence. The only other Archimedean copula in our ‘toolbox’ that also lacks lower tail dependence is the Ali-Mikhail-Haq (AMH) copula. Yet this copula seems not to give a good fit when we look at its weighted log-likelihood value. The explanation for that, is that the AMH copula can only model dependence relationships for which Kendall’s tau is between 0 and $\frac{1}{3}$.

Kendall’s taus from the data for all bivariate margins are: 0.45 (GBP-DEM), 0.16 (GBP-JPY), 0.44 (GBP-CHF), 0.18 (DEM-JPY), 0.79 (DEM-CHF), and 0.20 (JPY-CHF). These values are not really close, and maybe this will be a problem for the Archimedean copulas, since they have all identical bivariate margins. Next, the parameters for the Gaussian and $t$-copula are also estimated. The estimate for $\Sigma$ in the Gaussian and $t$-copula is (GBP, DEM, JPY, CHF):

$$
\begin{bmatrix}
1 & 0.655 & 0.244 & 0.641 \\
0.655 & 1 & 0.282 & 0.948 \\
0.244 & 0.282 & 1 & 0.310 \\
0.641 & 0.948 & 0.310 & 1
\end{bmatrix}
$$

Estimating the matrix $\Sigma$ for the Gaussian and $t$-copula is done using the technique described in Sections 6.2 and 6.3. The estimate for $\nu$, the number of degrees of freedom, in the $t$-copula is 7.14. This estimate is obtained by maximizing the $t$-copula likelihood with fixed $\Sigma$. Looking at this low estimate for the degrees of freedom parameter, it could be the case that a Gaussian copula is unable to give a good description for the dependence structure. But on the
other hand, maybe we actually only need either upper or lower tail dependence and not both. In that situation maybe the Gaussian copula could give more accurate VaR and CVaR estimates. Estimating the parameters for these two copulas took about 15 seconds.

Assume now that we are holding an equally weighted portfolio of USD 40,000. We are interested in the Value at Risk (VaR) and the Conditional Value at Risk (CVaR) of our portfolio. Define $V_t$ to be the value of our portfolio at time $t$, expressed in US Dollars. Now the VaR is defined as the maximum portfolio loss $V_{t+1} - V_t$ that will not be exceeded with a probability of 95% or 99%. In statistical terms it is the lower 5% or 1% quantile value of the return distribution. A CVaR is defined as the expected value of the change in portfolio value, given that we are dealing with observations from the 1% or 5% worst cases. In statistical terms that is the expected value of the distribution in its 5% or 1% lower tail.

Using the estimated copula parameters, how can we simulate data and calculate a VaR and CVaR from that? We have modeled the return of the exchange rates. So we can proceed in the following way:

1. Simulate data $u_{GBP}, u_{DEM}, u_{JPY}, u_{CHF}$ from the copula $C$ of our choice.
2. Transform all of these uniform variates using the estimated marginal distribution functions $\tilde{F}_{GBP}, \tilde{F}_{DEM}, \tilde{F}_{JPY}, \tilde{F}_{CHF}$. So $(r_{GBP}, r_{DEM}, r_{JPY}, r_{CHF}) = \left( \tilde{F}_{GBP}^{-1}(u_{GBP}), \tilde{F}_{DEM}^{-1}(u_{DEM}), \tilde{F}_{JPY}^{-1}(u_{JPY}), \tilde{F}_{CHF}^{-1}(u_{CHF}) \right)$. These data will be the estimates for the returns on the currencies.
3. Use the four simulated returns $r_{GBP}, r_{DEM}, r_{JPY}, r_{CHF}$ to calculate the value of the portfolio on time $T + 1$. So $\tilde{V}_{T+1} = 10000(1 + r_{GBP}) + 10000(1 + r_{DEM}) + 10000(1 + r_{JPY}) + 10000(1 + r_{CHF})$. The change in portfolio value is equal to $\tilde{V}_{T+1} - V_T = \tilde{V}_{T+1} - 40,000$.
4. Perform steps 1, 2 and 3 a lot of times, say 10,000 or more times, and calculate from this dataset the VaR and the CVaR. After that ‘translate’ this VaR into a percentage of portfolio value and check how many times this percentage was exceeded in realized loss.

**Step 1:** Using the algorithms given in the section about simulation this is an easy task. Note however that the simulations for the Gaussian, t-, and Cook and Johnson copulas will give accurate results, but the simulations for the other Archimedean copulas are not completely reliable. The dependence structure will not be as strong as we desired (See Section 7). Because we are interested in the VaR and the CVaR of a portfolio, this will especially influence the results for the Archimedean copulas that have lower tail dependence. This is in particular the Type I copula. Unfortunately there is no way of telling

\[35\] Also the Type III Copula has lower tail dependence, but that is not depending on the value for $\alpha$. 

how much the results are influenced by the simulation inaccuracy.

**Step 2:** In Table 5 the Jarque-Bera statistic and the corresponding p-value for each of the currency series is also reported. This is a test based on the difference between the third and fourth moment statistics of the empirical distribution function and the Gaussian distribution. Based on this test the null hypothesis of normality of the returns can be rejected on a significance level of 1%. The marginals are modeled using the method described in Mashal and Naldi 2002. They use a $t$-distribution that is shifted and scaled. The returns that we observe, $\{r_t\}_{t=1}^T$, are assumed to follow a standard $t_\nu$-distribution if we scale and shift them in the following way:

$$\tilde{r}_t = \frac{r_t - m}{\sqrt{H}} \sim t_\nu$$

where $m$ is called the shift-parameter and $H$ is called the scaling factor.

We want to obtain estimates for $m$, $H$, and $\nu$ (the number of degrees of freedom in the $t$-distribution) simultaneously in a Maximum Likelihood procedure. To be able to do that, we have to calculate the Likelihood Function for $r_t$. This is done using the fact that:

$$r_t = \tilde{r}_t\sqrt{H} + m$$

The 'Change of Variables' technique tell us that:

$$f_r(r_t) = f_{\tilde{r}}(g^{-1}(r_t)) \cdot |g^{-1}(r_t)|$$

where $f_r(\cdot)$ denotes the (yet unknown) density function of $r_t$, $f_{\tilde{r}}(\cdot)$ is the density function of $\tilde{r}_t$ that we had assumed to be standard $t_\nu$ and $g(x) = x\sqrt{H} + m$ is the transformation function. Straightforward calculations yield:

$$f_r(r_t) = t_\nu \left( \frac{r_t - m}{\sqrt{H}} \right) \cdot \frac{1}{\sqrt{H}}$$

where $t_\nu(\cdot)$ is the density function of a $t$-distribution with $\nu$ degrees of freedom. Now we can find the Maximum Likelihood Estimator $\tilde{\theta}_{MLE} = (\tilde{m}, \tilde{H}, \tilde{\nu})$ by solving:

$$\tilde{\theta}_{MLE} = \arg \max_{(m,H,\nu)} \sum_{t=1}^T \left[ \log \left( t_\nu \left( \frac{r_t - m}{\sqrt{H}} \right) \right) + \log \left( \frac{1}{\sqrt{H}} \right) \right]$$

For each of the currency return series the estimates are given in Table 7.
Table 7: MLEs for the shift and scale parameter and the degrees of freedom.

Now the transformation of our simulated uniform variates into return estimates is done by

\[(r_{GBP}, r_{DEM}, r_{JPY}, r_{CHF}) = (\hat{F}^{-1}_{GBP}(u_{GBP}), \hat{F}^{-1}_{DEM}(u_{DEM}), \hat{F}^{-1}_{JPY}(u_{JPY}), \hat{F}^{-1}_{CHF}(u_{CHF}))\].

So \(r_{GBP} = \sqrt{\hat{H}^*} T^{-1}_{\nu}(u_{GBP}) + \hat{m}\), where \(\hat{m}, \hat{H}, \hat{\nu}\) are the numbers in Table 7 corresponding to the GBP series and \(T^{-1}_{\nu}()\) is the inverse of the \(t\)-distribution function. The other series are transformed in similar fashion.

**Step 3 + Step 4:** At time \(T\), in this case June 19th 2003, the value of the currency portfolio expressed in USD is equal to: \(V_T = 40,000\). Using 1,000,000 simulations (taking about 9 minutes) we estimate the VaR and the CVaR of our currency portfolio, assuming that the dependency structure is given by a Gaussian or a \(t\)-copula. For comparison we also look at the results using the traditional assumption of multivariate normality. A multivariate normal distribution is a Gaussian copula with all univariate Gaussian marginals. The estimated VaRs and CVaRs are in Table 8.

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
<th>95% CVaR</th>
<th>99% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mult. Norm.</td>
<td>321.76</td>
<td>456.07</td>
<td>404.20</td>
<td>523.40</td>
</tr>
<tr>
<td>Gaussian</td>
<td>321.42</td>
<td>489.46</td>
<td>427.53</td>
<td>599.55</td>
</tr>
<tr>
<td>(t)</td>
<td>320.21</td>
<td>502.19</td>
<td>435.75</td>
<td>628.18</td>
</tr>
</tbody>
</table>

Table 8: VaR and CVaR in USD for the currency portfolio. Gaussian / \(t\)-copula.

Next, the same analysis was carried out, this time assuming an Archimedean copula as underlying dependence structure. These calculations were based on 50,000 simulations and took about 40 minutes. The results are:

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
<th>95% CVaR</th>
<th>99% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cook and Johnson</td>
<td>360.22</td>
<td>584.22</td>
<td>502.90</td>
<td>734.29</td>
</tr>
<tr>
<td>Ali-Mikhail-Haq</td>
<td>293.38</td>
<td>420.70</td>
<td>373.24</td>
<td>488.17</td>
</tr>
<tr>
<td>Gumbel-Hougaard</td>
<td>319.20</td>
<td>469.81</td>
<td>413.97</td>
<td>562.73</td>
</tr>
<tr>
<td>Frank</td>
<td>328.90</td>
<td>444.32</td>
<td>402.50</td>
<td>511.05</td>
</tr>
<tr>
<td>Type I</td>
<td>364.55</td>
<td>602.77</td>
<td>513.29</td>
<td>758.79</td>
</tr>
<tr>
<td>Type II</td>
<td>351.52</td>
<td>548.95</td>
<td>474.83</td>
<td>679.18</td>
</tr>
<tr>
<td>Type III</td>
<td>341.57</td>
<td>550.05</td>
<td>476.37</td>
<td>700.62</td>
</tr>
</tbody>
</table>

Table 9: VaR and CVaR in USD for the currency portfolio. Archimedean copula.
Now it is interesting to see the large differences in results between the different copulas. This is, amongst others, a result of the fact that some copulas do have lower tail dependence and others don’t. The results for the copulas that have lower tail dependence, the Cook and Johnson, the Type I and Type III and the $t$-copula predict higher ‘possible’ losses than the others copulas, especially on the 99% VaR and CVaR level. If we compare the results for the Gaussian and $t$-copula to those of the Archimedean copulas then we see that the estimates for the ‘possible’ losses in case of a Cook and Johnson, a Type I, Type II and Type III copula are higher than the ones predicted by the two elliptical copulas. The estimated losses assuming a Ali-Mikhail-Haq, a Gumbel-Hougaard or a Frank copula are below the losses for the elliptical copulas.

The VaR estimates for the traditional assumption of multivariate normality are approximately equal on the 95% level to the estimates of the Gaussian and $t$-copula. On a 99% level they are lower. Note that the multivariate normal distribution is equivalent to using a Gaussian copula with all univariate Gaussian marginals. So the results coincide with our expectations. If we have fatter tailed margins, then the VaR and CVaR will be higher if one is ‘far enough in the tail’. The higher predicted 99% VaR level for the Gaussian copula is completely due to the fatter-tailed marginal distributions that were used there.

In order to see how well the different copulas can model the VaR and CVaR, we will also observe the empirical results from the dataset. This gives:

<table>
<thead>
<tr>
<th>Empirical Results</th>
<th>95% VaR</th>
<th>99% VaR</th>
<th>95% CVaR</th>
<th>99% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>312.56</td>
<td>453.03</td>
<td>409.26</td>
<td>547.24</td>
<td></td>
</tr>
</tbody>
</table>

Table 10: Empirical VaR and CVaR in USD for the currency portfolio.

Comparing these empirical results with the model predictions we see that the best (i.e., closest to the empirical results) copula model outcomes are given by the Gumbel-Hougaard and the Frank copula. The results for the Gaussian and $t$-copula seem to overestimate the losses, but the empirical numbers are based on a small number of observations (in this case 11 observations for the 99% empirical VaR and CVaR). It is interesting to see that in RANK 2000 a portfolio of two currency returns is also modeled using a Gumbel-Hougaard copula, but unfortunately the author does not show us how he made his choice or what other kind of copulas he included in this procedure. The results based on the assumption of multivariate normality seem to predict the VaR and CVaR levels remarkably well. Possibly in this case it is not a serious problem to model the marginals by univariate Gaussian distributions.

