Thierry Ané
University of Lausanne

Cécile Kharoubi
University Paris Dauphine

Dependence Structure and Risk Measure*

Understanding the relationships among multivariate assets would help one greatly about how best to position one's investments and enhance one's financial risk protection. We present a new method to model parametrically the dependence structure of stock index returns through a continuous distribution function, which links an *n*-dimensional density to its one-dimensional margins. The resulting multivariate model could be used in a wide range of financial applications. Focusing on risk management, we show that a misspecification of the dependence structure introduces, on average, an error in Value-at-Risk estimates.

I. Introduction

The regulatory environment and the need for controlling risk in the financial community have provided incentives for banks to develop proprietary risk measurement models. To allow an optimal capital allocation, banks are urged to build sound interval measures of market risks for all their activities. The challenge lies in the difficulty of having a complete picture of risks (or assets) joint distribution. Although the understanding of the relationships among multivariate outcomes would help one greatly on how best to position one's investments and enhance one's financial risk protection, the modeling of joint distributions remains a difficult problem in statistical science.

Not surprisingly, considering its mathematical tractability, the normal distribution has long dominated the multivariate analysis. Multivariate normal distributions are appealing because (*a*) the marginal distributions are normal, as in almost all leading financial models, and (*b*) the association between any two ran-

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dom outcomes can be fully described knowing only the marginal distributions and one additional parameter, the correlation coefficient. The assumption of multivariate normal returns lies at the heart of the modern portfolio theory that establishes the variance as the risk measure and the correlation between asset returns as the measure of dependence that determines portfolio diversification and hedging performance.

In response to the existence of skewness and leptokurtosis in individual asset returns, however, an impressive body of recent literature has investigated new probabilistic models for univariate returns. However, one has to recognize the paucity of distributions that can be used as an aid for modeling the structure of multivariate data.

Historically, most of the alternative multivariate distributions have been developed as immediate extensions of their univariate counterparts. This approach suffers from many drawbacks. In particular, it requires that the marginal distributions of the multivariate vector under consideration all belong to the same (univariate) distribution family. Another important theoretical drawback is that the measures of dependence (i.e., the parameters characterizing the dependence between the components of the multivariate vector) often appear in the marginal distributions. For financial modeling purposes, the analyst will need more flexibility in specifying the joint distribution than is allowed by these well-known functional forms. In particular, it should be possible to evaluate the marginal distributions separately. From an economic viewpoint, it will give the analyst the opportunity to use different marginal distributions to account for the diversity in financial risks (or assets). From a modeling viewpoint, the lower the dimensionality of a model or the cardinality of its parameters, the higher the reliability of the estimates. In this sense, any approach that could decompose the multivariate estimation procedure into the separate specification of the marginal behaviors and the functional form of the dependence between the risks would effectively increase the reliability of the estimation process. The problem is, then, to define a measure of association between the financial risks consistent with this framework.

In this article, we present a new approach to the construction of multivariate densities that overcomes the limitations of the traditional multivariate models and brings the flexibility necessary to risk management models. The method relies on the copula function. A copula is a function that links an n-dimensional distribution function to its one-dimensional margins and is itself a continuous distribution function characterizing the model's dependence structure. The approach is designed specifically to use subjective judgments of marginal distributions, leaving all the information about the dependence structure, as represented by the copula function, to be estimated separately. In order not to distract the reader's attention with the estimation of marginal distributions, we adopt a nonparametric approach to the modeling of margins and focus on the problem of parametric estimation for the dependence structure.

The technique should be used in a wide range of financial applications. We apply our methodology to investigating the dependence structure in international stock market indices and illustrate its utility for risk measure calculations. Despite its numerous detractors, Value-at-Risk (VaR) remains the most widely used summary measure of risk. We show that it allows us to take into account various kinds of cross-dependence between asset returns, as well as fat-tail and non-normality effects, when it is used with a "coherent multivariate distribution." In performing Monte Carlo VaR calculations, we prove (*a*) that copula functions provide easily simulated multivariate distributions and show (*b*) the importance of a well-specified dependence structure in risk measurements.

The remainder of the article is organized as follows: Section II introduces the copulas and discusses some properties relevant to financial modeling. Section III is dedicated to the empirical analysis of the dependence structure of six international stock market indices. In Section IV, we explain how to select a parametric form for the copula function. Section V presents an application to risk management with VaR estimates. Section VI contains the concluding remarks and directions for future research.

II. Copula Representation of Multivariate Distributions

A. Definitions and Basic Properties of Copula Functions

Although seldom used in finance and economics, copula functions have been extensively studied in recent years. Joe (1997) and Nelsen (1999) present a comprehensive discussion of their mathematical properties. In this section, we only describe the basic properties useful to our study and show that copula functions provide a simple and convenient answer to the problem of specifying a joint distribution with given margins while controlling for the dependence structure.

A copula function $C(u_1, u_2, \dots, u_n)$ is defined as a cumulative distribution function for a multivariate vector with support in $[0, 1]^n$ and uniform marginals. Denoting $(U_1, U_2, \dots, U_n)$ the corresponding multivariate vector, the copula function is defined as

$$C(u_1, u_2, \dots, u_n) = P(U_1 \leq u_1, U_2 \leq u_2, \dots, U_n \leq u_n). \tag{1}$$

If we now select arbitrary marginal distribution functions,

$$F_i(x_i) = P(X_i \leq x_i) \text{ for } 1 \leq i \leq n, \tag{2}$$

and use the transformations $U_i = F_i(X_i)$, one can easily check that the function

$$F(x_1, x_2, \dots, x_n) = C[F_1(x_1), F_2(x_2), \dots, F_n(x_n)] \tag{3}$$

defines a new multivariate distribution, evaluated at $x_1, x_2, \dots, x_n$, with marginals $F_i, i = 1, 2, \dots, n$. Sklar (1973) shows the converse, namely, that any multivariate distribution function F can be written in terms of its marginals using a copula representation, as in equation (3). Moreover, if we further assume that the marginal distributions F_i are all continuous, then F has a unique copula C.[1]

The definition of the copula function in equation (3) is given in terms of cumulative distribution functions. If we further assume that each F_i and C are differentiable, the joint density $f(x_1, x_2, \ldots, x_n)$ takes the form

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) \times f_2(x_2) \times \cdots \times f_n(x_n) \times c[F_1(x_1), F_2(x_2), \ldots, F_n(x_n)], \quad (4)$$

where $f_i(x_i)$ is the density corresponding to F_i, and where

$$c(u_1, u_2, \ldots, u_n) = \frac{\partial^n C(u_1, u_2, \ldots, u_n)}{\partial u_1 \partial u_2 \ldots \partial u_n} \quad (5)$$

is the density of the copula. By contrast to the traditional modeling approach that decomposes the joint density as a product of marginal and conditional densities, equation (4) states that, under appropriate conditions, the joint density can be written as a product of the marginal densities and the copula density. From equation (4), it is clear that the density $C(u_1, u_2, \ldots, u_n)$ encodes information about the dependence structure among the X_i's while the f_i's describe the marginal behaviors. It thus shows that copulas represent a way of trying to extract the dependence structure from the joint distribution and to extricate the dependence and marginal behaviors.

B. Classical Copula Functions and Their Asymptotic Behavior

Although many copula families are available, flexibility and analytical tractability represent two fundamental features in the search for an appropriate parametric form for the dependence structure. Table 1 provides a brief description of the four families most commonly used in biostatistics, actuarial science, or even management science.

Li (2000) introduces the Gaussian copula in finance to calibrate default correlation. It is called the Gaussian copula because it encodes dependence in precisely the same way as the normal distribution does, using only pairwise correlations among the variables, but it does so for variables with arbitrary marginals. In univariate analysis, the search for an appropriate distribution often starts with the assessment of the departure from normality of the empirical asset returns. In the same way, the Gaussian copula will serve as a benchmark to test whether the dependence structure in index returns behaves as the Gaussian framework implies.

