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Mathematical Statistics

Pricing Bermudan swap options using the BGM model with arbitrage-free discretisation and boundary based option exercise

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Abstract

This paper studies the practical pricing of Bermudan swap options, attempting to find both lower and upper bounds for the option price. It uses the BGM model with three driving factors and Monte Carlo simulation for determining the evolution of forward interest rates. A discretisation proposed by Glasserman is used, as an alternative to direct discretisation of the forward rates. Two sub-optimal exercise strategies using exercise boundaries proposed by Andersen are evaluated for finding the lower bound for the option price. A perfect foresight strategy is evaluated for finding an upper bound. This paper also studies the systematic errors in the forward rate evolution and discusses simple measures for reducing their impact on the option pricing.

Keywords: Bermudan swap options, multi-factor BGM model, arbitrage-free discretisation, exercise strategies

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Contents

1 Introduction

An interest rate swap is a contract between two parties of exchanging cash flows at a fixed interest rate and cash flows at a floating interest rate. A Bermudan swap option is an option to enter into an interest rate swap at a specified set of dates, provided this option has not already been used. These Bermudan swap options are useful for example for hedging callable bonds, i.e. bonds with built-in options to cancel the contract before maturity. Pricing these Bermudan swap options can be done in several ways using different models for the interest rates.

The BGM model (Brace, Gatarek, Musiela 1997) provides a model for the evolution of market observable forward rates, the same forward rates that are used for pricing swaps. It incorporates correlations between forward rates and can be calibrated to price simple interest rate options like caps according to a market standard Black-Scholes type formula. The main disadvantage is that due to its large number of state variables it has to be implemented using Monte Carlo simulation. The two main drawbacks of the Monte Carlo simulation are slow convergence and the difficulty to incorporate early exercise.

At the time when the BGM model was proposed the mathematical framework used was that of the HJM model (Heath, Jarrow, Morton 1992), but since then the BGM model has been derived separately.

This paper considers practical aspects of pricing Bermudan swap options by Monte Carlo simulation. The aim is to describe the logic from mathematical assumptions to a product usable for interest rate traders. It incorporates a refined discretisation of artificial simulation variables based on the forward rates (Glasserman, Zhao 2000), and an exercise boundary based suboptimal exercise strategy (Andersen 1999).

2 Framework

2.1 Complete arbitrage-free market

The notation that will be used for the functions of continuous time are $0 \leq t \leq T \leq \tau$, where T is usually a stopping time for a stochastic variable, for example exercise date for an option, and τ is the last date covered by the market defined below.

Definition 1 (market) *On a complete filtered probability space* $(\Omega, \mathcal{F}_t, \mathbf{Q})$*, let* ε *be the set of all continuous semi-martingales. Let* $\varepsilon_+ = \{Y \in \varepsilon | Y > 0\}$ *and let* $\varepsilon^n = \{Y | Y = (Y_1, \ldots, Y_n), Y_i \in \varepsilon\}$*. Define a market as a price system* $B \in \varepsilon^n$

Definition 2 (arbitrage) *The market* $B \in \varepsilon^n$ *is said to be arbitrage free if there exists a* $\xi \in \varepsilon_+$ *with* $\xi(0) = 1$ *such that* ξB_i *are martingales for all* $1 \leq i \leq n$*. The process* ξ *is called state price deflator.*

Definition 3 (self-financing trading strategies) *Let* $\theta = (\theta_1, \ldots, \theta_n)$ *be a vector of adapted* B_i*-integrable processes* θ_i *on a market* $B \in \varepsilon^n$*. The pair* (θ, B) *is called a self-financing trading strategy (SFTS) if* $\theta_t B_t = \theta_0 B_0 + \int_0^t \theta_s \, \mathrm{d}B_s$ *for all* $t > 0$*.*

Definition 4 (contingent claim) *A contingent claim* C_T *is an* $\mathcal{F}_T$*-adapted stochastic variable.*

Definition 5 (complete market) *The market $B \in \varepsilon^n$ is called complete if for any claim $C_T \in \mathcal{F}_T$ there exists an SFTS (θ, B) such that $\theta_T B_T = C_T$.*

Theorem 1 (completeness) *An arbitrage free market $B \in \varepsilon^n$ is complete if and only if there exists exactly one $\xi \in \varepsilon_+$ with $\xi(0) = 1$ such that ξB_i are martingales for all $1 \leq i \leq n$.*

Proof 1 *The proof is not presented here but is covered by fundamental works, for example by Harrison and Pliska [5].*

Introducing a d-dimensional Wiener process as the only source of randomness for the market variables and assuming they can be described in the form of drifted geometric Brownian motions

$$\frac{dB_i}{B_i} = \mu_i \,\mathrm{d}t + \sigma_i \,\mathrm{d}W, \tag{1}$$

and

$$\frac{d\xi_i}{\xi_i} = -r \,\mathrm{d}t - \varphi \,\mathrm{d}W, \tag{2}$$

where $\mu_i(t, \omega)$ and $r(t, \omega)$ are scalar processes and $\sigma_i(t, \omega)$ and $\varphi(t, \omega)$ are d-dimensional vector processes, adapted to the filtration of the Wiener process.