Modeling a distribution over the complete outcome space is different from modeling tail events. Therefore, although the Gaussian copula is equal to a $t$-copula with $\nu \to \infty$, the $t$-copula does not automatically always give better predictions for tail events. The ‘overall fit’ of the $t$-copula is however better, due
to the extra parameter that it has in comparison with the Gaussian copula. The two copulas that seem to perform best, the Gumbel-Hougaard and the Frank copula, do not have lower tail dependence. An extra thing to note is, that even though the Gumbel-Hougaard and the Frank copulas produce results that are close to the empirical VaR and CVaR, the simulations are not to be trusted as much as those for the Gaussian and $t$-copula. The simulation results of the Gaussian, the $t$, and the Cook and Johnson copula can be trusted, but the VaR and CVaR numbers of the other Archimedean copulas will be affected by the fact that the simulation technique that is used for those copulas is not really accurate.

The Archimedean copulas that do have lower tail dependence do not produce VaR and CVaR estimates close to the empirical numbers. Especially on a 99% level they overestimate 'possible' losses. The Ali-Mikhail-Haq copula cannot give an accurate description of the dependence structure because it is limited in modeling events with a Kendall's tau between 0 and $\frac{1}{3}$. It is somewhat surprising, that even though the dependence relations between the different series are of very different 'orders of strength', the Gumbel-Hougaard and Frank copula produce such good results compared to the Gaussian and $t$-copula. The Archimedean copulas only have one single parameter and all bivariate margins are exactly equal.

Another thing worth looking at is the fact, that even though the average return was positive for three out of four series, the estimates for the shift parameter $m$ were negative for all four return series. Maybe this is a result of the fact that the empirical distribution of the returns is skewed. Three out of four series have a negative median return and the other series has a 0 median return.

So far we have only compared VaR levels, but another check that can be done is a count on the number of 'violations' of the VaR estimates for all copulas and the multivariate normal method. I.e., count how many times the realized loss was higher than the VaR level that was estimated by that particular copula. The predicted VaR levels are first translated into a percentage of the portfolio level. We hope to find approximately 5% of realized losses higher than the estimated 95% VaR level, and for the 1% of realized losses higher than the estimated 99% VaR level. The number of in-sample 'violations' and the percentages are in Table 11.
Based on these outcomes the Ali-Mikhail-Haq, the Gumbel-Hougaard and the Frank copula look superior to the other copulas. A good performance for this portfolio is also given by the multivariate normal method. We can test if these values could be the outcomes of a binomial experiment with \( p = 0.05 \) for the 95% VaR and \( p = 0.01 \) for the 99% VaR. For the 95% VaR we reject the hypothesis of a Cook and Johnson, the Type I and Type II copula on a significance level of 5%. For the 99% VaR we reject the hypothesis (also using a significance level of 5%) of a Cook and Johnson, Type I, Type II, Type III copula and a \( t \)-copula.

So the Gaussian, Ali-Mikhail-Haq, Gumbel-Hougaard and the Frank copula can’t be rejected based on the number of violations of their estimated VaR levels. In this case the copula method does not seem to have an advantage over the assumption of a multivariate normal distribution. Although it must be said that the multivariate normal distribution is embedded into the copula approach by estimating a Gaussian copula with all Gaussian marginals. Comparing the empirical results to the Gaussian copula estimates and the estimates stemming from the multivariate normal distribution it looks as if the marginal distributions are better fitted for our purpose by Gaussians then by the shifted and scaled \( t \)-distribution.

We could also perform this copula technique ‘rolling’. Each day we estimate the VaR levels again based on the most recent \( T \) realized return observations. We assume that the dependence relationships are given by a Gaussian or a \( t \)-copula. But we have to make a choice for the number of observations that we use to estimate the copula parameter(s) as well as for the number of observations that we use to estimate the margins. For estimation of the copula parameter(s) we take 252 trading days (the equivalent of one trading year) and for estimation of the margins we take 63 trading days (the equivalent of three

---

### Table 11: Number and percentage of violations of the estimated VaR levels.

<table>
<thead>
<tr>
<th>Copula Type</th>
<th>Violations 95% VaR</th>
<th>Violations 99% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mult. Norm.</td>
<td>45 (4.01%)</td>
<td>10 (0.89%)</td>
</tr>
<tr>
<td>Gaussian</td>
<td>46 (4.10%)</td>
<td>7 (0.62%)</td>
</tr>
<tr>
<td>( t )</td>
<td>46 (4.10%)</td>
<td>4 (0.36%)</td>
</tr>
<tr>
<td>Cook and Johnson</td>
<td>34 (3.03%)</td>
<td>1 (0.09%)</td>
</tr>
<tr>
<td>Ali-Mikhail-Haq</td>
<td>58 (5.17%)</td>
<td>15 (1.34%)</td>
</tr>
<tr>
<td>Gumbel-Hougaard</td>
<td>46 (4.10%)</td>
<td>8 (0.71%)</td>
</tr>
<tr>
<td>Frank</td>
<td>44 (3.93%)</td>
<td>12 (1.07%)</td>
</tr>
<tr>
<td>Type I</td>
<td>35 (2.94%)</td>
<td>1 (0.09%)</td>
</tr>
<tr>
<td>Type II</td>
<td>41 (3.66%)</td>
<td>2 (0.18%)</td>
</tr>
<tr>
<td>Type III</td>
<td>42 (3.75%)</td>
<td>2 (0.18%)</td>
</tr>
</tbody>
</table>

\( ^{36} \)To get accurate VaR results we have to perform about 50,000 simulations. For the Archimedean copulas, with the exception of the Cook and Johnson copula, simulating 50,000 values from a four-dimensional copula takes about 6 minutes. Besides that, a new estimate for the copula parameter has to be made for each day.
trading months). The margins were again all taken to be shifted and scaled t-distributions. This seemed to work better than taking all Gaussian margins\(^\text{17}\).

In Figure 1 the estimated VaR levels as well as the realized losses were plotted. The bars represent the realized losses.

Now we can count the number of violations of the estimated VaR levels. This means that we count the number of times that the realized loss was higher than the predicted VaR levels. The total number of days that we check the estimated VaR levels is 858. These results can be found in Table 12.

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian copula</td>
<td>47 (5.41%)</td>
<td>16 (1.84%)</td>
</tr>
<tr>
<td>t-copula</td>
<td>48 (5.53%)</td>
<td>14 (1.61%)</td>
</tr>
</tbody>
</table>

Table 12: Realized losses (out of 858 obs.) outside the estimated VaR levels.

\(^{17}\) More discussion on the choice of the margins and the number of observations to use for estimation can be found in the subsection on bonds.
We can now test whether or not these number of violations are likely to be the outcome of a binomial trial with \( p = 0.05 \) for the 95% VaR level and \( p = 0.01 \) for the 99% VaR level. A two-sided test on a significance level of 5% cannot reject the 95% VaR estimates, but for the 99% VaR estimates the test rejects the case of the Gaussian copula.

It seems that the in-sample results for these copulas differ quite a lot from the out-of-sample predictions that were made ‘rolling’. The Gaussian and \( t \)-copula overestimated the VaR in the in-sample, static, predictions. In the out-of-sample, ‘rolling’ predictions they tend to underestimate the risk, even when the margins were taken to be shifted and scaled \( t \)-distributions. It would be interesting to see how the Archimedean copula perform on this test.
8 Applications

8.2 Hedge Fund Indices

In this section we will model the monthly returns on two different portfolios consisting of four hedge funds indices each. We will try to estimate the Value at Risk and the Conditional Value at Risk of equally weighted portfolios of these hedge fund indices. The data start January 31st 1994 and the last observation is from the May 31st 2003, so there are 113 observations. We will observe the behavior of two different kind of portfolios, one homogeneous portfolio and one heterogeneous portfolio. Just as in the previous paragraph the portfolio weights are chosen to be to be equal. The investments in each fund are USD 1,000,000.

The homogeneous portfolio consists of four hedge fund indices that are called CSFB/Tremont Hedge Fund Index (CSFB), Convertible Arbitrage (CA), Equity Neutral (EN) and Risk Arbitrage (RA). These hedge fund indices were chosen in such a way that they have a 'common degree of correlation', i.e., the bivariate margins in this portfolio look similar. One would expect that Archimedean copulas could be very useful in modeling this portfolio. For comparison we will again look at the multivariate normal estimates for the VaR. Hedge fund returns are often said to behave in a non-normal way and we would therefore expect to find better results using copulas than the multivariate normal estimates are able to give. This would be because the marginals have fatter tails than the Gaussian distribution.

The heterogeneous portfolio consists of four hedge fund indices that are called CSFB/Tremont Hedge Fund Index (CSFB), Convertible Arbitrage (CA), Dedicated Short (DS), and Long Short Equity (LSE). This portfolio is selected in such a way that the correlation coefficients among these funds are very diverse. Intuitively one would expect to find that Archimedean copulas are not suited to model this portfolio distribution.

Note that the CSFB/Tremont Hedge Fund Index is a weighted index, which contains, amongst others, all of the other hedge fund indices used here. The steps that were performed in the previous paragraph will be repeated for each of the two portfolios. Here is a brief overview of the steps:

1. Simulate uniform data from the copula $C$ of our choice.
2. Transform these uniform data into return estimates by using the estimated marginal distributions.
3. Calculate the portfolio value for one period ahead using the estimated returns.
4. Perform steps 1 up until 3 a lot of times and use these predicted values to calculate the VaR and CVaR of our portfolio.

Unfortunately the limited number of available observations leaves us no room for making the estimates 'rolling'.
8 Applications

8.2.1 A Homogeneous Portfolio

As mentioned before, our portfolio now consists of investments worth USD 1,000,000 each, in four different hedge fund indices: CSFB/Tremont Hedge Fund Index (CSFB), Convertible Arbitrage (CA), Equity Neutral (EN) and Risk Arbitrage (RA). To get an idea of the marginal distributions of each hedge fund index we look at several descriptive statistics for the return series (returns are in %):

<table>
<thead>
<tr>
<th></th>
<th>CSFB</th>
<th>CA</th>
<th>EN</th>
<th>RA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.90</td>
<td>0.86</td>
<td>0.87</td>
<td>0.67</td>
</tr>
<tr>
<td>Median</td>
<td>0.79</td>
<td>1.12</td>
<td>0.82</td>
<td>0.63</td>
</tr>
<tr>
<td>Maximum</td>
<td>8.53</td>
<td>3.57</td>
<td>3.26</td>
<td>3.81</td>
</tr>
<tr>
<td>Minimum</td>
<td>-7.55</td>
<td>-4.68</td>
<td>-1.15</td>
<td>-6.15</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>2.52</td>
<td>1.39</td>
<td>0.91</td>
<td>1.32</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.09</td>
<td>-1.61</td>
<td>0.16</td>
<td>-1.27</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>4.37</td>
<td>7.03</td>
<td>3.95</td>
<td>8.46</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>8.99</td>
<td>125.41</td>
<td>0.52</td>
<td>170.59</td>
</tr>
<tr>
<td>Probability</td>
<td>0.01</td>
<td>0.00</td>
<td>0.77</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 13: Descriptive statistics for the returns (in %) on the hedge fund index series.

The Kendall’s taus for the bivariate margins are very close. They are 0.21 (CSFB-CA), 0.21 (CSFB-EN), 0.24 (CSFB-RA), 0.29 (CA-RN), 0.23 (CA-RA), and 0.20 (EN-RA). These numbers are in a relatively small interval, and we would expect some Archimedean copulas are able to model this portfolio well.

Also for these series we perform a Dickey Fuller test to see if the series are indeed stationary because this is an implicit assumption that we make. The hypothesis of a unit root in the series was rejected on a 1% significance level for all series. The tables can be found in appendix C, Tables C5 until C10 (This includes the results for the two hedge fund indices that are in the heterogeneous, but not in the homogeneous portfolio.).

Before the process of simulation can start, first the copula parameters have to be estimated. Using the plug-in estimator, the estimate for the parameter Σ in the Gaussian and t-copula is (CSFB, CA, EN, RA):

\[
\begin{bmatrix}
1 & 0.325 & 0.328 & 0.366 \\
0.325 & 1 & 0.442 & 0.351 \\
0.328 & 0.442 & 1 & 0.315 \\
0.366 & 0.351 & 0.315 & 1
\end{bmatrix}
\]

The number of degrees of freedom of the t-copula is estimated to be 6.14. This suggests that there is either upper or lower tail dependence present in the
data. In appendix A, Figure A15, there is a plot of the bivariate margins, the CSFB return versus the CA return. It looks as if there is some degree of lower tail dependence present.