In addition to this benchmark dependence structure, we investigate the Gumbel (1960), Frank (1979), and Cook-Johnson (1981) copulas. Among their numerous applications, let us mention their use in Yi and Bier (1998)

1. Note that in the case of independent random variables, $F(x_1, x_2, \ldots, x_n) = F_1(x_1) \times F_2(x_2) \times \cdots \times F_n(x_n)$. Hence, the copula characterizing independence is simply $C(u_1, u_2, \ldots, u_n) = u_1 \times u_2 \times \cdots \times u_n$.

TABLE 1 Four Parametric Copula Families

Family	Copula $C(u_1, u_2, \dots, u_n)$	Comments
Frank	$C(u_1, u_2, \dots, u_n) = -\frac{1}{\alpha}\ln\left[1 + \frac{(e^{-\alpha u_1} - 1)(e^{-\alpha u_2} - 1)\cdots(e^{-\alpha u_n} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right]$	$\alpha > 0$
Gumbel	$C(u_1, u_2, \dots, u_n) = \exp\left(-\{[\ln(u_1)]^{1/\alpha} + [-\ln(u_2)]^{1/\alpha} + \cdots + [-\ln(u_n)]^{1/\alpha}\}^{\alpha}\right)$	$0 < \alpha \leq 1$
Cook-Johnson	$C(u_1, u_2, \dots, u_n) = (u_1^{-\alpha} + u_2^{-\alpha} + \cdots + u_n^{-\alpha} - n + 1)^{-1/\alpha}$	$\alpha > 0$
Gaussian	$C(u_1, u_2, \dots, u_n) = \Phi_\rho^n[\Phi^{-1}(u_1), \Phi^{-1}(u_2), \dots, \Phi^{-1}(u_n)]$ Φ_ρ^n is an n-variate normal distribution, ρ is the correlation matrix, Φ^{-1} is the inverse of the univariate standard normal distribution	$-1 \leq \alpha \leq 1$

NOTE.—For information on copulas, see Gumbel (1960), Frank (1979), and Cook-Johnson (1981). This table summarizes the different copula families used in this study.

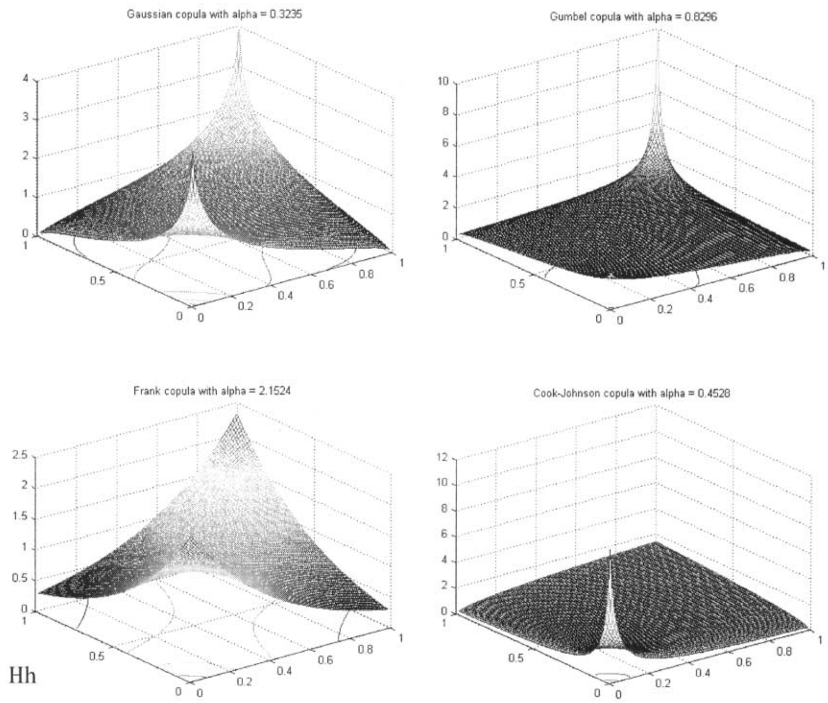

FIG. 1.—Four families of copula density functions. To illustrate the diversity of shapes obtained with the copula families presented in Sec. II, we show here the bivariate copula density functions obtained with the parameters estimated on one of the pairs presented in Sec. III, namely, the FTSE 100–NASDAQ.

for accident precursor analysis as well as in Frees and Valdez (1998) for the estimation of joint life mortality and multiple decrement models. As an illustration of the diversity of shapes obtained with these families, we show in figure 1 the corresponding four bivariate copula density functions with the copula parameters estimated on one of the pairs discussed in Section III, namely, the FTSE 100–NASDAQ.[2]

Recently, much effort has been made to extend the univariate extreme value theory for applications in a multivariate context. A complete picture of assets joint tail behavior during extreme periods will improve hedging performance as well as risk measurements. Poon, Rockinger, and Tawn (2000) empirically examine whether extreme stock market returns exhibit asymptotic dependence or asymptotic independence. When multivariate distributions are represented through a copula function, the ability for large positive and negative returns

2. Let us note that these four bivariate distribution functions have the same linear correlation as well as the same Spearman’s rho and Kendall’s tau, showing the limits of simple scalar measures of dependence.

TABLE 2 **Tail Dependence**

	Upper Tail Dependence (γ_U)	Lower Tail Dependence (γ_L)	Conditions
Gaussian	0	0	Asymptotic independence if and only if $-1 < \rho < 1$
Frank (1979)	0	0	Asymptotic independence
Gumbel (1960)	$2 - 2^{\alpha}$	0	Asymptotic upper dependence if $\alpha < 1$
Cook-Johnson (1981)	0	$2^{-1/\alpha}$	Asymptotic lower dependence if $\alpha > 0$

NOTE.—Table 2 follows Joe (1997) to define the tail dependence of the four copula families. Let us first define $C(u,u) = 1 - 2u + C(u,u)$ as the survival copula. Assume that a bivariate copula C is such that $\lim[C(u,u)]/(1-u) = \lambda_U$ exists when $u \to 1^-$. Then, C is said to have upper tail dependence if $\lambda_U \in (0;1]$, whereas C has no upper tail dependence if $\lambda_U = 0$. An upper tail dependence means that the extreme positive returns are correlated. In a symmetric way, lower tail dependence is tantamount to correlation of extreme negative returns. If a copula is such that $\lim[C(u,u)]/u = \lambda_L$ exists, when $u \to 0^+$, then C has lower tail dependence if $\lambda_L \in (0;1]$ and has no lower tail dependence if $\lambda_L = 0$.

to be correlated is an asymptotic copula property. Table 2 provides a summary of the asymptotic behavior of the copula families introduced in the previous subsection. The Gaussian and Frank (1979) copulas do not exhibit tail dependence. This means, for instance, that even in the case of bivariate normal distribution with a correlation coefficient of 90%, the extreme upper and lower returns are asymptotically independent. On the other hand, the large positive returns are correlated if the dependence structure belongs to the Gumbel (1960) family. Finally, the Cook-Johnson (1981) family presents lower tail dependence, indicating correlation in extreme negative events.

C. Copula Functions and Traditional Dependence Measures

The limits of Pearson's linear correlation as a measure of dependence outside the Gaussian framework have a long history in the financial literature. However, as noted by Embrechts, McNeil, and Straumann (1999), "one does not have to search far in the literature of financial risk management to find misunderstanding and confusion about correlation" wrongly used as a dependence measure outside of its legitimate framework. Indeed, it is known that linear correlation cannot capture the nonlinear dependence relationships exhibited by many financial series. Practitioners are also aware that correlation is not invariant under strictly increasing transformations of the risks. Few people know, however, that the possible values of correlation depend on the marginal distributions of risks (assets). Hence, given marginal distributions F_1 and F_2 for two risks X_1 and X_2, all linear correlations between -1 and $+1$ cannot be attained through a suitable specification of the joint distribution F. It implies that perfectly positively (respectively, negatively) correlated risks do not necessarily have a correlation coefficient of $+1$ (respectively, -1).

In view of the previous shortcomings, some alternative measures of dependence, known as rank correlations, have been introduced in statistics. Unlike Pearson's product-moment correlation, rank-order correlations do not depend on the marginal distributions and avoid many drawbacks of the classical linear measure. The most famous rank correlations include Spearman's rho and Kendall's tau.