The solutions to the stochastic differential equations above are

$$B_i(t) = B_i(0) \exp\left(\int_0^t (\mu_i - \frac{1}{2}|\sigma_i|^2) \,\mathrm{d}s + \int_0^t \sigma_i \,\mathrm{d}W\right), \tag{3}$$

and

$$\xi(t) = \xi(0) + \exp\left(\int_0^t (-r - \frac{1}{2}|\varphi|^2) \,\mathrm{d}s + \int_0^t \varphi \,\mathrm{d}W\right). \tag{4}$$

Multiplying (3) by (4) yields that ξB_i is a martingale whenever $-\frac{1}{2}|\sigma_i - \varphi|^2 = \mu_i - r - \frac{1}{2}|\sigma_i|^2 - \frac{1}{2}|\varphi|^2$. This can also be expressed as

$$\mu_i = r - \sigma_i \varphi, \tag{5}$$

which produces a useful relation between drift and volatility of different assets, namely $\mu_i + \sigma_i \varphi = \mu_j + \sigma_j \varphi$, or

$$\varphi = \frac{\mu_i - \mu_j}{\sigma_i - \sigma_j}, \tag{6}$$

The market is arbitrage free if there exist r and φ such that μ_i and σ_i satisfy (5) for every i.

2.2 Pricing numeraires

Assume claim $C_T \in \mathcal{F}_T$ such that there exists an SFTS (θ, B) where $\theta_T B_T = C_T$. As $(\theta, \xi B)$ is also an SFTS then

$$\xi_T C_T = \theta_T \cdot \xi_T B_T = \theta_0 \cdot \xi_0 B_0 + \int_0^T \theta_s \cdot \,\mathrm{d}(\xi_s B_s). \tag{7}$$

Taking the expectations of both sides and using the martingale property of the SFTS

$$\begin{aligned}\mathbf{E}[\xi_T C_T|\mathcal{F}_t] = \mathbf{E}[\theta_0 \cdot \xi_0 B_0 + \textstyle\int_0^T \theta_s \cdot \mathrm{d}(\xi_s B_s)] = \\ \theta_0 \cdot \xi_0 B_0 + \textstyle\int_0^t \theta_s \cdot \mathrm{d}(\xi_s B_s) = \theta_t \cdot \xi_t B_t,\end{aligned} \tag{8}$$

thus

$$\theta_t \cdot B_t = \xi_t^{-1}\mathbf{E}[\xi_T C_T|\mathcal{F}_t], \tag{9}$$

which in a complete market becomes

$$C_t = \xi_t^{-1}\mathbf{E}[\xi_T C_T|\mathcal{F}_t]. \tag{10}$$

Definition 6 (numeraire measure) *Let ξA be a martingale, $B \in \varepsilon^n$ the usual market and ξ its state price deflator. Define the A numeraire measure $\mathbf{Q}_A$ by $\frac{\mathrm{d}\mathbf{Q}_A}{\mathrm{d}\mathbf{Q}} = \frac{\xi A}{A(0)}$.*

Theorem 2 *If ξA and ξY are martingales then Y/A is a $\mathbf{Q}_A$ martingale.*

Proof 2 *For $s > 0$*

$$\begin{aligned}\mathbf{E}_A[\frac{Y(t+s)}{A(t+s)}|\mathcal{F}_t] &= \frac{\mathbf{E}[\frac{\xi(t+s)A(t+s)}{A(0)} \cdot \frac{Y(t+s)}{A(t+s)}|\mathcal{F}_t]}{\mathbf{E}[\frac{\xi A}{A(0)}|\mathcal{F}_t]} \\ &= \frac{\mathbf{E}[\xi(t+s) \cdot \frac{Y(t+s)}{A(0)}|\mathcal{F}_t]}{\frac{\xi(t)A(t)}{A(0)}} \\ &= \frac{Y(t)}{A(t)}.\end{aligned} \tag{11}$$

The advantage of using a pricing numeraire is that there is no need to explicitly use a state price deflator ξ, as

$$C_t = \xi_t^{-1}\mathbf{E}[\xi_T C_T|\mathcal{F}_T] = B_i(t)\mathbf{E}_{B_i}[\frac{C_T}{B_i(T)}|\mathcal{F}_T]. \tag{12}$$

2.3 Interest rate market

The interest rate market considered has a tenor structure $0 \leq T_0 < \ldots < T_N$ with $T_{i+1} - T_i = \delta$, pricing only instruments with cash flows on these dates.