Also the parameters of the Archimedean copulas are estimated. The estimates as well as the weighted log-likelihood are in Table 14.

![Table 14: Estimation of $\alpha$ for the different Archimedean copula functions.](image)

As explained earlier, these estimates are weighted averages of the estimates for each bivariate margin. As expected, all of those estimates were close and as a result it seems that the Archimedean copulas are suited to model this portfolio. Exceptions are the Type I and Type III copulas who perform bad and give estimates that are on their parameter boundary. This is because the Type I and Type III copula are unable to model margins which have a Kendall's tau of less than $\frac{1}{3}$. The Ali-Mikhail-Haq copula gave one estimate that was on the upper boundary. That was for the CSFB-CA margin. Now the four-step plan can be started again:

**Step 1**: Simulating $(u_{CSFB}, u_{CA}, u_{EN}, u_{RA})$ using the copula parameters that were estimated above. Note again that the simulation methods for the Gaussian, $t$ and the Cook and Johnson copula are robust, but the other copula simulations produce data that has a less strong dependence relationship than we imply to give.

**Step 2**: Looking at the Jarque-Bera statistic in Table 13, the decision was made to model the marginal distributions again by the transformed and shifted $t$-distribution, except for the series EN, which will be fitted by a Gaussian distribution. The excess kurtosis that is present in the CSFB, the CA and the RA return series could be captured by working with the $t$-distribution, but the skewness in the series CA and RA will not be accounted for.

The estimates for the parameters in the Gaussian distribution for EN are $\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t = 0.87$ and $\hat{\sigma}^2 = \frac{1}{T-1} \sum_{t=1}^{T} (r_t - \hat{\mu})^2 = 1.94$. The estimates for the parameters $m$, $H$, and $\nu$ are given by:
Now transforming the uniform simulations into returns is done by 
\[(r_{\text{CFSB}}, r_{\text{CA}}, r_{\text{EN}}, r_{\text{RA}}) = (\hat{F}_{\text{CFSB}}^{-1}(u_{\text{CFSB}}), \hat{F}_{\text{CA}}^{-1}(u_{\text{CA}}), \hat{F}_{\text{EN}}^{-1}(u_{\text{EN}}), \hat{F}_{\text{RA}}^{-1}(u_{\text{RA}})).\]
So 
\[r_{\text{CFSB}} = \sqrt{H} \cdot T^{-1}_{\nu}(u_{\text{CFSB}}) + \hat{m},\]
where \(T^{-1}_{\nu}\) is the inverse \(t_{\nu}\)-distribution function. 
\(u_{\text{CA}}\) and \(u_{\text{RA}}\) are transformed similarly. For \(u_{\text{EN}}\) transformation is carried out by 
\[r_{\text{EN}} = \sigma \cdot \Phi^{-1}(u_{\text{EN}}) + \bar{\mu},\]
where \(\Phi^{-1}\) is the inverse standard Gaussian distribution function.

**Step 3 + 4:** At time \(T\), in this case May 31th 2003, the value of the hedge fund index portfolio expressed in USD is equal to: 
\[V_T = 4,000,000.\] Using 1,000,000 simulations (taking about 9 minutes) we can now estimate the VaR and the CVaR of our hedge fund index portfolio, assuming that the dependency structure is given by a Gaussian or a \(t\)-copula. Again, the VaR and CVaR under the assumption of multivariate normality are also reported:

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
<th>95% CVaR</th>
<th>99% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mult Norm</td>
<td>41,009</td>
<td>71,689</td>
<td>59,765</td>
<td>86,851</td>
</tr>
<tr>
<td>Gaussian</td>
<td>41,877</td>
<td>98,408</td>
<td>81,832</td>
<td>163,160</td>
</tr>
<tr>
<td>(t)</td>
<td>40,185</td>
<td>101,940</td>
<td>85,224</td>
<td>180,970</td>
</tr>
</tbody>
</table>

Table 16: VaR and CVaR in USD for the hom. portfolio. Gaussian / \(t\)-copula.

The same analysis is done once again, this time assuming an Archimedean copula as underlying dependence structure. These calculations were based on 50,000 simulations and took about 40 minutes. The VaR and CVaR for the 95% and 99% levels are:

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
<th>95% CVaR</th>
<th>99% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cook and Johnson</td>
<td>45,374</td>
<td>116,206</td>
<td>94,659</td>
<td>195,480</td>
</tr>
<tr>
<td>Ali-Mikhail-Haq</td>
<td>42,306</td>
<td>93,468</td>
<td>81,156</td>
<td>163,160</td>
</tr>
<tr>
<td>Gumbel-Hougaard</td>
<td>34,910</td>
<td>83,125</td>
<td>69,740</td>
<td>141,970</td>
</tr>
<tr>
<td>Frank</td>
<td>37,141</td>
<td>83,971</td>
<td>72,263</td>
<td>147,770</td>
</tr>
<tr>
<td>Type I</td>
<td>52,713</td>
<td>137,837</td>
<td>111,620</td>
<td>233,810</td>
</tr>
<tr>
<td>Type II</td>
<td>40,675</td>
<td>101,069</td>
<td>84,390</td>
<td>175,320</td>
</tr>
<tr>
<td>Type III</td>
<td>50,741</td>
<td>131,651</td>
<td>106,300</td>
<td>220,770</td>
</tr>
</tbody>
</table>

Table 17: VaR and CVaR in USD for the hom. portfolio. Archimedean copula.

Once again we see that the copula types that have lower tail dependence produce higher VaRs and CVaRs. Again the 95% VaR estimate multivariate
normality is about equal to the estimates produced by the Gaussian and \( t \)-
copulas. But the estimate for the 99% VaR is much lower, in fact its estimate
is lower than the 99% VaR estimate of any of the copulas. This leads to the
conclusion that probably this time the marginals behave non-normal, just as we
expected them to be. The empirical 95% VaR and CVaR are reported in Table
18, but note that we only have 113 observations.

<table>
<thead>
<tr>
<th>Empirical Results</th>
<th>95% VaR</th>
<th>95% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>39,991</td>
<td>87,743</td>
</tr>
</tbody>
</table>

Table 18: Empirical VaR and CVaR in USD for the hom. portfolio.

It seems that the Archimedean copulas with lower tail dependence do not
give a very accurate description of the tail of this loss distribution. Also it is dif-
ficult to tell which of the resulting copulas do give a good description of the tail
distribution, because of the small amount of data that we have. The Gaussian,
\( t \)-, Ali-Mikhail-Haq and the Frank copula seem to produce nice results based on
a comparison with the empirical numbers.

Just as in the currency example, the number of in-sample violations of the
estimated VaR levels has been counted for all copulas. The results can be found
in Table 19.

<table>
<thead>
<tr>
<th></th>
<th>violations 95% VaR</th>
<th>violations 99% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mult Norm</td>
<td>4 (3.54%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>Gaussian</td>
<td>4 (3.54%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>( t )</td>
<td>4 (3.54%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>Cook and Johnson</td>
<td>4 (3.54%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>Ali-Mikhail-Haq</td>
<td>4 (3.54%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>Gumbel-Hougaard</td>
<td>7 (6.19%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>Frank</td>
<td>7 (6.19%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>Type I</td>
<td>2 (1.77%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>Type II</td>
<td>4 (3.54%)</td>
<td>1 (0.88%)</td>
</tr>
<tr>
<td>Type III</td>
<td>2 (1.77%)</td>
<td>1 (0.88%)</td>
</tr>
</tbody>
</table>

Table 19: Number and percentage of violations of the estimated VaR levels.

Because of the small number of observations that we have we cannot reject
any of the estimates based on a significance level of 5%. Keep in mind that this is
only an in-sample test. A real check on the accuracy of the VaR estimates should
be based on an out-of-sample test. Due to the small number of observations it
was not possible however to perform such a test.
8.2.2 A Heterogeneous Portfolio

The second portfolio that we will look at is the one consisting of the hedge fund indices: CSFB/Tremont Hedge Fund Index (CSFB), Convertible Arbitrage (CA), Dedicated Short (DS), and Long Short Equity (LSE). Two of the hedge fund indices, CSFB and CA were also in the previous portfolio, but for the sake of completeness Table 20 also includes their statistics. The methods used in this paragraph are similar to the ones used in the previous paragraph, so the results will be given without explaining all the used methods again.

<table>
<thead>
<tr>
<th></th>
<th>CSFB</th>
<th>CA</th>
<th>DS</th>
<th>LSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.90</td>
<td>0.86</td>
<td>0.06</td>
<td>0.98</td>
</tr>
<tr>
<td>Median</td>
<td>0.79</td>
<td>1.12</td>
<td>-0.19</td>
<td>0.77</td>
</tr>
<tr>
<td>Maximum</td>
<td>8.53</td>
<td>3.57</td>
<td>22.71</td>
<td>13.01</td>
</tr>
<tr>
<td>Minimum</td>
<td>-7.55</td>
<td>-4.68</td>
<td>-8.69</td>
<td>-11.43</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>2.52</td>
<td>1.39</td>
<td>5.26</td>
<td>3.27</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.09</td>
<td>-1.61</td>
<td>0.86</td>
<td>0.23</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>4.37</td>
<td>7.03</td>
<td>4.89</td>
<td>5.86</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>8.99</td>
<td>125.41</td>
<td>30.74</td>
<td>39.55</td>
</tr>
<tr>
<td>Probability</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 20: Descriptive statistics for the returns on the hedge fund index series.

The Kendall’s taus for the bivariate margins are not very close. They are 0.21 (CSFB-CA), -0.35 (CSFB-DS), 0.58 (CSFB-LSE), -0.15 (CA-DS), 0.16 (CA-LSE), and -0.56 (DS-LSE). These numbers are far apart, and consequently we would expect that the Archimedean copulas are not able to model this portfolio appropriately.

Also for both new series, DS and LSE, the Dickey Fuller test rejects the null hypothesis of a unit root on a significance level of 1%. Again the marginal distribution of the hedge fund index returns will be modeled using the shifted and scaled $t$-distribution. The parameter estimates for $m$, $H$, and $\nu$ are:

<table>
<thead>
<tr>
<th></th>
<th>$\hat{m}$</th>
<th>$\hat{H}$</th>
<th>$\hat{\nu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSFB</td>
<td>0.82</td>
<td>3.14</td>
<td>3.44</td>
</tr>
<tr>
<td>CA</td>
<td>1.15</td>
<td>0.45</td>
<td>1.90</td>
</tr>
<tr>
<td>DS</td>
<td>-0.16</td>
<td>21.66</td>
<td>10.06</td>
</tr>
<tr>
<td>LSE</td>
<td>0.84</td>
<td>5.38</td>
<td>3.82</td>
</tr>
</tbody>
</table>

Table 21: MLEs for the shift and scale parameter and the degrees of freedom.

The estimate for the parameter $\Sigma$ in the Gaussian and $t$-copula is:

$$
\begin{bmatrix}
1 & 0.325 & -0.526 & 0.791 \\
0.325 & 1 & -0.241 & 0.242 \\
-0.526 & -0.241 & 1 & -0.767 \\
0.791 & 0.242 & -0.767 & 1 \\
\end{bmatrix}
$$
The estimate for the degrees of freedom parameter in the $t$-copula is 4.78. The parameter estimates for the Archimedean copulas as well as the weighted average of the log-likelihood are:

<table>
<thead>
<tr>
<th>Copula</th>
<th>Parameter estimate</th>
<th>Weighted Log-Lik.</th>
<th>Kendall’s tau</th>
<th>Lower Tail Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cook and Johnson</td>
<td>0.51</td>
<td>9.92</td>
<td>0.20</td>
<td>0.26</td>
</tr>
<tr>
<td>Ali-Mikhail-Haq</td>
<td>0.43</td>
<td>7.96</td>
<td>0.11</td>
<td>0</td>
</tr>
<tr>
<td>Gumbel-Hougaard</td>
<td>1.29</td>
<td>9.81</td>
<td>0.22</td>
<td>0</td>
</tr>
<tr>
<td>Frank</td>
<td>1.85</td>
<td>9.68</td>
<td>0.20</td>
<td>0</td>
</tr>
<tr>
<td>Type I</td>
<td>1.10</td>
<td>-14.60</td>
<td>0.39</td>
<td>0.53</td>
</tr>
<tr>
<td>Type II</td>
<td>2.06</td>
<td>10.38</td>
<td>0.23</td>
<td>0</td>
</tr>
<tr>
<td>Type III</td>
<td>1.00</td>
<td>-17.69</td>
<td>0.33</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 22: Estimation of $\alpha$ for the different Archimedean copula functions.