Let X_1 and X_2 be random variables with distribution functions F_1 and F_2 and joint distribution F. The Spearman's rank correlation is given by

$$\rho_S(X_1, X_2) = \rho[F_1(X_1), F_2(X_2)], \tag{6}$$

where $\rho(. , .)$ is the usual linear correlation.

If we now let (X_1^a, X_2^a) and (X_1^b, X_2^b) be two independent pairs of random variables from F, the Kendall's tau rank correlation is given by

$$\begin{aligned} \rho_\tau(X_1, X_2) = {} & P[(X_1^a - X_1^b)(X_2^a - X_2^b) > 0] \\ & - P[(X_1^a - X_1^b)(X_2^a - X_2^b) < 0]. \end{aligned} \tag{7}$$

Both ρ_s and ρ_τ are measures of the degree of monotonic dependence between X_1 and X_2, whereas the linear correlation measures the degree of linear dependence. As such, they are invariant under increasing transformations of risks. But the main advantage of rank correlations over a linear measure is undoubtedly the sensible handling of perfect dependence: for arbitrary marginals F_1 and F_2, a bivariate distribution F can be found with any rank correlation in the interval $[-1, +1]$.

Unfortunately, however, the distribution F is not uniquely defined by the marginals and the rank correlation. If we are to find a unique multivariate representation for financial risks (or assets), simple scalar measurements of dependence (whether by linear or rank correlations) are of limited use. It is the purpose of this article to show that the unique multivariate function defining the copula of F conveys all the information about the dependence of risks. Then, once a multivariate distribution has been uniquely specified by its marginals F_1 and F_2 and its copula function $C(u_1, u_2)$, let us note that, being invariant under strictly increasing transformations, the usual Spearman's rho and Kendall's tau can be written in terms of the copula $C(u_1, u_2)$ associated with F as follows:

$$\rho_S(X_1, X_2) = 12 \int_0^1 \int_0^1 [C(u_1, u_2) - u_1 u_2] du_1 du_2, \tag{8}$$

$$\rho_\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1. \tag{9}$$

III. The Behavior of International Index Returns: Empirical Evidence

A. Presentation of the Data

We examine daily index returns over the period January 2, 1987, through December 31, 2000, for five countries: Germany (Dax 30), Hong Kong (Hang Seng), Japan (Nikkei 225), the United Kingdom (FTSE 100), and the United States (S&P 500 and NASDAQ). The usual descriptive statistics of the six series are displayed in the upper part of table 3. The lower part gives the correlation matrix.

We first observe that the indices present very different statistical characteristics. However, exhibiting a significant negative skewness and a kurtosis larger than 3, all series fail to pass the Jarque-Bera normality test. The correlation coefficients also cover a wide range of values revealing the variety of dependence relationships. To investigate the dependence structure further, we display in figure 2 the scatterplots of two pairs of index returns among the 15 pairs that can be formed. On the same graph, we plot the confidence contour for the bivariate Gaussian distribution with the same mean vector and correlation matrix as the pair under consideration. The existence of numerous outliers[3] as well as the departure from the ellipsoid shape are evidence against the use of multivariate Gaussian distributions.

Finally, we use the nonparametric kernel density estimation to obtain the univariate density distributions of the six index returns. The kernel density estimator of the random variable X_i is defined by Silverman (1986):

$$\hat{f}_i(x) = \frac{1}{nh}\sum_{t=1}^{T} K\left(\frac{x - x_i^t}{h}\right), \tag{10}$$

where

$$\left(x_i^t\right)_{t=1}^{T}$$

is the *i*th observed return series. The results presented in this article use a Gaussian kernel, and the optimal bandwidth h is found by likelihood cross-validation. To ease comparison, all the estimated densities are displayed on the same graph. The distributional diversity in index returns revealed by figure 3 is a strong evidence of the impossibility of finding a parametric family rich enough to proxy all the series.

3. Under the null hypothesis of a bivariate Gaussian distribution, one expects observations outside the confidence contour. The actual number of observations outside this range is for the FTSE 100–NASDAQ pair and for the Dax-NASDAQ pair, representing 5.20% and 4.44%, respectively, of the observations. The existence of so many outliers supports the rejection of a bivariate Gaussian model.

TABLE 3 **Descriptive Statistics of the Data**

	Dax 30	FTSE 100	Hang Seng	NASDAQ	Nikkei 225	S&P 500
Descriptive statistics of the six index return series:						
Mean	4.1136E-04	3.5870E-04	4.8498E-04	5.3603E-04	−8.5247E-05	4.6439E-04
SD	1.3076E-02	9.9160E-03	1.8551E-02	1.3293E-02	1.4035E-02	1.0592E-02
Median	3.9476E-04	2.4356E-04	5.7686E-05	9.8321E-04	.0000E + 00	3.2170E-04
Minimum	−1.3710E-01	−1.3029E-01	−4.0542E-01	−1.4002E-01	−1.6135E-01	−2.2833E-01
Maximum	7.2875E-02	7.5970E-02	1.7247E-01	9.9636E-02	1.2430E-01	8.7089E-02
Skewness	−.7942	−1.1061	−3.6234	−.8605	−.1354	−3.0717
Kurtosis	11.8899	18.8664	80.3499	15.7447	12.2775	68.0143
Jarque-Bera	12,409.82	39,051.40	918,405.00	25,166.56	13,108.44	648,930.09
Correlation matrix of the index returns:						
Dax 30	1	.5387	.3720	.3413	.2755	.3182
FTSE 100	.5387	1	.3162	.3892	.2908	.4081
Hang Seng	.3720	.3162	1	.1730	.2928	.1665
NASDAQ	.3413	.3892	.1730	1	.1200	.7611
Nikkei 225	.2755	.2908	.2928	.1200	1	.1022
S&P 500	.3182	.4081	.1665	.7611	.1022	1

NOTE.—Descriptive statistics of the six index return series over the period January 2, 1987–December 31, 2000, are presented here. The number of observations for the period of analysis is $T = 3{,}652$. All the series exhibit both significant skewness and kurtosis. In the same way as the skewness and kurtosis reveal important distributional differences between the indices, the pairwise correlations suggest diversity in the dependence structure of bivariate series.

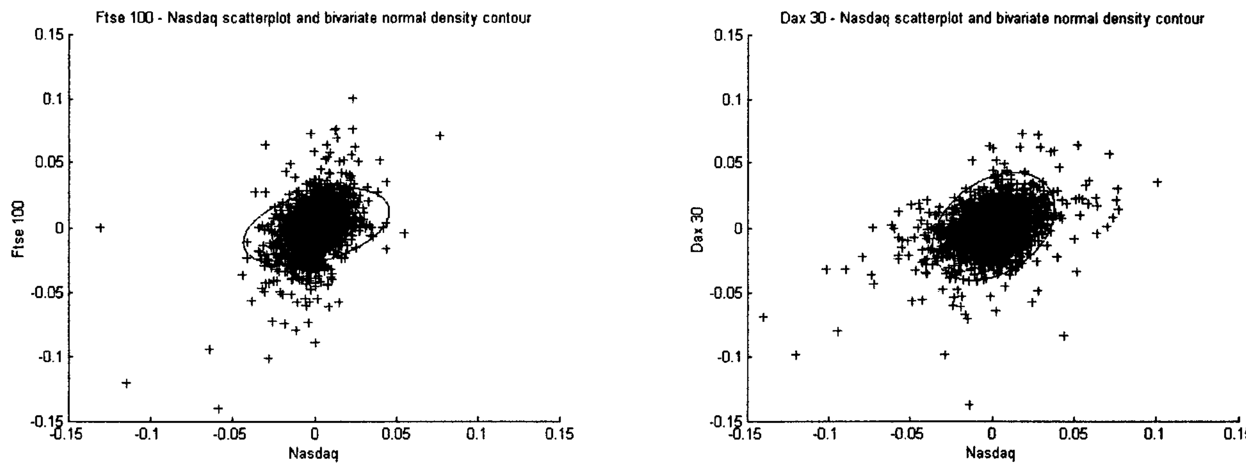

FIG. 2.—Scatterplots obtained for two pairs of index returns among the 15 pairs summarized in table 4. These graphs highlight the complexity of pairwise dependence in our series. The contours in solid line represent the 99% confidence intervals for bivariate Gaussian distributions, with mean vectors and correlation matrices corresponding to the empirical values. The expected number of observations outside the contour is 36 under the null hypothesis of a bivariate Gaussian distribution. The actual number of observations outside the confidence contour is 190 for the FTSE 100–NASDAQ and 162 for the Dax-NASDAQ, respectively, corresponding to a sample proportion of 5.20% and 4.44%. The existence of numerous amounts of outliers supports the rejection of a multivariate Gaussian model.