It is important in this case to look not only at the final estimates for $\alpha$, but also at the individual estimates for the bivariate margins. If they are close, then that is an indication that the particular Archimedean copula will probably be a good choice in modeling this portfolio. If the individual estimates for $\alpha$ for the bivariate margins are far apart, then one should be aware that the copula will probably not give a good description for the dependence structure of the portfolio. In this case the estimates for all the bivariate margins that include DS are equal to their lower boundary value.

For instance, the estimates for the Cook and Johnson copula look like this:

<table>
<thead>
<tr>
<th>margin</th>
<th>estimate for $\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSFB-CA</td>
<td>0.66</td>
</tr>
<tr>
<td>CSFB-DS</td>
<td>0.00</td>
</tr>
<tr>
<td>CSFB-LSE</td>
<td>2.02</td>
</tr>
<tr>
<td>CA-DS</td>
<td>0.00</td>
</tr>
<tr>
<td>CA-LSE</td>
<td>0.35</td>
</tr>
<tr>
<td>DS-LSE</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 23: Estimates for $\alpha$ for each bivariate margin for a Cook and Johnson copula.

Why is it difficult for the Archimedean copulas to model this portfolio? The reason can be found in Theorem (53). As a consequence of that theorem, all of the bivariate margins of these Archimedean copulas will have a Kendall’s tau that is larger than or equal to 0. As seen in the beginning of this section, Kendall’s tau for the data is negative for all bivariate margins containing DS. Consequently, the fit that is given by the Archimedean copulas will not be very good.
We estimate once again the VaR and CVaR of this portfolio on a 95% and 99% level. For the Gaussian and $t$-copula, 1,000,000 simulations were used, resulting in the following numbers:

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
<th>95% CVaR</th>
<th>99% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mult Norm</td>
<td>45,870</td>
<td>76,720</td>
<td>64,856</td>
<td>92,136</td>
</tr>
<tr>
<td>Gaussian</td>
<td>47,762</td>
<td>95,701</td>
<td>83,656</td>
<td>160,870</td>
</tr>
<tr>
<td>$t$</td>
<td>45,326</td>
<td>102,950</td>
<td>87,952</td>
<td>180,160</td>
</tr>
</tbody>
</table>

Table 24: VaR and CVaR in USD for the het. portfolio. Gaussian / $t$-copula.

We can also assume that the dependence structure of the portfolio can be described by an Archimedean copula. Then the estimated VaR’s and CVaR’s for the heterogeneous portfolio are:

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
<th>95% CVaR</th>
<th>99% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cook and Johnson</td>
<td>128,489</td>
<td>240,310</td>
<td>206,290</td>
<td>355,470</td>
</tr>
<tr>
<td>Ali-Mikhail-Haq</td>
<td>101,019</td>
<td>167,083</td>
<td>147,310</td>
<td>232,980</td>
</tr>
<tr>
<td>Gumbel-Hougaard</td>
<td>108,261</td>
<td>182,646</td>
<td>157,900</td>
<td>248,350</td>
</tr>
<tr>
<td>Frank</td>
<td>110,328</td>
<td>178,995</td>
<td>156,000</td>
<td>240,710</td>
</tr>
<tr>
<td>Type I</td>
<td>144,731</td>
<td>290,599</td>
<td>242,430</td>
<td>429,340</td>
</tr>
<tr>
<td>Type II</td>
<td>120,441</td>
<td>215,035</td>
<td>184,100</td>
<td>302,920</td>
</tr>
<tr>
<td>Type III</td>
<td>138,814</td>
<td>266,597</td>
<td>227,090</td>
<td>398,800</td>
</tr>
</tbody>
</table>

Table 25: VaR and CVaR in USD for the het. portfolio. Archimedean copula.

The large differences between the Gaussian and $t$-copula on one hand and the Archimedean copulas on the other hand are due to the fact that the Archimedean copulas are unable to handle the negative Kendall’s tau that is present in the data. Just as in previous examples the 95% VaR estimate using the multivariate normality assumption is approximately equal to that of the Gaussian and $t$-copula estimate but the 99% VaR estimate is lower.

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>95% CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empirical Results</td>
<td>61,023</td>
<td>111,744</td>
</tr>
</tbody>
</table>

Table 26: Empirical VaR and CVaR in USD for the het. portfolio.

Looking at the empirical 95% VaR and CVaR estimate it appears as if the $t$-copula produces the best results. Because of the obvious inability of the Archimedean copulas to model this portfolio we will not look at the number of ‘violations’ of the estimated VaR levels. For the Gaussian and $t$-copula and the multivariate normal approach the results are:
The 95% VaR level was underestimated by all three approaches, but looking at the 99% VaR the Gaussian copula with fatter tailed margins (the scaled and shifted $t$-distributions) and the $t$-copula both look reasonably accurate. Once again the number of observations is however too low to reject any of the estimates on a significance level of 5%.

<table>
<thead>
<tr>
<th></th>
<th>violations 95% VaR</th>
<th>violations 99% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mult Norm</td>
<td>9 (7.96%)</td>
<td>3 (2.65%)</td>
</tr>
<tr>
<td>Gaussian</td>
<td>8 (7.08%)</td>
<td>2 (1.77%)</td>
</tr>
<tr>
<td>$t$</td>
<td>9 (7.96%)</td>
<td>1 (0.88%)</td>
</tr>
</tbody>
</table>

Table 27: Number and percentage of violations of the estimated VaR levels.
8 Applications

8.3 Bonds

In this section the copula techniques will be applied to model the return on a portfolio of 10 corporate bonds, as well as a portfolio consisting of 5 corporate bonds. We will try to estimate the Value at Risk of these two portfolios. All estimates will be made in a 'rolling' framework. The Gaussian and t-copula will be used to model the dependence structure. Due to the slower simulation procedure for the Archimedean copulas it would take a lot of time to generate results for the Archimedean copulas. Besides that, one would also expect a portfolio of 10 assets to be too large for an Archimedean copula to model. This is because an Archimedean copula only has one parameter and all margins are identical. The portfolio consists of bonds AES, AT&T, Boeing, Enron, Lucent Technologies, Petroleum Geoservices, General Motors, Coca Cola, Dell and Johnson & Johnson. Assuming that we possess 100 of each of these bonds we want to know what the 95% and 99% Value at Risk for this portfolio is.

The portfolio is seen from the point of view of an American investor, i.e., returns are assumed to be realized on American trading days only. We have a collection of daily prices for all of these bonds from January 2nd 2001 until July 3rd 2003. Starting on January 2nd 2002 we will estimate the 95% and 99% VaR of this portfolio on a daily basis. Next day we can check whether the realized loss was higher than the estimated VaRs or not. This we will do until July 3rd 2003, and then we have backtested our VaR estimates 376 times.

The reason that this specific portfolio was chosen, is to see how the VaR will respond to very large losses. At the beginning of the sample period the bonds were all priced in between USD 93.97 (Lucent) and USD 113.31 (Enron). At the end of the sample period a few of them have led severe losses. For instance, AES is then worth USD 72.25, Enron USD 18.00 and Petroleum Geoservices is worth USD 61.00.

The VaR estimates will be based upon two important assumptions. The first being that the dependence structures of the 10 returns is given by a Gaussian or a t-copula. The second critical assumption is that the complete dependence structure can be described by taking into account the last 252 trading days. The first case that we will consider is one in which all marginals are estimated using the last 252 trading days. All of them will have equal weight.

Possible other approaches that might be interesting to look at in the future could be for instance exponential weighting for the observations. In that way the more recent information is given more importance than information in the far past. Another interesting idea was first seen in Patton 2001. In that paper the correlation parameter in the Gaussian copula is modeled by a scaled ARMA process, and so it is made time varying in a different way.

This time-varying copula parameter modeling is relatively unexplored so far.
One could also apply the technique that is used in this thesis (equally weighted periods) in combination with a test described in Dias and Embrechts 2002. With a Generalized Likelihood Ratio test they try to track the most recent point in history when a change in dependence structure occurred. In that way one could see how many of the most recent observations should be taken into account in the estimation procedure, instead of just taking 252 as we will do here.

On one hand we want to have a description of the dependence structure that is up to date (i.e., not using info from the far past) while on the other hand we want reliable parameter estimates (and this requires a lot of observations). Without an obvious solution for this problem we will just try to take 252 equally weighted trading days as a starting point for further research.

The difficulty with modeling the marginal distributions of the returns on bonds, is that bond prices can become very illiquid after a drop in price. This happened to 5 out of 10 bonds in this portfolio. The most severe case of illiquidity happened for AES, whose bond price remained unchanged for more than 6 months. Some bonds start to behave 'normal\footnote{In this case meaning nothing more than that the bond price becomes fluid again.}' again after a drop and a period of non-trading, but others remain illiquid. Let’s first try to model the marginals by Gaussian distributions. The parameters estimates are given by the sample averages and the sample variances over the most recent 252 trading days.

The technique used to estimate the value for the 95% and 99% VaR has been described in the sections about the currency and the hedge fund portfolios. The number of simulations used to generate the VaR and CVaR estimates is 100,000 per day. In Figure 2 the estimates for the 95% and 99% VaR using a Gaussian copula, and a \( t \)-copula are plotted. All marginals are univariate Gaussian, so in this case the VaR estimates using the Gaussian copula are equal to the classical VaR estimates under assumption of multivariate normality. The realized losses (profits have not been shown) are shown by the bars. The highest realized loss occurred on July 31st 2002, mainly because the bond price of Petroleum Geoservices dropped by 44%. After that, the estimated VaR levels have raised somewhat.

As can be seen in the picture, the 95% VaR for the Gaussian and the \( t \)-copula are very close. For the 99% level the \( t \)-copula obviously predicts higher 'possible' losses. Strange is the fact that after the 178th trading day in 2002, September 13th 2002, there is a very sudden drop in the estimates for the VaR levels. How is that to be explained? First we have to realize that the estimates for the matrix \( \Sigma \) in the copulas is not likely to change a lot in one day. So the reason for this sudden decline in estimated VaR must lie in one of the marginal distributions. Let's look at the estimated means and variances for the bond returns on September 13th, 16th and 17th 2002 (Table 28):
Figure 2: Estimated VaR levels and realized losses for the bond portfolio.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>AES</td>
<td>-0.118</td>
<td>1.89</td>
<td>-0.118</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>-0.037</td>
<td>0.23</td>
<td>-0.038</td>
</tr>
<tr>
<td>Boeing</td>
<td>0.005</td>
<td>1.24</td>
<td>0.011</td>
</tr>
<tr>
<td>Enron</td>
<td>-0.705</td>
<td>29.93</td>
<td>-0.674</td>
</tr>
<tr>
<td>Lucent T.</td>
<td>-0.067</td>
<td>2.42</td>
<td>-0.082</td>
</tr>
<tr>
<td>Petroleum G.</td>
<td>-0.243</td>
<td>17.97</td>
<td>-0.234</td>
</tr>
<tr>
<td>General Motors</td>
<td>0.000</td>
<td>0.06</td>
<td>-0.001</td>
</tr>
<tr>
<td>Coca Cola</td>
<td>0.021</td>
<td>2.85</td>
<td>0.088</td>
</tr>
<tr>
<td>Dell</td>
<td>0.037</td>
<td>0.17</td>
<td>0.037</td>
</tr>
<tr>
<td>Johnson &amp; J.</td>
<td>0.036</td>
<td>0.49</td>
<td>0.037</td>
</tr>
</tbody>
</table>

Table 28: Estimated mean and variance for the bond returns on three trading days.

The estimated returns do not give an explanation for the sudden decline in VaR estimates. But looking at the variance for the Coca Cola bond we see that according to the analysis the Coca Cola bond is one of the largest risk factor in
our portfolio on September 13th, but two days later it is the least volatile bond in our portfolio.

This sudden drop in volatility is a result of the effect of taking the unweighted sample variance over the past 252 trading days as an estimate for the variance of the bond return. On September 14th 2001 the Coca Cola bond dropped 17% in value and one trading day later, September 17th 2001 the Coca Cola bond raised 21% in value. After 252 trading days those two observations disappear from our ‘window’ and this results in a much lower estimate for the volatility of the bond return. This is a disadvantage of using the observations equally weighted.

We have 376 out-of-sample checks for the 95% and 99% VaR levels. In Table 29 the number and percentage of realized losses are shown that were outside of the estimated VaR levels.