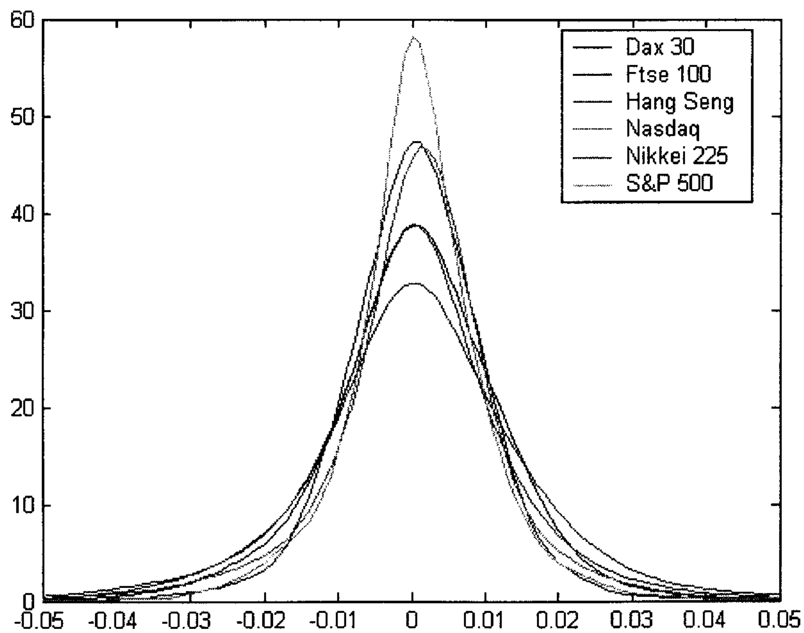

FIG. 3.—Estimated densities of stock index returns. This figure represents the density distributions of the six index returns. They are reconstructed through the kernel method. All distributions are more peaked than the Gaussian one and exhibit different degrees of asymmetry and fat tails. The diversity of shapes betrays the distributional differences between these indices.

B. Log-Likelihood Estimation of the Copula Functions

In this subsection, we explain how the estimation of a parametric copula function can be carried out independently from the estimation of the marginal distributions. We use this approach to calibrate the four copula families on the 15 pairs of index returns.[4] Suppose we assume a parametric form for the marginal distributions of the X_i's and the copula C and denote θ the vector of parameters of the margins and the copula. The calibration of the multivariate model could be obtained by classically maximizing the log-likelihood:

$$l(\theta) = \sum_{t=1}^{T} \ln f(x_1^t, x_2^t, \dots, x_n^t; \theta). \tag{11}$$

4. However, the estimation does not increase in complexity with the dimensionality of the multivariate model. We choose to present bivariate results since they allow simple graphic representations, which help us form an opinion on the impact of the different dependence structures introduced in this article.

Using equation (4), the log-likelihood can also be written

$$l(\theta) = \sum_{t=1}^{T} \ln c[F_1(x_1^t), F_2(x_2^t), \ldots, F_n(x_n^t); \theta] + \sum_{t=1}^{T} \sum_{i=1}^{n} \ln f_i(x_i^t; \theta). \quad (12)$$

Although possible in theory, this direct estimation procedure may prove difficult in practice when the dimensionality of the problem increases. In particular, risk-management applications involve computations on large portfolios depending on numerous risk factors.

Joe and Xu (1996) note, however, that the copula representation splits the parameters into specific parameters for the marginal distributions and common parameters for the dependence structure. Let us denote θ_i the vector of parameters of the marginal distribution of the random variable X_i. The distribution function in equation (2) takes the form $F_i(\theta_i; \theta_i)$, while $c(u_1, u_2, \ldots, u_n; \alpha)$[5] denotes the parametric counterpart of the copula density in equation (5). The maximum likelihood approach reduces to the maximization of the following quantity:

$$l(\theta) = \sum_{t=1}^{T} \ln c[F_1(x_1^t; \theta_1), F_2(x_2^t; \theta_2), \ldots,$$

$$F_n(x_n^t; \theta_n); \alpha] + \sum_{t=1}^{T} \sum_{i=1}^{n} \ln f_i(x_i^t; \theta_i). \quad (13)$$

To this point, using the kernel density estimates $\hat{f}_i$ from equation (10) and a Gaussian quadrature to compute the empirical cumulative distribution functions $\hat{F}_i$, we transform the realizations of the X_i's representing the index returns into realizations of uniform random variables U_i. We thus obtain series

$$\left[\hat{u}_i^t = \hat{F}_i(x_i^t)\right]_{t=1}^{T}$$

for $1 \le i \le n$. The parametric estimation of the four copula functions is then obtained as follows:

$$\hat{\alpha} = \arg\max(\alpha) = \arg\max \sum_{t=1}^{T} \ln c(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n; \alpha). \quad (14)$$

It corresponds to step 2 in the inference functions for the margins method introduced in Joe and Xu (1996).[6] Table 4 gives the parameters obtained for each pair of returns and each copula function together with the standard error of the estimates.

5. Note that in the bivariate case, the correlation matrix of the Gaussian copula contains a single parameter. In what follows, we will denote it α as it conveys the information of the dependence structure in the same way as the parameter α that appears in the other three families.

6. It can be shown that, like the classical maximum likelihood estimate, the vector of parameters $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n, \hat{\alpha})$ obtained with the inference function method verifies the property of asymptotic normality. See Joe and Xu (1996) for the covariance matrix.

TABLE 4 **Estimated Parameters of the Copula Functions**

	Frank	Gaussian	Gumbel	Cook-Johnson
Dax 30–FTSE 100	3.8857	.5626	.6475	.9975
	(.1188)	(.0105)	(.0090)	(.0350)
Dax 30–Hang Seng	2.1743	.3445	.805	.4791
	(.1102)	(.0143)	(.0102)	(.0271)
Dax 30–NASDAQ	1.6333	.2635	.865	.3266
	(.1017)	(.0138)	(.0093)	(.0228)
Dax 30–Nikkei 225	1.5921	.264	.8529	.3515
	(.1037)	(.0151)	(.00102)	(.0250)
Dax 30–S&P 500	1.9431	.3377	.7967	.4803
	(.1150)	(.0160)	(.0109)	(.0295)
FTSE 100–Hang Seng	1.872	.2995	.8369	.4171
	(.1083)	(.0148)	(.0103)	(.0262)
FTSE 100–NASDAQ	2.1524	.3235	.8296	.4528
	(.1020)	(.0130)	(.0930)	(.0248)
FTSE 100–Nikkei 225	1.3223	.2324	.8783	.3147
	(.1017)	(.0140)	(.0100)	(.0244)
FTSE 100–S&P 500	2.658	.4273	.7455	.6694
	(.1136)	(.0138)	(.0102)	(.0312)
Hang Seng–NASDAQ	.7907	.1201	.9446	.157
	(.0944)	(.0135)	(.0080)	(.0182)
Hang Seng–Nikkei 225	1.5988	.2704	.8559	.3475
	(.0976)	(.0137)	(.0093)	(.0229)
Hang Seng–S&P 500	.6238	.1095	.9465	.1535
	(.1054)	(.0170)	(.0100)	(.0221)
NASDAQ–Nikkei 225	.7094	.107	.9509	.1589
	(.0907)	(.0133)	(.0077)	(.0181)
NASDAQ–S&P 500	6.884	.7295	.526	1.5968
	(.1371)	(.0062)	(.0069)	(.0408)
Nikkei 225–S&P 500	.6707	.1117	.9448	.1534
	(.0994)	(.0164)	(.0098)	(.0224)

Note.—For information on copulas, see Gumbel (1960), Frank (1979), and Cook-Johnson (1981). We use the empirical distribution functions $\hat{F}_i$ to transform the return series into uniform variables $[\hat{u}_i^t = \hat{F}_i(x_i^t)]_{t=1}^T$. Through this transformation, the parameter α can be estimated by classical maximum likelihood method as explained in eq. (14). The estimated parameters for each copula function and each pair of returns are summarized in this table. In parentheses we provide the standard errors of the parameters.

IV. How to Select the Dependence Structure

A. Deheuvels's Empirical Copula Function

Joint distributions with the same marginals and correlation coefficients may exhibit fundamentally different properties depending on the copula family selected to represent their dependence structure. To give an intuitive feel of this, we use the four copulas of figure 1 (i.e., the same parameters) estimated on the FTSE 100–NASDAQ pair together with standard Gaussian marginals to graph the corresponding bivariate densities in figure 4. Obviously, the choice of the copula significantly affects the resulting multivariate distribution. Surprisingly, however, the few applications presented in the literature do not discuss this fundamental issue and rely on an arbitrary choice for the copula. In the same way as Durrleman, Nikeghbali, and Roncalli (2000), we use the empirical copula function introduced by Deheuvels (1979, 1981) to avoid this problem.

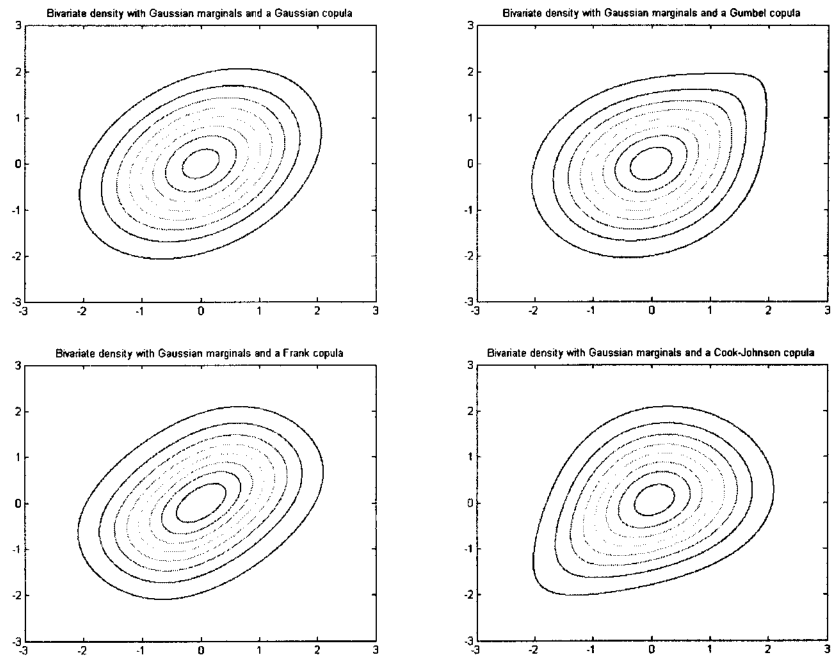

FIG. 4.—Joint distributions with the same marginals and different copulas. To measure the sensitivity of a joint distribution to the choice of a specific copula family, we use the same two standard Gaussian distributions with density functions ϕ_1 and ϕ_2 and cumulative distributions Φ_1 and Φ_2, together with the copulas presented in fig. 1 (estimated on the pair FTSE 100–NASDAQ) to graph the corresponding bivariate densities given by $f(x,y) = \phi_1(x) \times \phi_2(y) \times c[\Phi_1(x), \Phi_2(y)]$.

Let $(x_1^t, x_2^t, \dots, x_n^t)_{t=1}^{T}$ be a sample of size T from a continuous n-variate distribution and denote $(r_1^t, r_2^t, \dots, r_n^t)$ the rank statistic of the sample. The empirical copula $\hat{C}_{(T)}$ as introduced by Deheuvels is defined on the lattice

$$L = \left[\left(\frac{t_1}{T}, \frac{t_2}{T}, \dots, \frac{t_n}{T}\right), t_i = 1, \dots, T, 1 \leq i \leq n\right]$$

by the following equation:

$$\hat{C}_{(T)}\left(\frac{t_1}{T}, \frac{t_2}{T}, \dots, \frac{t_n}{T}\right) = \frac{1}{T}\sum_{t=1}^{T}\prod_{i=1}^{n} 1_{r_i^t \leqslant t_i}. \tag{15}$$

It can be shown that, in the same spirit as Sklar's (1959) theorem, the empirical distribution function $\hat{F}$ corresponding to F is uniquely and reciprocally defined both by the univariate empirical distribution functions $\hat{F}_i$ and the values of the empirical copula $\hat{C}_{(T)}$ on the set £. Moreover, Deheuvels's copula, $\hat{C}_{(T)}$, converges to C as T increases.

The empirical copula density can be obtained from the empirical copula distribution function, with the relation (see Nelsen 1999):

$$\hat{c}_{(T)}\left(\frac{t_1}{T}, \dots, \frac{t_i}{T}, \dots, \frac{t_n}{T}\right) = \sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} (-1)^{j_1+j_2+\dots+j_n} \hat{C}_{(T)} \left(\frac{t_1 - j_1 + 1}{T}, \dots, \frac{t_i - j_i + 1}{T}, \dots, \frac{t_n - j_n + 1}{T}\right). \tag{16}$$

Figure 5 provides an illustration of these empirical estimates for the FTSE 100–NASDAQ pair. The left-hand-side graph shows the empirical copula distribution function. The middle graph represents the contour plot of this empirical copula. The last graph of figure 6 displays the empirical copula density.

B. Some Selection Criteria

The appropriate dependence structure may be selected by minimizing the distance between a parametric copula and the empirical copula. Since the empirical copula is only defined on a lattice £, we define our distance with discrete norms.

We first measure the goodness of fit obtained with each parametric copula with the Anderson-Darling test, which emphasizes deviations in the tails

$$AD = \max_{1 \le t_1 \le T,\, 1 \le t_2 \le T} \frac{\left|\hat{C}_{(T)}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) - C_{\text{parametric}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right)\right|}{\sqrt{C_{\text{parametric}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right)\left[1 - C_{\text{parametric}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right)\right]}}. \tag{17}$$

Moreover, to have a more global measure we also compute the Integrated Anderson-Darling (IAD):

$$IAD = \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} \frac{\left[\hat{C}_{(T)}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) - C_{\text{parametric}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right)\right]^2}{C_{\text{parametric}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right)\left[1 - C_{\text{parametric}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right)\right]}. \tag{18}$$

Finally, we also measure the relevance of each copula using the concept of entropy.[7] This measure of information defines the uncertainty of a distribution $f(x_1,x_2)$ with the quantity:

$$E[f(x_1,x_2)] = -\int\int f(x_1,x_2)\ln[f(x_1,x_2)]dx_1 dx_2. \tag{19}$$

If we use a copula representation of the bivariate density, as in equation (4),

7. In addition to being a more general measure of relevance, the entropy offers a distance relying on the copula density, whereas the first two measures are computed with cumulative distribution functions.

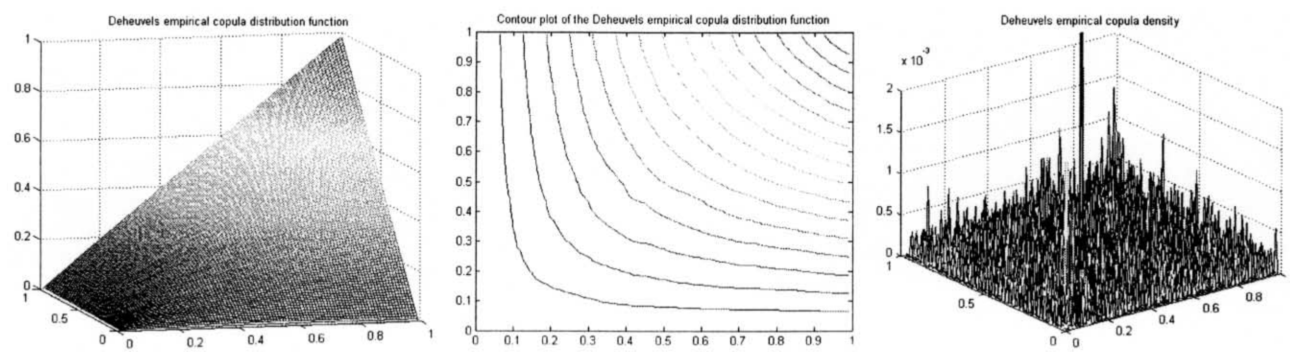

FIG. 5.—The empirical copula introduced by Deheuvels (for the pair FTSE100–NASDAQ). On the left, we graph the empirical copula distribution function defined in eq. (15). We also represent in the middle graph the contour plot of this empirical distribution function. The graph on the right displays the empirical copula density as described in eq. (16).