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian copula</td>
<td>7 (1.8%)</td>
<td>3 (0.8%)</td>
</tr>
<tr>
<td>t-copula</td>
<td>7 (1.8%)</td>
<td>3 (0.8%)</td>
</tr>
</tbody>
</table>

Table 29: Realized losses (out of 376 obs.) outside the estimated VaR levels.

Testing the number of exceedings of the estimated VaR levels using a significance level of 5% we cannot reject the 99% VaR estimates given by the Gaussian and t-copula. Both the 95% VaR estimates are rejected however.

After having seen previous results we analyzed the cause of the strange returns on the Coca Cola on September 14th and September 17th 2001. It was found out that the bond price of September 14th 2001 was an error in the database. In reality on this day an increase of less than 1% in the Coca Cola bond price had taken place. So the whole analysis was performed once again, this time using the adjusted bond price. This results in a new plot, Figure 3.

In Table C11 (appendix C) some characteristics of the return series on the bonds are given. The checks on the estimated vs. realized loss gives:

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian copula</td>
<td>11 (2.9%)</td>
<td>5 (1.3%)</td>
</tr>
<tr>
<td>t-copula</td>
<td>11 (2.9%)</td>
<td>4 (1.06%)</td>
</tr>
</tbody>
</table>

Table 30: Realized losses (out of 376 obs.) outside the estimated VaR levels.

This time none of the estimated VaR levels can be rejected when testing on a significance level of 5%. The 99% VaR estimates seem to perform well, but the 95% levels are still low.
Probably the biggest problem in modeling the return of this portfolio is the fact that some bonds become illiquid at some periods in time. If we have this 252 trading day window, then there is a lot of influence of old information on our VaR estimates. Also the single error in the Coca Cola bond price has shown, that outliers have a large influence on the estimates and, even more important, they stay in our window for a long period. The long lasting effect of such outliers can be reduced by reducing the number of observations used to estimate the margins. Observing Figures 2 and 3, we see that the VaR estimates are very static. They do react to very large losses with an increase in the VaR estimates, but the change is not much.

This will be different when we decide to take fewer observations into account when estimating the margins. But how many trading days should we take then? After seeing the results for 21 trading days (1 trading month), 63 trading days (3 trading months) and 126 trading days (6 trading months), it looked like 63 trading days would be the best choice. Taking 21 trading days into account gives estimates that fluctuate very much, while taking 126 trading days into account gives us estimates that behave very static.
Another adjustment to the estimation procedure could be using the shifted and scaled $t$-distributions to model the data. Unfortunately, when trying this out, the estimates for the degrees of freedom parameter sometimes were 0.5 or even lower. This gave some numerical problems when using the inverse $t$-distribution function. This 'routine' was instable when the number of degrees in the $t$-distribution was less than 1. Two possible solution methods have been tested. The first approach was to try to model the margins by the shifted and scaled $t$-distribution, but instead using a Gaussian distribution if the estimate for the number of degrees of freedom in the $t$-copula was less than 1. The second possible approach was to perform a restricted optimization procedure for the shifted and scaled $t$-distribution, the restriction being that the number of degrees of freedom is at least equal to 1.

Unfortunately both procedures result in VaR estimates with a lot of jumps. These jumps occur in the first approach whenever there is a change in the type of distribution that is used. So when we use the shifted and scaled $t$-distribution on one day and on the next day a Gaussian distribution, or the other way around. In the second approach they occur whenever the margin estimate changes from an unrestricted into a restricted solution, or the other way around. The big problem is, that while we want to have a distribution with very fat tails (degrees of freedom parameter in the $t$-distribution below 1), we actually use a thinner tailed distribution (Gaussian distribution, or a shifted, scaled $t$-distribution with a higher number of degrees of freedom) a lot of times. For the sake of completeness both graphs are in appendix A, Figures A19 and A20.

This thesis is about copulas and how they perform in risk measurement. If we assume that we can model the individual bond behavior, then we could really see how the Gaussian and $t$-copula perform. Therefore a new portfolio will be formed by 100 units each of 5 bonds, chosen in such a way that the marginals are easier to model. These bonds have no illiquidity problem in this dataset, and based on the Jarque-Bera statistic they are closest to the Gaussian distribution. The bonds are Boeing, General Motors, Coca Cola, Dell and Johnson & Johnson.

We use 252 trading days to estimate the copula parameters, just as before. For modeling the margins we have tried to use 1 month, 3 months and 6 months, respectively 21, 63 and 126 trading days. When using only 21 observations the estimates for the VaR levels become very volatile. This is because one outlier can have huge consequences now for that particular margin. On the other hand with 126 trading days the VaR is changing very slowly. Just as in Figure 2 the realized losses/profits seem to have a small effect on the estimated VaR levels in the near future. Therefore we will use 3 months of observations, 63 trading days, to model the margins. The margins are first estimated by Gaussians. The analysis was performed on this small bond portfolio and the VaR estimates and losses are shown in Figure 4.
This seems to be a much better behavior of the VaR estimates than we had before. They react more strongly on the realized losses. But how do they perform on the out-of-sample test?

None of the estimated VaR levels could be rejected based on the number of violations (using a significance level of 5%). But maybe we can improve the 99% VaR estimate by using the scaled and shifted $t$-distribution for the margins. The calculations were performed once again, this time using the shifted and scaled $t$-distribution for the margins. The resulting graph with realized losses and predicted VaR levels is shown as Figure 5.
Clearly the 99% VaR estimates have increased and it gets a better performance based on the number of violations. But the 95% estimates have dropped a little on average. As one can see in Table 32 this has a serious consequence for the accuracy of the 95% VaR estimates. The number of violations has gone up.

<table>
<thead>
<tr>
<th></th>
<th>95% VaR</th>
<th>99% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian copula</td>
<td>30 (7.98%)</td>
<td>5 (1.33%)</td>
</tr>
<tr>
<td>t-copula</td>
<td>32 (8.51%)</td>
<td>5 (1.33%)</td>
</tr>
</tbody>
</table>

Table 32: Realized losses (out of 376 obs.) outside the estimated VaR levels.

Testing on a significance level of 5% results in rejecting both 95% estimates for the VaR level. The 99% VaR estimates are not rejected.
Conclusions

Theory

Sklar’s theorem proves the one-to-one relationship that exists between the joint distribution function on one hand, and the copula function combined with all marginal distribution functions on the other hand. For any continuous joint distribution function the univariate margins and the dependence structure, the copula, can be separated. This makes copulas great modeling ‘tools’. A copula function completely describes the dependence structure that exists between random variables. Using copulas in constructing multivariate distributions has the big advantage that there exists a substantial amount of flexibility in the modeling phase. All of the marginal distributions can be imposed independently of each other and of the copula function. Another advantage is the increased knowledge about the characteristics of the dependence structure.

Closely related to copulas are association concepts like Kendall’s tau, Spearman’s rho, and lower and upper tail dependence. Kendall’s tau and Spearman’s rho are measures range between -1, meaning perfect negative dependence, and 1, meaning perfect positive dependence. Contrary to the often-used linear correlation coefficient, Kendall’s tau and Spearman’s rho are not influenced by a choice of margins or increasing monotone transformations on one or more variables. They are based upon the rank numbers of observations, and they can be expressed in copula terms. Kendall’s tau and Spearman’s rho are measures that describe the ‘general dependence’ that exists between variables. If one is interested in the joint extreme behavior of the variables, then the lower - and upper tail dependence coefficients are convenient to report. Upper tail dependence arises when extremely high values for two random variables tend to occur jointly. Lower tail dependence arises when extremely low values for two random variables tend to occur jointly. Both are measures between 0, meaning independence of extreme events, and 1, meaning there is perfect dependence between extreme events. One of the advantages of using copulas when modeling risk measures, such as the Value at Risk, is that copulas give a lot of insight in the tail behavior.

In this thesis two important classes of parametric copulas are treated. Those are:

- The class of elliptical copulas, and
- The class of Archimedean copulas.

Elliptical copulas are copulas derived from elliptical distributions. In practice only the Gaussian copula and the t-copula are used. This class of copulas has at least one parameter for each bivariate margin, combined in the correlation matrix. For the Gaussian and the t-copula there exists a very specific relationship between each of the correlation coefficients and Kendall’s tau for
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the variables. The $t$-copula has one extra parameter, which is the degrees of freedom parameter. If we take a $t$-copula, and let the degrees of freedom go to infinity, then this will converge to a Gaussian copula. The advantage of a $t$-copula over a Gaussian copula is that the tail dependence (lower- and upper-tail dependence are necessarily equal for an elliptical copula) is dependent on the degrees of freedom parameter. For the Gaussian copula the tail dependence is zero by definition. Extensions of bivariate elliptical copulas to multivariate copulas are straightforward to make.

Archimedean copulas are copulas ‘generated’, and characterized, by a function with only one parameter. Archimedean copulas can have unequal coefficients of upper-, and lower tail dependence. Archimedean copulas have bivariate margins that are all exactly identical. In this thesis we look at seven specific Archimedean copulas. The seven are a subset of the Archimedean copulas in Nelsen 1999. They are called the Cook and Johnson copula (also called Clayton), the Ali-Mikhail-Haq copula, the Gumbel-Hougaard copula, the Frank copula, and three copulas without a name, which are labeled Type I, Type II, and Type III copula.

Advantages of elliptical copulas over Archimedean copulas are:

1. Elliptical copulas have at least one parameter per bivariate margin. On the other hand, Archimedean copulas have only one parameter in total, and as a consequence of this fact all bivariate margins are identical. This is the same as saying that all dependence structures between combinations of two variables are exactly the same. In a lot of cases this will not be realistic to assume. An advantage of only having one parameter in the dependence structure is that the estimation uncertainty is much smaller.

2. Elliptical copulas can have Kendall’s taus over the complete interval $[-1, 1]$. Multivariate Archimedean copulas used in practice can only have a Kendall’s tau that is non-negative. This means that they cannot describe negative dependence relationships.

3. Elliptical copulas can be extended to more dimensions in a straightforward way. This is not the case for Archimedean copulas. Technical conditions have to be satisfied in order to make an Archimedean copula a valid copula for all dimensions $n \geq 2$.

4. Simulating from elliptical copulas is easy, and fast. Despite what is claimed in a lot of papers, simulating from Archimedean copulas in general is NOT simple. Simulating from bivariate Archimedean copulas however is an exception to this.

Advantages of Archimedean copulas over elliptical copulas are:

1. The class of Archimedean copulas allows for a great variety of different dependence structures. Important is the allowed asymmetry of upper and lower tail.
Conclusions

2. Archimedean copulas have closed form expressions. This in contrast to the elliptical copulas, that are defined in terms of integral formulas, or in terms of distribution functions and inverses thereof. The closed form expressions and their description in terms of the generating function often simplifies calculations for the Archimedean copulas.

Parameter estimation for copulas is often done with methods based on the Maximum Likelihood Principle, and in this thesis the focus is on those methods. Distinction is made between three approaches:

- **Exact Maximum Likelihood (EML)**
  In this method the copula idea of separation of margins from the dependence structure is not used. After making parametric assumptions about the margins, the margin parameters are estimated simultaneously with the copula parameter(s). This method is therefore computationally intensive, and the copula parameters are dependent on the assumptions that were made about the marginal distributions.

- **Inference Functions for Margins (IFM)**
  This method is a two-step procedure. First one makes parametric assumptions about the margins, and their parameters are estimated. In the second step the margin parameter estimates are used in the Maximum Likelihood procedure for determining the copula parameter estimates. Also for this method the copula parameters will be dependent on the assumptions that were made about the marginal distributions.

- **Canonical Maximum Likelihood (CML)**
  This procedure estimates the copula parameters without making any assumption about the margins. The margins are transformed into uniform variates using rank numbers (or equivalently using the empirical distribution function). After that, the copula parameters can be estimated margin-free (margin-independent) with the Maximum Likelihood Principle.

In this thesis an explanation is given how to use the CML method to estimate the copula parameters for the Gaussian copula, the $t$-copula, and the Archimedean copulas. Also a method is proposed to estimate the copula parameters in the case of missing values for one or more variables. The method is based on making separate parameter estimates for all bivariate margins (i.e., combinations of two variables).

Practice

The main focus of this thesis was copulas of order higher than two. For the Gaussian and $t$-copula a lot of results have been derived in the literature, and this is a result of the fact that it is straightforward to extend an elliptical copula
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to more dimensions. When studying papers about Archimedean copulas however, one has to be aware of the fact that extending a bivariate Archimedean copula to a multidimensional one is not straightforward, and that some of the results derived for bivariate Archimedean copulas might not hold for multidimensional Archimedean copulas.