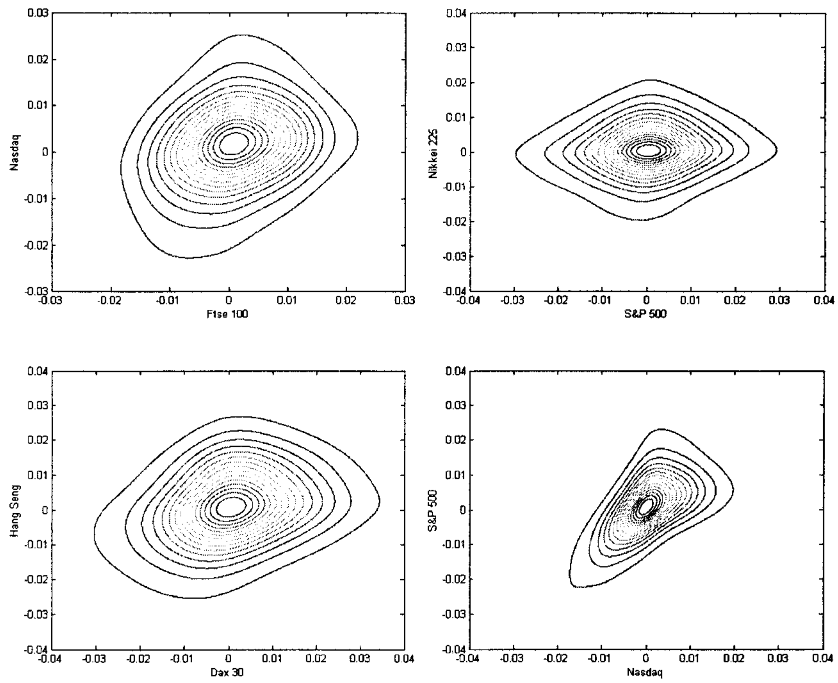

FIG. 6.—The semiparametric bivariate distributions for four pairs of index returns. This figure displays the semiparametric densities obtained with our nonparametric marginals and the estimated Cook-Johnson (1981) copula for four pairs of index returns.

the entropy of $f(x_1, x_2)$ is equal to the sum of the entropies[8] of the individual marginal distributions plus the entropy of the copula distribution function:

$$E[f(x_1, x_2)] = E[f_1(x_1)] + E[f_2(x_1)] + E\{c[F_1(x_1), F_2(x_2)]\}. \tag{20}$$

Since the marginal distributions f_i are estimated by the kernel method and hold constant during our search for the optimal parametric copula, the only quantity that affects the entropy of the bivariate distribution $f(x_1, x_2)$ in equation (20) is the entropy of the copula function. We thus compute the discrete entropy

8. The entropy of a univariate distribution $f_1(x_1)$ is defined by

$$E[f(x_1)] = -\int f(x_1)\ln[f(x_1)]dx_1.$$

obtained with the different parametric copulas on the lattice £ with the formula:

$$E\{c[F_1(x_1), F_2(x_2)]\} = \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} c\left(\frac{t_1}{T}, \frac{t_2}{T}\right) \ln\left[c\left(\frac{t_1}{T}, \frac{t_2}{T}\right)\right]. \tag{21}$$

We then select the most informative copula that is the parametric form that minimizes the entropy, as described in equation (21).

The values obtained for the different distance measures are presented in table 5. The IAD test ranks the Cook-Johnson (1981) copula first in 12 out of 15 cases. In the three remaining cases, no clear-cut ranking is obtained with the remaining parametric families. Moreover, the Cook-Johnson always provides distance values close to that of the best copula. The Anderson-Darling test as well as the entropy identifies the Cook-Johnson copula as the best parametric form in 100% of the cases. In addition, in our n-variate ($n > 2$) investigations, the Cook-Johnson copula always ranks first. We thus conclude that the Cook-Johnson copula represents an appropriate parametric form to capture the dependence structure in stock index returns.

As a final goodness-of-fit test, we use the simulation methodology discussed in the appendix to simulate values for each pair of index returns using the estimated Cook-Johnson (1981) copula and the empirical marginal distributions. We then compute the first four moments as well as the cross-moments up to order four. The obtained values are then compared with the empirical moments and cross-moments that were computed on the original index return series.[9] The good adequacy enables us to conclude that the Cook-Johnson copula provides an excellent fit for the dependence structure of stock index returns. We show in figure 6 the semiparametric densities obtained with our nonparametric marginals and the estimated Cook-Johnson copula for four pairs of index returns.

In table 2, we study the asymptotic behavior of the dependence structure implied by the different copula families and conclude that the Cook-Johnson (1981) copula provides no positive extreme correlation but an asymptotic lower tail dependence. Our calibration results suggest that the dependence structure of international stock index returns is accurately captured by a Cook-Johnson copula. It implies that the correlation of large positive returns beyond a given threshold will asymptotically go to zero as the threshold level increases. This is not the case, however, for large negative returns whose correlation does not converge to zero but tends to increase with the threshold. Hence, our findings are perfectly in line with those of Longin and Solnik (2001), who study the conditional correlation structure of international equity markets and conclude that correlation increases in bear markets, but not in bull markets.

9. To save space, we do not include the results in this article. Tables, however, are available on request.

TABLE 5 Goodness-of-Fit Measures for the Parametric Copula Functions

Portfolios	Gauss	Gumbel	Cook-Johnson	Frank
Dax 30–FTSE 100:				
AD	.9493	.1546	**.0566**	.2327
IAD	57.1814	2.9174	**2.1512**	3.0517
Entropy	−1.5031	−.7889	**−4.1341**	−.7546
Dax 30–Hang Seng:				
AD	.8966	.5677	**.0426**	.1667
IAD	9.1620	.6396	**.0687**	.4862
Entropy	−.5779	−.3031	**−2.0471**	−.3721
Dax 30–NASDAQ:				
AD	.8957	.1492	**.0435**	.1452
IAD	8.9813	.4607	**.1142**	.4423
Entropy	−.3799	−.1834	**−1.2081**	−.2571
Dax 30–Nikkei 225:				
AD	.8821	.1073	**.0470**	.1042
IAD	1.5198	.0549	**.0194**	.0644
Entropy	−.3625	−.2064	**−.9459**	−.2640
Dax 30–S&P 500:				
AD	.8970	.1272	**.0520**	.2187
IAD	9.1693	.3133	**.1858**	.7456
Entropy	−.4837	−.2955	**−1.6269**	−.3143
FTSE 100–Hang Seng:				
AD	.8885	.4055	**.0362**	.1273
IAD	9.0633	.4944	**.1366**	.3584
Entropy	−.4502	−.2388	**−1.4298**	−.3184
FTSE 100–NASDAQ:				
AD	.9063	.1588	**.0393**	.1449
IAD	1.5351	.1312	**.0081**	.1444
Entropy	−.5644	−.2637	**−1.0638**	−.3776
FTSE 100–Nikkei 225:				
AD	.8765	.0991	**.0373**	.1098
IAD	8.9046	.3269	**.0990**	.3183
Entropy	−.3103	−.1656	**−1.1249**	−.2171
FTSE 100–S&P 500:				
AD	.9132	1.0270	**.0548**	.1638
IAD	9.4845	1.0985	**.1531**	.444
Entropy	−.7754	−.4322	**−2.5726**	−.4772
Hang Seng–NASDAQ:				
AD	.8584	.1713	**.0401**	.1034
IAD	8.6584	.4327	**.1053**	.2758
Entropy	−.1308	−.0640	**−.3806**	−.1124
Hang Seng–Nikkei 225:				
AD	.8833	.3193	**.0366**	.1050
IAD	8.9904	.3644	**.0910**	.3224
Entropy	−.4191	−.2147	**−1.1434**	−.2771
Hang Seng–S&P 500:				
AD	.8571	.1563	**.0398**	.1069
IAD	8.6296	.4074	**.0710**	.3051
Entropy	−.1079	−.0603	**−.3011**	−.0879
NASDAQ–Nikkei 225:				
AD	.8684	.1290	**.0454**	.0845
IAD	1.4794	**.0150**	.0193	.0158
Entropy	−.1172	−.0575	**−.3418**	−.1029
NASDAQ–S&P 500:				
AD	.9655	.1458	**.1057**	.5246
IAD	10.3787	**.1067**	.1432	1.2368
Entropy	−4.2645	−1.9323	**−8.9936**	−1.6315

TABLE 5 *(Continued)*

Portfolios	Gauss	Gumbel	Cook-Johnson	Frank
Nikkei 225–S&P 500:				
AD	.8574	.1474	**.0466**	.0768
IAD	8.6520	.5014	.1879	**.1581**
Entropy	−.1135	−.0639	**−.2397**	−.0984

NOTE.—For each pair of index returns and each distance measure, we put in bold characters the best copula. For information on copulas, see Gumbel (1960), Frank (1979), and Cook-Johnson (1981). In order to select among the different copula families, we suggest to study the distance between each parametric copula and the empirical copula. This table gives three different goodness-of-fit measures for the estimated copula functions. For each pair of index returns and each parametric copula, the first row provides the Anderson-Darling statistic measuring the closeness of the estimated copula to the empirical one with a special emphasis on the tails. The integrated Anderson-Darling measure, presented in the second row, provides a more general assessment of the fit. The last row gives the entropy. All three measures support the choice of the Cook-Johnson (1981) copula.

V. Dependence Structure and Risk Measures: An Application to VaR

A. Value-at-Risk, Leptokurtosicity, and Tail Dependence

As trading activities have become more complex, it has become more difficult for senior management to obtain a useful yet practical measure of market risk. The most widely used summary measure is the so-called VaR. For a given time horizon T and a confidence level p, the VaR is the loss in market value that is exceeded over this time horizon with probability $1 - p$. Hence, the VaR of a portfolio is simply an estimate of a specific percentile of the probability distribution of the portfolio's value change over a given holding period. The specific percentile is usually in the lower tail of the distribution, for example, the ninety-fifth or ninety-ninth percentile.

Calculation of portfolio VaR is often based on the variance-covariance approach and makes the assumption, among others, that returns follow a multivariate normal distribution. At this point of our analysis, we know that this assumption both implies that (*a*) each marginal distribution representing an asset return (or risk factor) is Gaussian, and that (*b*) the dependence structure of the asset returns is given by a Gaussian copula. Empirical studies show, however, that financial asset returns tend to be negatively skewed and leptokurtic. The assumption of Gaussian marginals is thus likely to underestimate the actual risk since the return series have tails that are "fatter" than those implied by normal distributions, especially the negative one. Moreover, when several risks are taken into consideration for the computation of VaR, an accurate measurement of diversification effects is also fundamental to an exact estimation of the VaR. We showed in the previous section that the dependence structure of stock index returns is appropriately encoded by the Cook-Johnson (1981) copula. This copula exhibits a lower tail dependence. Hence, the Gaussian dependence structure implicitly assumed in traditional multivariate normal analysis is likely to overestimate the diversification effect in a given portfolio and lead to an underestimation of the VaR.

B. Monte Carlo Analysis of Risk Measures

To check our conjectures, we calculate the VaR of different portfolios using Monte Carlo simulations (see the appendix) of the following models:

1. We use a multivariate normal distribution for the stock index returns, that is, Gaussian marginals with a Gaussian copula function;
2. We also compute the VaR when stock index returns are appropriately modeled: for each stock index, we use the nonparametric marginal distribution computed with the kernel method together with the estimated Cook-Johnson (1981) copula to encode the dependence structure.

To separate the effects of an appropriate modeling of the marginals and of the dependence structure on VaR estimates, we also study the following simulations:

3. We use the nonparametric marginal distributions together with the Gaussian copula. Compared with the multivariate normal distribution, these simulations will reveal the sensitivity of the VaR to the choice of marginal distributions;
4. We also look at the separate impact of the dependence structure on VaR by simulating stock index returns with Gaussian marginals and the appropriate Cook-Johnson (1981) copula;
5. Finally, to assess the departure from historical values, we also compute the empirical VaR for each portfolio using historical returns.

We compute the VaR estimates with the above approaches for five two-stock-index portfolios obtained by varying the weight of each index from 0% to 100% by steps of 25%. The extent of discrepancy from using one model over another is depicted in figure 7, which shows the loss functions for three portfolios based on the FTSE 100–NASDAQ. In addition, table 6 provides a comparison of VaR estimates for the five portfolios formed with the FTSE 100–NASDAQ and Dax 30–Nikkei 225 paris, both at 95% and 99% confidence levels.

Value-at-Risk by definition should be very sensitive to the degree to which the distributions are fat tailed. The structure of the difference between the empirical VaR and the two parametric models using Gaussian marginals (but different dependence structures) is indeed what would be expected for fat-tailed distributions. At low probability levels, the two parametric Gaussian VaR overestimate the actual VaR, and then, as we move to higher probability levels, the parametric Gaussian approaches underestimate the VaR.

As predicted, the difference becomes much smaller when kernel densities are used instead of Gaussian margins. The large discrepancy between the two approaches (kernel vs. Gaussian margins) shows how important it is to find as accurate a model as possible for each marginal distribution. In a multivariate setting, we explain that only the copula approach brings an entire flexibility, in the sense that we model the marginals independently from the dependence

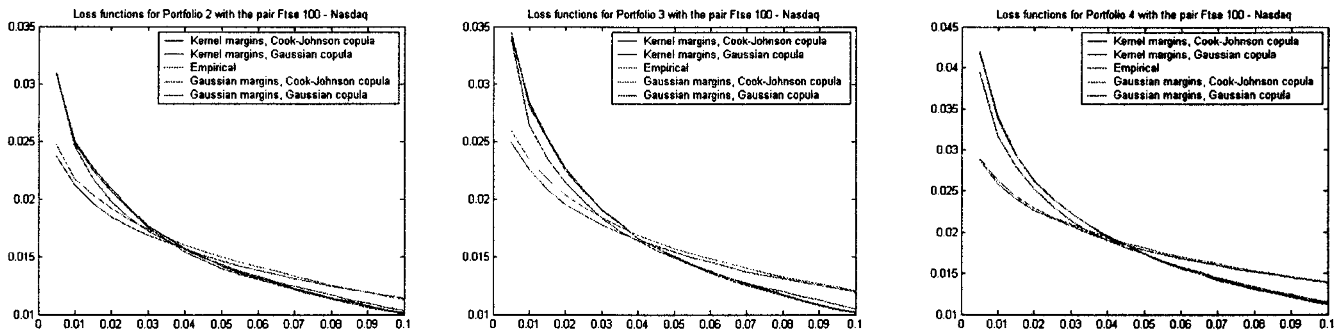

FIG. 7.—Loss functions for three portfolios based on the pair FTSE 100–NASDAQ. This figure depicts by how much the various (semi-) parametric Value-at-Risk estimates differ from the empirical Value-at-Risk for three portfolios formed with the pair FTSE 100–NASDAQ and over a large range of confidence levels.