Some papers claim that simulating from Archimedean copulas is easy. This is however only the case for bivariate Archimedean copulas. In some exceptional cases, like for the Cook and Johnson copula, a fast and accurate simulation algorithm is known, but for most cases there is not. The algorithm that is used in this thesis for simulating from Archimedean copulas is a recursive algorithm. It has the drawback that any numerical inaccuracy will make the dependence relation between the simulated variables weaker than desired. The more dimensions the Archimedean copula has, the worse the simulations will be. This has put an important restriction on the copula dimension size that has been used in this thesis.

In the Applications section the focus is on the calculation of the Value at Risk (VaR) and Conditional Value at Risk (CVaR) of a portfolio. This is done by simulation. The most important test on the methods that are used is the out-of-sample test. This is done by estimating the VaR for next day, and repeating this procedure day after day. Next day one can check whether or not the VaR estimate was ‘violated’ or not. Due to the fact that this simulation procedure takes an enormous amount of time, this procedure could only be used for the Gaussian and $t$-copula. Note that although the $t$-copula is a class that implicitly encloses the Gaussian copula, this does not mean that the VaR estimates assuming a $t$-copula are better than the estimates assuming a Gaussian copula. First of all because ‘fitting’ a complete distribution, which is done in the estimation procedure, is different from ‘fitting’ tail events, which is what the VaR is. Secondly because we are fitting parameters in-sample, but the important tests are out-of-sample tests. All tests are performed on a 5% significance level. The areas of application and the results are:

1. Currencies
   A portfolio of four currencies (the British Pound, the German Mark, the Japanese Yen, and the Swiss Franc) is created (daily returns). All copulas are used to model this portfolio. The margins are approximated by shifted, scaled $t$-distributions. The in-sample results indicate that the Gaussian, the Ali-Mikhail-Haq, the Gumbel-Hougaard, and the Frank copula are reasonable. \textsc{Rank} 2000 also uses the Gumbel-Hougaard copula to describe a portfolio of two currencies, but no explanation is given why he does so. But even better in-sample results are given by the simple assumption of multivariate normality. The out-of-sample results compare a Gaussian and a $t$-copula, both with shifted, scaled $t$-distributions as margins. This time the Gaussian copula is rejected, because it has too many violations of the 99% VaR.
2. Hedge fund indices

Two portfolios of hedge funds are constructed (monthly returns). One portfolio is homogeneous, meaning that the dependence structures between the funds are 'similar'. This implicates that the Kendall's tau for the bivariate margins are close. The margins are modeled by the shifted, scaled t-distributions. Because of the small number of observations available only in-sample tests have been performed. None of the copulas could be rejected based on the number of violations.

The second portfolio is a heterogeneous portfolio, a portfolio for which the Kendall's taws for the bivariate margins are far apart. Once again the margins are modeled by shifted, scaled t-distributions. As expected the Archimedean copulas are not able to model this portfolio appropriately. The assumption of multivariate normality, the Gaussian copula and the t-copula cannot be rejected based on the number of violations.

3. Bonds

In this section two portfolios with bonds are constructed (daily returns).

The first one consists of ten bonds (AES, AT&T, Boeing, Enron, Lucent, Petroleum Geoservices, General Motors, Coca Cola, Dell, and Johnson & Johnson), several of which have dropped in price during the data period (e.g., Enron). All estimates are made in a 'rolling' framework for the bonds. The problem with modeling these bonds, is that they can become illiquid all of a sudden, and then the margins are difficult to model. It turns out that the number of observations to use for the estimation of the margins should not be too large. Taking a too long period for estimation of the margins causes the VaR estimates to become very 'static'. Estimating the margins by shifted, scaled t-distributions does not give desirable results, due to the illiquidity problem.

The second portfolio is a subset of the first one (Boeing, General Motors, Coca Cola, Dell, and Johnson & Johnson), and is selected in such a way that the margins are easier to model. Taking Gaussians as marginal distributions results in not rejecting any of the VaR estimates. But for both the Gaussian and the t-copula it looks as if the estimates for the 99% VaR are low. Changing the margins into shifted, scaled t-distributions indeed results in higher 99% VaR estimates, but this time the 95% VaR estimates are rejected.

Another general conclusion from the applications is, that due to the fact that we estimate the margins using all observations with equal weight, a single outlier will have a long lasting influence on our VaR estimates.

So now the question: "Why use copulas?". The answer is that it gives more insight in the dependence structure and it is a very flexible way of creating multivariate distribution functions. The question: "Which copula to use?" is much more difficult, and this thesis only provides the intuition for choosing between the Gaussian copula, the t-copula, and a few Archimedean copulas.
Basically of course, this depends on the problem that we are dealing with. But some intuition can be given. An asymmetric dependence structure suggests that the class of Archimedean copulas is certainly worth looking at. If on the other hand the dependence structure that we want to model is (almost) symmetric, then the advantage of having one parameter per bivariate margin for the Gaussian and $t$-copulas should be recognized. In the literature some people have suggested (O’Kane and Schloegl 2002 and Mashal et al. 2003) that in a lot of situations the Gaussian copula can be rejected in favor of the $t$-copula. Looking at the out-of-sample tests (currency portfolio and the bond portfolios) in this thesis it was only possible once to reject the Gaussian copula model when the $t$-copula model could not be rejected. This was the case for the currency portfolio.

Suggestions for further Research

Since 1999 copulas have become a popular topic for financial research. New research is published almost on a daily basis. Giving an answer to the question: ”What are the most interesting or promising branches of newly developed copula theory?” is therefore a sheer impossible task.

The three most important follow-ups to the research carried out in this thesis are:

1. Finding an accurate, and preferably fast, simulation algorithm for the Archimedean copulas. The algorithm presented in this thesis for simulation from Archimedean copulas with the number of dimensions larger than two is inaccurate. The two most straightforward approaches would be by either using the conditional sampling method, or by using the method in Marshall and Olkin 1988. For software that is not able to take derivatives analytically it might be possible to use numerical derivatives in the conditional sampling method. If the method from Marshall and Olkin 1988 is used, then one has to find a way to simulate a variable for which only its characteristic function is known.

2. A problem to be solved is how to model the marginal distributions for certain variables, see for example Section 8.3 about bond prices.

3. In this thesis we constantly fit a whole distribution, but we actually use the estimates to find the Value at Risk for portfolios. Maybe we could try to use a weighted estimation procedure, putting more weight on observations in the lower tail.

Two important, more advanced, topics that can be found in the literature about copula theory:

- Time-dependent copulas as seen in Patton 2001.
Conclusions

- The introduction of the class of generalized Archimedean copulas in Rogge and Schönbucher 2002. In this paper the class of Archimedean copulas is generalized in such a way that the bivariate margins are not identically any more. Note that before using this class one first has to make sure that an accurate simulation algorithm for the Archimedean copulas is present.
A 3D-graphs and Contour Plots

Figure A1: 3-D graph of the bivariate $C^+$ copula.

Figure A2: 3-D graph of the bivariate $C^-$ copula.
Figure A3: Contour plot of the bivariate $C^+$ copula.

Figure A4: Contour plot of the bivariate $C^-$ copula.
Figure A5: 3-D graph of the bivariate $C^\perp$ copula.

Figure A6: Contour plot of the bivariate $C^\perp$ copula.
Figure A7a: Contour plot of the density for a bivariate Gaussian copula with $\rho = 0.2$.

Figure A7b: Contour plot of the density for a bivariate Gaussian copula with $\rho = 0.5$. 
Figure A7c: Contour plot of the density for a bivariate Gaussian copula with $\rho = 0.8$.

Figure A8a: Contour plot of the density for a bivariate $t_8$ copula with $\rho = 0.2$. 
Figure A8b: Contour plot of the density for a bivariate $t_8$ copula with $\rho = 0.5$.

Figure A8c: Contour plot of the density for a bivariate $t_8$ copula with $\rho = 0.8$. 
Figure A9a: Density of the CML-estimator for ρ in an elliptical copula. $T = 50$ with 1000 replications.
Data: Gaussian. Estimation: Gaussian.

Data: t\(10\). Estimation: t.

Data: t\(10\). Estimation: Gaussian.

Figure A9b: Density of the CML-estimator for \(\rho\) in an elliptical copula. \(T = 100\) with 1000 replications.
Figure A9c: Density of the CML-estimator for $\rho$ in an elliptical copula. $T = 200$ with 1000 replications.
Figure A9d: Density of the CML-estimator for $\rho$ in an elliptical copula. $T = 500$ with 1000 replications.
Figure A10a: Density of the transformed Kendall’s tau estimator for $\rho$ in an elliptical copula. $T = 50$ with 1000 replications.

Figure A10b: Density of the transformed Kendall’s tau estimator for $\rho$ in an elliptical copula. $T = 100$ with 1000 replications.
Figure A10c: Density of the transformed Kendall’s tau estimator for $\rho$ in an elliptical copula. $T = 200$ with 1000 replications.

Figure A10d: Density of the transformed Kendall’s tau estimator for $\rho$ in an elliptical copula. $T = 500$ with 1000 replications.
Appendices

Figure A11a: Gaussian copula with two standardnormal margins.

Figure A11b: Gaussian copula with two $t_8$ margins.
Figure A11c: $t_4$ copula with two standardnormal margins.

Figure A11d: $t_4$ copula with two $t_8$ margins.
Figure A11e: $t_4$ copula with one standard normal and one $t_8$ margin.

Figure A12a: Cook and Johnson copula ($\alpha = 1$) with two standard Gaussian margins.
Figure A12b: Ali-Mikhail-Haq copula ($\alpha = 0.99$) with two standard Gaussian margins.

Figure A12c: Gumbel-Hougaard copula ($\alpha = 1.5$) with two standard Gaussian margins.
Appendices

Figure A12d: Frank copula ($\alpha = 3.41$) with two standard Gaussian margins.

Figure A12e: Type I copula ($\alpha = 1$) with two standard Gaussian margins.
Figure A12f: Type II copula ($\alpha = 2.80$) with two standard Gaussian margins.

Figure A12g: Type III copula ($\alpha = 1$) with two standard Gaussian margins.
Figure A13a: Gumbel-Hougaard copula ($\alpha = 1.71$) with two standard Gaussian margins.

Figure A13b: Type I copula ($\alpha = 1.71$) with two standard Gaussian margins.
Figure A13c: Type III copula ($\alpha = 1.71$) with two standard Gaussian margins.

Figure A14: Type I copula ($\alpha = 1$) with one standard Gaussian and one $t_6$ margin.
Figure A15: Scatter plot of the monthly returns on the CSFB and CA Hedge Funds.

Figure A16: Estimated VaR levels and realized losses for the bond portfolio (margins a combination of $t$ and Gaussian).
Figure A17: Estimated VaR levels and realized losses for the bond portfolio (margins restricted $t$).
### B Archimedean Copulas

**Clayton/Cook and Johnson**

<table>
<thead>
<tr>
<th>Generating function</th>
<th>Inverse function</th>
<th>Copula</th>
<th>Domain</th>
<th>Kendall’s tau</th>
<th>Tail dependence</th>
<th>Special cases</th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_\alpha (t) = \frac{1}{\alpha} (t^{-\alpha} - 1) )</td>
<td>( \varphi^{-1}_\alpha (s) = (\alpha s + 1)^{-1/\alpha} )</td>
<td>( C^\text{Cl}<em>\alpha (u_1, \ldots, u_n) = \left( \sum</em>{j=1}^{n} u_j^{-\alpha} - n + 1 \right)^{-1/\alpha} )</td>
<td>( \alpha \in (0, \infty) )</td>
<td>( \tau_{X,Y} = \frac{\alpha}{\alpha + 1} )</td>
<td>( \lambda_u = 0, \lambda_l = 2^{-1/\alpha} )</td>
<td>( \lim_{\alpha \to 0} C^\text{Cl}_\alpha = C^\perp )</td>
<td>positively ordered</td>
</tr>
</tbody>
</table>

**Ali-Mikhail-Haq**

<table>
<thead>
<tr>
<th>Generating function</th>
<th>Inverse function</th>
<th>Copula</th>
<th>Domain</th>
<th>Kendall’s tau</th>
<th>Tail dependence</th>
<th>Special cases</th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_\alpha (t) = \ln \frac{1-\alpha(1-t)}{1-t} )</td>
<td>( \varphi^{-1}_\alpha (s) = \frac{1}{\frac{1}{s}-\alpha} )</td>
<td>( C^\text{AMH}<em>\alpha (u_1, \ldots, u_n) = \frac{(1-\alpha) \prod</em>{j=1}^{n} u_j}{\prod_{j=1}^{n} (1-\alpha(1-u_j))^{-\alpha}} )</td>
<td>( \alpha \in [0, 1) )</td>
<td>( \tau_{X,Y} = 1 - \frac{2^{\alpha+1}}{\alpha(\alpha^2-2\alpha+1)\ln(1-\alpha)} )</td>
<td>( \lambda_u = 0, \lambda_l = 0 )</td>
<td>( C^\text{AMH}_0 = C^\perp )</td>
<td>positively ordered</td>
</tr>
</tbody>
</table>

**Gumbel-Hougaard**

<table>
<thead>
<tr>
<th>Generating function</th>
<th>Inverse function</th>
<th>Copula</th>
<th>Domain</th>
<th>Kendall’s tau</th>
<th>Tail dependence</th>
<th>Special cases</th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi_\alpha (t) = (-\ln t)^\alpha )</td>
<td>( \varphi^{-1}_\alpha (s) = \exp (-s^{1/\alpha}) )</td>
<td>( C^\text{GH}<em>\alpha (u_1, \ldots, u_n) = \exp \left[ - \left( \sum</em>{j=1}^{n} (-\ln u_j)^\alpha \right)^{1/\alpha} \right] )</td>
<td>( \alpha \in [1, \infty) )</td>
<td>( \tau_{X,Y} = 1 - \frac{\alpha}{\alpha^2} )</td>
<td>( \lambda_u = 2 - 2^{1/\alpha}, \lambda_l = 0 )</td>
<td>( C^\text{GH}_1 = C^\perp )</td>
<td>positively ordered</td>
</tr>
</tbody>
</table>

**Ordering**

- **positively ordered**
Frank

Generating function $\varphi_\alpha (t) = - \ln \frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}$

Inverse function $\varphi_\alpha^{-1} (s) = - \frac{1}{\alpha} \ln [e^{-s} (e^{-\alpha} - 1) + 1]$

Copula $C_\alpha^F (u_1, \ldots, u_n) = - \frac{1}{\alpha} \ln \left[ 1 + \frac{1}{e - 1} \prod (e^{-\alpha u_j} - 1) \right]$

Domain $\alpha \in (0, \infty)$

Kendall’s tau $\tau_{X,Y} = 1 - \frac{2}{\alpha} \left[ 1 - D_1 (\alpha) \right]$ where $D_1 (\alpha)$ is the Debye function.

Tail dependence $\lambda_u = 0 \quad \lambda_l = 0$

Special cases

$I$

Generating function $\varphi_\alpha (t) = \left( \frac{1}{\alpha} - 1 \right)^\alpha$

Inverse function $\varphi_\alpha^{-1} (s) = \frac{1}{s^{1/\alpha} + 1}$

Copula $C_\alpha^I (u_1, \ldots, u_n) = \left[ \sum_{j=1}^{n} \left( \frac{1}{u_j} - 1 \right) \right]^{1/\alpha} \left[ 1 + \frac{1}{e - 1} \prod (e^{-\alpha u_j} - 1) \right]^{-1}$

Domain $\alpha \in [1, \infty)$

Kendall’s tau $\tau_{X,Y} = 1 - 2^{1/\alpha}$ where $\tau_{X,Y} \in [\frac{1}{2}, 1)$

Tail dependence $\lambda_u = 2 - 2^{1/\alpha} \quad \lambda_l = 2^{-1/\alpha}$

Special cases $\lim_{\alpha \to \infty} C_\alpha^I = C^+$

Ordering positively ordered

---

$^{39}$ Where $D_k (x)$ is the Debye function, defined for any positive integer $k$ by

$$D_k (x) = \frac{k}{x^k} \int_0^\infty \frac{t^k}{e^t - 1} dt$$
II

Generating function \( \varphi_\alpha(t) = (1 - \ln t)^\alpha - 1 \)
Inverse function \( \varphi_\alpha^{-1}(s) = \exp \left( 1 - (s + 1)^{1/\alpha} \right) \)
Copula \( C_\alpha^{II}(u_1, \ldots, u_n) = \exp \left\{ 1 - \left[ \sum_{j=1}^{n} (1 - \ln u_j)^\alpha - n + 1 \right]^{1/\alpha} \right\} \)
Domain \( \alpha \in [1, \infty) \)
Kendall’s tau \( \tau_{X,Y} : \text{no explicit formula} \)
Tail dependence \( \lambda_u = 0 \quad \lambda_l = 0 \)
Special cases \( C_1^{II} = C^\perp \quad \lim_{\alpha \to \infty} C_\alpha^{II} = C^+ \)
Ordering \( \text{positively ordered} \)

III

Generating function \( \varphi_\alpha(t) = (t^{-1/\alpha} - 1)^\alpha \)
Inverse function \( \varphi_\alpha^{-1}(s) = (s^{1/\alpha} + 1)^{-\alpha} \)
Copula \( C_\alpha^{III}(u_1, \ldots, u_n) = \left[ \sum_{j=1}^{n} (u_j^{-1/\alpha} - 1)^\alpha + 1 \right]^{-\alpha} \)
Domain \( \alpha \in [1, \infty) \)
Kendall’s tau \( \tau_{X,Y} = 1 - \frac{2}{\tau^{2\alpha}} \quad \tau_{X,Y} \in [\frac{1}{3}, 1) \)
Tail dependence \( \lambda_u = 2 - 2^{1/\alpha} \quad \lambda_l = \frac{1}{2} \)
Special cases \( \lim_{\alpha \to \infty} C_\alpha^{III} = C^+ \)
Ordering \( \text{positively ordered} \)
Appendices

# C Tables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>GBP(-1)</td>
<td>-1.006120</td>
<td>0.042274</td>
<td>-23.79972</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(GBP(-1))</td>
<td>0.009773</td>
<td>0.029943</td>
<td>0.326369</td>
<td>0.7442</td>
</tr>
<tr>
<td>C</td>
<td>0.002151</td>
<td>0.014410</td>
<td>0.149298</td>
<td>0.8813</td>
</tr>
</tbody>
</table>

R-Squared       | 0.498107    | Mean dependent var | -0.000249 |
Adjusted R-squared | 0.497207 | S.D. dependent var | 0.679778 |
S.E. of regression        | 0.482016   | Akaike info criterion | 1.380999 |
Sum squared resid         | 259.2909   | Schwarz criterion | 1.39458 |
Log likelihood             | -769.6690  | F-statistic | 553.7899 |
Durbin-Watson stat        | 1.999870   | Prob(F-statistic) | 0.000000 |

Table C1: Dickey Fuller test for the return on the British Pound.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEM(-1)</td>
<td>-0.997756</td>
<td>0.041946</td>
<td>-23.78645</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(DEM(-1))</td>
<td>0.011901</td>
<td>0.029893</td>
<td>0.398119</td>
<td>0.6906</td>
</tr>
<tr>
<td>C</td>
<td>0.002375</td>
<td>0.019168</td>
<td>0.123885</td>
<td>0.9014</td>
</tr>
</tbody>
</table>

R-Squared       | 0.493671    | Mean dependent var | 0.000724 |
Adjusted R-squared | 0.492764 | S.D. dependent var | 0.900279 |
S.E. of regression        | 0.641183   | Akaike info criterion | 1.951675 |
Sum squared resid         | 458.8057   | Schwarz criterion | 1.965134 |
Log likelihood             | -1088.962  | F-statistic | 544.0512 |
Durbin-Watson stat        | 1.997712   | Prob(F-statistic) | 0.000000 |

Table C2: Dickey Fuller test for the return on the German Mark.
ADF Test Statistic -24.09890  1% Critical Value -3.4390
10% Critical Value -2.5684

MacKinnon critical values for rejection of hypothesis of a unit root

Augmented Dickey-Fuller Test Equation
Included observations: 1119 after adjusting endpoints

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>JPY(-1)</td>
<td>-1.033345</td>
<td>0.042879</td>
<td>-24.09890</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(JPY(-1))</td>
<td>0.005268</td>
<td>0.029855</td>
<td>0.176442</td>
<td>0.8600</td>
</tr>
<tr>
<td>C</td>
<td>-0.002469</td>
<td>0.020260</td>
<td>-0.121880</td>
<td>0.9030</td>
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</table>

R-Squared 0.514919  Mean dependent var 0.001030
Adjusted R-squared 0.514050  S.D. dependent var 0.972185
S.E. of regression 0.677711  Akaike info criterion 2.062487
Sum squared resid 512.5707  Schwarz criterion 2.075946
Log likelihood -1150.961  F-statistic 592.3292
Durbin-Watson stat 1.993489  Prob(F-statistic) 0.000000

Table C3: Dickey Fuller test for the return on the Japanese Yen.

ADF Test Statistic -23.54957  1% Critical Value -3.4390
10% Critical Value -2.5684

MacKinnon critical values for rejection of hypothesis of a unit root

Augmented Dickey-Fuller Test Equation
Included observations: 1119 after adjusting endpoints

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHF(-1)</td>
<td>-1.008617</td>
<td>0.042830</td>
<td>-23.54957</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(CHF(-1))</td>
<td>-0.018137</td>
<td>0.029901</td>
<td>-0.606566</td>
<td>0.5443</td>
</tr>
<tr>
<td>C</td>
<td>0.006596</td>
<td>0.019779</td>
<td>0.333468</td>
<td>0.7388</td>
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</table>

R-Squared 0.514911  Mean dependent var 0.001077
Adjusted R-squared 0.514041  S.D. dependent var 0.949034
S.E. of regression 0.677718  Akaike info criterion 2.014300
Sum squared resid 488.4573  Schwarz criterion 2.027559
Log likelihood -1124.001  F-statistic 592.3038
Durbin-Watson stat 1.998655  Prob(F-statistic) 0.000000

Table C4: Dickey Fuller test for the return on the Swiss Franc.
### ADF Test Statistic

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSFB(-1)</td>
<td>-0.865264</td>
<td>0.128802</td>
<td>-6.717774</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(CSFB(-1))</td>
<td>-0.026776</td>
<td>0.097704</td>
<td>-0.274054</td>
<td>0.7846</td>
</tr>
<tr>
<td>C</td>
<td>0.754432</td>
<td>0.268417</td>
<td>2.810668</td>
<td>0.0059</td>
</tr>
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</table>

R-Squared: 0.445247
Adjusted R-squared: 0.434974
S.E. of regression: 2.541334
Sum squared resid: 697.5046
Log likelihood: -259.5100
F-statistic: 43.34061
Durbin-Watson stat: 1.995322
Prob(F-statistic): 0.000000

Table C5: Dickey Fuller test for the return on the CSFB/Tremont Hedge Fund Index.

### ADF Test Statistic

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CA(-1)</td>
<td>-0.363253</td>
<td>0.088308</td>
<td>-4.113468</td>
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</tr>
<tr>
<td>D(CA(-1))</td>
<td>-0.161240</td>
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<td>-1.698985</td>
<td>0.0922</td>
</tr>
<tr>
<td>C</td>
<td>0.300542</td>
<td>0.133361</td>
<td>2.253592</td>
<td>0.0262</td>
</tr>
</tbody>
</table>

R-Squared: 0.236955
Adjusted R-squared: 0.222824
S.E. of regression: 1.150141
Sum squared resid: 142.8649
Log likelihood: -171.5087
F-statistic: 16.76905
Durbin-Watson stat: 1.935771
Prob(F-statistic): 0.000000

Table C6: Dickey Fuller test for the return on the Convertible Arbitrage.
### Table C7: Dickey Fuller test for the return on the Equity Neutral hedge fund index.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>EN(-1)</td>
<td>-0.610738</td>
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<td>-5.341412</td>
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</tr>
<tr>
<td>D(EN(-1))</td>
<td>-0.117963</td>
<td>0.096362</td>
<td>-1.224176</td>
<td>0.2235</td>
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<tr>
<td>C</td>
<td>0.525588</td>
<td>0.130422</td>
<td>4.029903</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

MacKinnon critical values for rejection of hypothesis of a unit root

Augmented Dickey-Fuller Test Equation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>RA(-1)</td>
<td>-0.810123</td>
<td>0.114908</td>
<td>-7.050160</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(RA(-1))</td>
<td>0.107251</td>
<td>0.095068</td>
<td>1.128151</td>
<td>0.2618</td>
</tr>
<tr>
<td>C</td>
<td>0.527415</td>
<td>0.143691</td>
<td>3.670485</td>
<td>0.0004</td>
</tr>
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</table>