TABLE 6 Comparison of Value-at-Risk Estimates

Margins	Copula	Portfolio 1 (100%, 0%)	Portfolio 2 (75%, 25%)	Portfolio 3 (50%, 50%)	Portfolio 4 (25%, 75%)	Portfolio 5 (0%, 100%)
		One Day Value-at-Risk at a 95% Confidence Level				
FTSE 100–NASDAQ:						
Nonparametric	Cook-Johnson	1.5285E-02	1.4228E-02	1.5045E-02	1.7391E-02	2.0705E-02
Nonparametric	Gaussian	1.5244E-02	1.4349E-02	1.4947E-02	1.7381E-02	2.0510E-02
Gaussian	Cook-Johnson	1.5881E-02	1.5021E-02	1.5735E-02	1.8132E-02	2.1494E-02
Gaussian	Gaussian	1.5848E-02	1.4678E-02	1.5431E-02	1.7820E-02	2.1490E-02
Historical Value-at-Risk		1.5092E-02	1.4015E-02	1.4985E-02	1.7324E-02	2.0626E-02
Dax 30–Nikkei 225:						
Nonparametric	Cook-Johnson	2.0684E-02	1.8125E-02	1.7648E-02	1.9483E-02	2.2892E-02
Nonparametric	Gaussian	2.0862E-02	1.8292E-02	1.7927E-02	1.9891E-02	2.3566E-02
Gaussian	Cook-Johnson	2.1271E-02	1.8776E-02	1.8241E-02	1.9823E-02	2.3174E-02
Gaussian	Gaussian	2.1164E-02	1.8044E-02	1.7496E-02	1.9553E-02	2.3203E-02
Historical Value-at-Risk		2.0507E-02	1.8004E-02	1.7498E-02	1.9398E-02	2.2834E-02

		One Day Value-at-Risk at a 99% Confidence Level				
FTSE 100-NASDAQ:						
Nonparametric	Cook-Johnson	2.6140E-02	2.5041E-02	2.8406E-02	3.4103E-02	4.0495E-02
Nonparametric	Gaussian	2.6297E-02	2.5041E-02	2.6403E-02	3.1603E-02	3.8426E-02
Gaussian	Cook-Johnson	2.2460E-02	2.1728E-02	2.3409E-02	2.6219E-02	3.0447E-02
Gaussian	Gaussian	2.2930E-02	2.1249E-02	2.2539E-02	2.5825E-02	3.0627E-02
Historical Value-at-Risk		2.6255E-02	2.4852E-02	2.8083E-02	3.3660E-02	4.0382E-02
Dax 30-Nikkei 225:						
Nonparametric	Cook-Johnson	3.6432E-02	3.2334E-02	3.1145E-02	3.29083E-02	3.8158E-02
Nonparametric	Gaussian	3.7334E-02	3.1825E-02	3.0726E-02	3.3015E-02	3.9288E-02
Gaussian	Cook-Johnson	3.0029E-02	2.7412E-02	2.6917E-02	2.8677E-02	3.2438E-02
Gaussian	Gaussian	3.0107E-02	2.6028E-02	2.4952E-02	2.7875E-02	3.3367E-02
Historical Value-at-Risk		3.6236E-02	3.2376E-02	3.1030E-02	3.2573E-02	3.7921E-02

NOTE.—For information on copulas, see Gumbel (1960), Frank (1979), and Cook-Johnson (1981). We use the four semiparametric models to obtain series of index returns for each pair by 20,000 Monte Carlo simulations. For a given pair, we form five portfolios by varying the weight of each index from 0% to 100% by steps of 25%. The Value-at-Risk estimates are then calculated for each portfolio using the empirical approach (historical data) and the four semiparametric approaches described in the article. Results at 95% and 99% confidence levels are presented here for two pairs: FTSE 100–NASDAQ and Dax 30–Nikkei 225.

TABLE 7 Average Absolute Errors

	Range of Probabilities (%)	
Average Absolute Difference between:	90–99.5	98.5–99.5
Nonparametric margins–Cook-Johnson copula and empirical values	.12	.26
Nonparametric margins–Cook-Johnson copula and Gaussian margins–Gaussian copula (1)	9.23	24.84
Nonparametric margins–Cook-Johnson copula and nonparametric margins–Gaussian copula (2)	1.97	4.06
Nonparametric margins–Gaussian copula and Gaussian margins–Gaussian copula (3)	8.45	18.47
Proportion of (1) explained by (2)	18.93	18.01
Proportion of (1) explained by (3)	81.07	81.99

NOTE.—For information on copula, see Cook-Johnson (1981). We use the loss functions obtained with all the bivariate portfolios to compute the average absolute percentage error between the nonparametric margins with a Cook-Johnson (1981) copula model and the bivariate normal model (1). We also decompose the effect resulting from the dependence structure (2) or the marginals (3). The proportion of (1) explained both by (2) and (3) is also provided. Finally, the discrepancy between the empirical Value-at-Risk estimates and those obtained with nonparametric margins and a Cook-Johnson copula is also calculated for two ranges of probabilities.

structure and independently from one another while still ending up with a multivariate distribution.

Although the choice of a specific dependence structure may seem quantitatively less important, it remains fundamental to ensure an accurate estimation of VaR. The loss function obtained when a Cook-Johnson (1981) copula is used in addition to kernel margins is virtually identical to the empirical one, while the model relying on a Gaussian copula clearly underestimates the VaR at high probability levels. Even when both copulas are applied to Gaussian margins, the discrepancy between the loss functions is apparent.

Finally, we use the loss functions to compute the average absolute percentage difference between the different models to assess the proportion of discrepancy resulting from marginal distributions or dependence structure. Results are summarized in table 7. For the whole range of probabilities covered by the loss functions (i.e., 90%–99.5%), the marginal distributions account, on average, for 81.91% of the error made in computing VaR estimates, while an error in the selection of the dependence structure explains, on average, 18.09%.

Since the 99% confidence interval is the level required by the Basle committee, we also compute the proportion of error explained by both components for a smaller range of probabilities, 98.5%–99.5%, centered around this 99% confidence interval. Results indicate that in this area, the choice of the marginal distributions explains 87.76% of the error while the role played by the dependence structure reduces to 12.24%. Although less important, the error resulting from a misspecification of the dependence structure remains largely significant and is, of course, magnified for the million-dollar positions that mutual funds, for example, typically hold.

VI. Conclusion

The purpose of this article is to investigate the dependence structure of international stock index returns. We show that it can be modeled separately from the marginal behaviors of index returns through a copula representation. While the choice of the copula may be important, the few applications presented in the literature do not discuss this fundamental issue. To avoid an ad hoc selection for the copula, we calibrate our parametric copula on Deheuvels's empirical copula function.

We then investigate the relative importance of marginal distributions and dependence structure in VaR estimation. We find that a misspecification of the dependence structure can account for up to 20% of the error in the estimation of VaR.

Beyond risk management, the possibility of an accurate and flexible modeling of the dependence structure between financial assets opens new avenues of research. Applications to the pricing of multiassets options as well as portfolio selection represent two of our current investigations.

Appendix

Simulation from a Bivariate Cook-Johnson Copula

Our VaR calculations involve the simulation of a bivariate vector (X_1, X_2) whose dependence structure obeys a Cook-Johnson copula (1981). Remembering that the variables $U_i = F_i(X_i)$ are uniformly distributed on [0, 1] once a random vector (U_1, U_2) has been simulated with the Cook-Johnson copula distribution function, the random vector, $F^{-1}(U_1), F^{-1}(U_1)$, has the desired multivariate distribution. Thus, the only problem lies in simulating realizations (u_1, u_2) from the Cook-Johnson copula $C(u_1, u_2) = (u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-(1/\alpha)}$. Following a classical procedure (see Frees and Valdez 1998), we first compute the bidimensional marginal:

$$C_2(u_2/u_1) = P(U_2 \leq u_2/U_1 = u_1)$$

$$= \frac{\partial C(u_1, u_2)}{\partial u_1} = [1 + u_1^{\alpha}(u_2^{-\alpha} - 1)]\infty.$$

Then, to simulate from the bivariate Cook-Johnson (1981) copula, use the following:

1. simulate U_i from $U_{[0,\,1]}$,
2. simulate q from $U_{[0,\,1]}$,
3. compute $u_2 = C_2^{-1}(q/u_1) = (\{q^{[-\alpha/(1+\alpha)]} - 1\}u_1^{-\alpha} + 1)^{(-1/\alpha)}$, and
4. the simulated vector is (u_1, u_2).

The simulated values from the bivariate distribution $F(X_1, X_2) = C[F_1(X_1), F_2(X_2)]$

are thus obtained using a root finding procedure to compute $(x_1, x_2) = [F_1^{-1}(u_1), F_2^{-1}(u_2)]$.

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