R-Squared 0.375551 Mean dependent var -0.008559
Adjusted R-squared 0.362969 S.D. dependent var 1.599922
S.E. of regression 1.276965 Akaike info criterion 3.353505
Sum squared resid 176.1091 Schwarz criterion 3.426735
Log likelihood -183.1195 F-statistic 32.33798
Durbin-Watson stat 2.009227 Prob(F-statistic) 0.000000

### Table C8: Dickey Fuller test for the return on the Risk Arbitrage hedge fund index.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>RA(-1)</td>
<td>-0.810123</td>
<td>0.114908</td>
<td>-7.050160</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(RA(-1))</td>
<td>0.107251</td>
<td>0.095068</td>
<td>1.128151</td>
<td>0.2618</td>
</tr>
<tr>
<td>C</td>
<td>0.527415</td>
<td>0.143691</td>
<td>3.670485</td>
<td>0.0004</td>
</tr>
</tbody>
</table>

R-Squared 0.375551 Mean dependent var -0.008559
Adjusted R-squared 0.362969 S.D. dependent var 1.599922
S.E. of regression 1.276965 Akaike info criterion 3.353505
Sum squared resid 176.1091 Schwarz criterion 3.426735
Log likelihood -183.1195 F-statistic 32.33798
Durbin-Watson stat 2.009227 Prob(F-statistic) 0.000000
### Table C9: Dickey Fuller test for the return on the Dedicated Short hedge fund index.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS(-1)</td>
<td>-1.007469</td>
<td>0.129328</td>
<td>-7.790058</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(DS(-1))</td>
<td>0.077953</td>
<td>0.094975</td>
<td>0.820775</td>
<td>0.4136</td>
</tr>
<tr>
<td>C</td>
<td>0.163993</td>
<td>0.500354</td>
<td>0.327754</td>
<td>0.7437</td>
</tr>
</tbody>
</table>

| MacKinnon critical values for rejection of hypothesis of a unit root |

Augmented Dickey-Fuller Test Equation  
Included observations: 111 after adjusting endpoints

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSE(-1)</td>
<td>-0.813337</td>
<td>0.124470</td>
<td>-6.534386</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(LSE(-1))</td>
<td>-0.032568</td>
<td>0.096346</td>
<td>-0.338031</td>
<td>0.7360</td>
</tr>
<tr>
<td>C</td>
<td>0.767242</td>
<td>0.333754</td>
<td>2.298827</td>
<td>0.0234</td>
</tr>
</tbody>
</table>

### Table C10: Dickey Fuller test for the return on the Long Short Equity hedge fund index.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSE(-1)</td>
<td>-0.813337</td>
<td>0.124470</td>
<td>-6.534386</td>
<td>0.0000</td>
</tr>
<tr>
<td>D(LSE(-1))</td>
<td>-0.032568</td>
<td>0.096346</td>
<td>-0.338031</td>
<td>0.7360</td>
</tr>
<tr>
<td>C</td>
<td>0.767242</td>
<td>0.333754</td>
<td>2.298827</td>
<td>0.0234</td>
</tr>
</tbody>
</table>

| MacKinnon critical values for rejection of hypothesis of a unit root |

Augmented Dickey-Fuller Test Equation  
Included observations: 111 after adjusting endpoints

- ADF Test Statistic: -7.790058  
  - 1% Critical Value: -3.4900  
  - 5% Critical Value: -2.8874  
  - 10% Critical Value: -2.5804

- ADF Test Statistic: -6.534386  
  - 1% Critical Value: -3.4900  
  - 5% Critical Value: -2.8874  
  - 10% Critical Value: -2.5804
### Table C11: Descriptive statistics for the bonds in the bond portfolio.

<table>
<thead>
<tr>
<th>Bond</th>
<th>Source</th>
<th>Mean (in %)</th>
<th>Median (in %)</th>
<th>Maximum (in %)</th>
<th>Minimum (in %)</th>
<th>St. Dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>AES</td>
<td>Bloomberg 00130HAD</td>
<td>-0.05</td>
<td>0.00</td>
<td>1.56</td>
<td>-19.05</td>
<td>0.89</td>
<td>-18.64</td>
<td>372.29</td>
<td>3.6E06</td>
<td>0.00</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>Bloomberg 001957AS</td>
<td>0.01</td>
<td>0.00</td>
<td>4.92</td>
<td>-3.38</td>
<td>0.49</td>
<td>2.44</td>
<td>35.54</td>
<td>2.8E04</td>
<td>0.00</td>
</tr>
<tr>
<td>Boeing</td>
<td>Datastream 610196</td>
<td>0.02</td>
<td>0.05</td>
<td>5.12</td>
<td>-6.44</td>
<td>0.91</td>
<td>-0.64</td>
<td>13.27</td>
<td>2.8E03</td>
<td>0.00</td>
</tr>
<tr>
<td>Enron</td>
<td>Datastream 251992</td>
<td>-0.22</td>
<td>0.00</td>
<td>28.89</td>
<td>-36.19</td>
<td>3.76</td>
<td>-2.11</td>
<td>31.65</td>
<td>2.2E04</td>
<td>0.00</td>
</tr>
<tr>
<td>Lucent</td>
<td>Datastream 243466</td>
<td>0.00</td>
<td>0.00</td>
<td>9.76</td>
<td>-18.18</td>
<td>2.16</td>
<td>-1.52</td>
<td>17.28</td>
<td>5.6E03</td>
<td>0.00</td>
</tr>
<tr>
<td>Petroleum Geos.</td>
<td>Datastream 816426</td>
<td>0.00</td>
<td>0.00</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>General Motors</td>
<td>Datastream 241302</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Coca Cola</td>
<td>Datastream 245535</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Dell</td>
<td>Datastream 246344</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Johnson &amp; J.</td>
<td>Datastream 241302</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**Notes:**
- **Mean:** Average of the bond rates.
- **Median:** Middle value of the bond rates.
- **Maximum:** Highest bond rate.
- **Minimum:** Lowest bond rate.
- **St. Dev.:** Standard deviation.
- **Skewness:** Measure of the asymmetry of the bond rates.
- **Kurtosis:** Measure of the peakedness of the bond rates.
- **Jarque-Bera:** Test for normality.
- **Probability:** Probability of normality.

### Table C12: Sources bond prices.

<table>
<thead>
<tr>
<th>Bond</th>
<th>Source</th>
<th>bond rate</th>
<th>expiration</th>
<th>Source ID</th>
</tr>
</thead>
<tbody>
<tr>
<td>AES</td>
<td>Bloomberg 00130HAD</td>
<td>10 ½%</td>
<td>July 15th 2006</td>
<td>Bloomberg 00130HAD</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>Bloomberg 001957AS</td>
<td>7%</td>
<td>May 15th 2005</td>
<td>Bloomberg 001957AS</td>
</tr>
<tr>
<td>Boeing</td>
<td>Datastream 610196</td>
<td>7 ½%</td>
<td>June 16th 2025</td>
<td>Datastream 610196</td>
</tr>
<tr>
<td>Enron</td>
<td>Datastream 251992</td>
<td>6.95%</td>
<td>July 15th 2028</td>
<td>Datastream 251992</td>
</tr>
<tr>
<td>Lucent</td>
<td>Datastream 243466</td>
<td>7 ½%</td>
<td>July 15th 2006</td>
<td>Datastream 243466</td>
</tr>
<tr>
<td>Petroleum Geos.</td>
<td>Datastream 816426</td>
<td>7 ½%</td>
<td>March 31st 2007</td>
<td>Datastream 816426</td>
</tr>
<tr>
<td>General Motors</td>
<td>Datastream 600630</td>
<td>6 ½%</td>
<td>January 5th 2005</td>
<td>Datastream 600630</td>
</tr>
<tr>
<td>Coca Cola</td>
<td>Datastream 245535</td>
<td>6 ½%</td>
<td>August 1st 2004</td>
<td>Datastream 245535</td>
</tr>
<tr>
<td>Dell</td>
<td>Datastream 246344</td>
<td>6.55%</td>
<td>April 15th 2008</td>
<td>Datastream 246344</td>
</tr>
<tr>
<td>Johnson &amp; J.</td>
<td>Datastream 241302</td>
<td>6.73%</td>
<td>November 15th 2023</td>
<td>Datastream 241302</td>
</tr>
</tbody>
</table>
D Description Programs

D.1 CMLArchimedean.ox

This program gives the CML estimates for the Archimedean copulas. It uses the 7 programs:

1. CMLCandJ.ox, which estimates the parameter in a Cook and Johnson copula,
2. CMLAMH.ox, which estimates the parameter in an Ali-Mikhail-Haq copula,
3. CMLGH.ox, which estimates the parameter in a Gumbel-Hougaard copula,
4. CMLF.ox, which estimates the parameter in a Frank copula,
5. CMLI.ox, which estimates the parameter in a Type I copula,
6. CMLII.ox, which estimates the parameter in a Type II copula, and
7. CMLIII.ox, which estimates the parameter in a Type III copula.

For each of the 7 Archimedean copulas the weighted CML estimate is printed. Also for each copula a matrix is printed that displays the estimator per bivariate margin, the corresponding log-likelihood value, and the number of observations that could be used in the estimation procedure per bivariate margin. A warning is displayed whenever the estimated weighted CML estimate for a copula is very close to a boundary value.

D.2 CMLElliptical.ox

This program gives estimates for the matrix $\Sigma$ in a Gaussian and $t$-copula and for $\nu$, the number of degrees of freedom in a $t$-copula. The program uses a point estimation matrix of transformed Kendall’s tau estimates. If it is not positive semi-definite it will be transformed using the eigenvalue method.

D.3 VaRArchimedean.ox

This program gives estimates for the 95% and 99% VaR and CVaR levels assuming that the return process is driven by an Archimedean copula. In the simulation procedure the following programs are used:

1. RandCandJFast.ox, which simulates data from a Cook and Johnson copula,
2. RandAMH.ox, which simulates data from an Ali-Mikhail-Haq copula,
3. RandGH.ox, which simulates data from a Gumbel-Hougaard copula,
4. RandF.ox, which simulates data from a Frank copula,
5. RandI.ox, which simulates data from a Type I copula,
6. RandII.ox, which simulates data from a Type II copula, and
7. RandIII.ox, which simulates data from a Type III copula.

The marginal distributions are modeled by a shifted and scaled $t$-distribution. Note that all parameters have to be given explicitly, they are not estimated within this program. This is done because in most cases only one or two Archimedean copulas are relevant to consider. The estimates for the margins can be found using the program "Marginalt.ox" and the Archimedean copula parameter can be estimated by using the program "CMLArchimedean.ox".

D.4 VaRElliptical.ox
This program gives estimates for the 95% and 99% VaR and CVaR levels assuming that the return process is driven by a Gaussian or a $t$-copula. In the simulation procedure the following programs are used:

1. RandGauss.ox, which simulates data from a Gaussian copula, and
2. Randt.ox, which simulates data from a $t$-copula.

The marginals are estimated by shifted and scaled $t$-distributions. The estimates for the marginals can be found by using the program "Marginalt.ox". Note that the copula parameters are estimated within this program, so there is no need to use "CMLElliptical.ox".

Also printed are the estimated VaR levels assuming multivariate normality for the returns. These estimations are simply done by taking a Gaussian copula and assuming all margins to be univariate Gaussian.

D.5 VaREllipticalRolling.ox
This program gives rolling estimates for the 95% and 99% VaR and CVaR level. The user can specify the number of observations that will be used to estimate the copula parameters. The number of observations that will be used each time to estimate the margin parameters can also be specified. The margins will be estimated by scaled and shifted $t$-distributions. If that is not possible, due to illiquidity of the return series, then a Gaussian distribution will be used.

After each estimate it is checked whether or not next day’s loss was larger than the estimated VaR levels. For each day the VaR and CVaR estimates are printed, as well as the number and percentage of violations for each estimated VaR level.
D.6 Marginalt.ox

This program gives estimates for the parameters in a shifted and skewed \( t \)-distribution. Due to the numerical optimization of the log-likelihood function this procedure may sometimes become instable when the data is of very small order of magnitude. Therefore the data can be scaled before starting the estimation procedure. Of course the final estimates are scaled back again.

Two starting values for the degrees of freedom parameter are chosen, 5 and 50, and it is tested whether the final estimates for the number of degrees of freedom, the shift parameter and the scale parameter are close.

D.7 Sim1.ox

This program tests the performance of the CML estimator for the correlation parameter in a bivariate Gaussian and \( t \)-copula. The density of the estimates is shown.

D.8 Sim2.ox

This program tests the performance of the transformed Kendall’s tau estimator for the correlation parameter in a Gaussian and \( t \)-copula.
References