Definition 7 (discount function) *Define $P(t,T)$ as the price at time t of one money unit at time T. It is known as the discount function, as it is used for discounting a future cash flow (at time T) into its present (time t) value.*

The interest rate market on the given tenor structure is a market as described earlier, with $(P_1, \ldots, P_N) \in \varepsilon^N$. The dynamics of P_i is described by

$$\frac{dP_i}{P_i} = \mu_i \,\mathrm{d}t + \sigma_i \,\mathrm{d}W, \tag{13}$$

with the solution

$$P_i(t) = P_i(0)\exp\left(\int_0^t (\mu_i - \frac{1}{2}|\sigma_i|^2)\,\mathrm{d}s + \int_0^t \sigma_i \,\mathrm{d}W\right), \tag{14}$$

Definition 8 (forward rate) *The simple forward rate $F_i(t)$ at time t for the period T_i to T_{i+1} is defined by*

$$F_i(t) = \frac{1}{\delta}\left(\frac{P_i(t)}{P_{i+1}(t)} - 1\right). \tag{15}$$

Definition (15) comes from the usual economic definition of a forward interest rate, that

$$1 + \delta F_i(t) = \frac{P_i(t)}{P_{i+1}(t)}.$$

Deriving the dynamics of the forward rate F_i from (14) and (15), using relation (5) towards the end, yields that

$$\mathrm{d}F_i = \tfrac{1}{\delta}\tfrac{P_i(0)}{P_{i+1}(0)}\,\mathrm{d}\exp\Big(\textstyle\int_0^t(\mu_i - \mu_{i+1} - \frac{1}{2}|\sigma_i|^2 + \frac{1}{2}|\sigma_{i+1}|^2)\,\mathrm{d}s + \int_0^t(\sigma_i - \sigma_{i+1})\,\mathrm{d}W\Big) =$$

$$\begin{aligned}
&\tfrac{1}{\delta}\tfrac{P_i(0)}{P_{i+1}(0)}\exp\left(\textstyle\int_0^t(\mu_i - \mu_{i+1} - \frac{1}{2}|\sigma_i|^2 + \frac{1}{2}|\sigma_{i+1}|^2)\,\mathrm{d}s + \int_0^t(\sigma_i - \sigma_{i+1})\,\mathrm{d}W\right)\cdot\\
&\qquad\cdot\big((\mu_i - \mu_{i+1} - \tfrac{1}{2}|\sigma_i|^2 + \tfrac{1}{2}|\sigma_{i+1}|^2) + (\sigma_i - \sigma_{i+1})\big) +\\
&+\tfrac{1}{2}\tfrac{1}{\delta}\tfrac{P_i(0)}{P_{i+1}(0)}\exp\left(\textstyle\int_0^t(\mu_i - \mu_{i+1} - \frac{1}{2}|\sigma_i|^2 + \frac{1}{2}|\sigma_{i+1}|^2)\,\mathrm{d}s + \int_0^t(\sigma_i - \sigma_{i+1})\,\mathrm{d}W\right)\cdot\\
&\qquad\cdot|\sigma_i - \sigma_{i+1}|^2\,\mathrm{d}t =
\end{aligned}$$

$$\tfrac{1}{\delta}(1 + \delta F_i)\Big(\big(\mu_i - \mu_{i+1} + \sigma_{i+1}(\sigma_{i+1} - \sigma_i)\big)\,\mathrm{d}t + (\sigma_i - \sigma i + 1)\,\mathrm{d}W\Big) =$$

$$\tfrac{1}{\delta}(1 + \delta F_i)(\sigma_i - \sigma_{i+1})\Big(\,\mathrm{d}W + (\varphi - \sigma_{i+1})\,\mathrm{d}t\Big). \tag{16}$$

By introducing absolute forward rate volatilities

$$\beta_i = \frac{1}{\delta}(1 + \delta F_i)(\sigma_i - \sigma_{i+1}) \tag{17}$$

and drifted Brownian motions

$$\mathrm{d}W^j = \,\mathrm{d}W + \big(\varphi - \sigma_j\big)\,\mathrm{d}t \tag{18}$$

the expression (16) simplifies to

$$\begin{aligned}
\mathrm{d}F_i &= \beta_i\,\mathrm{d}W^{i+1}\\
&= \beta_i\big(\,\mathrm{d}W + (\varphi - \sigma_n)\,\mathrm{d}t - \sum_{j=i+1}^{n-1}(\sigma_j - \sigma_{j+1})\,\mathrm{d}t\big)\\
&= -\sum_{j=i+1}^{n-1}\beta_i(\sigma_j - \sigma_{j+1})\,\mathrm{d}t + \beta_i\,\mathrm{d}W^n\\
&= -\sum_{j=i+1}^{n-1}\frac{\delta\beta_i\beta_j}{(1 + \delta F_j)}\,\mathrm{d}t + \beta_i\,\mathrm{d}W^n,
\end{aligned} \tag{19}$$

where expression (19) directly relates forward rates to different numeraire measures.

Using relative forward rate volatilities $\gamma_i = \beta_i/F_i$ the relations become

$$\begin{aligned} \mathrm{d}F_i &= \gamma_i F_i \,\mathrm{d}W^{i+1} \\ &= -\sum_{j=i+1}^{n-1} \frac{\delta\gamma_i\gamma_j F_i F_j}{(1+\delta F_j)}\,\mathrm{d}t + \gamma_i F_i\,\mathrm{d}W^n, \end{aligned} \tag{20}$$

or

$$\begin{aligned} \frac{\mathrm{d}F_i}{F_i} &= \gamma_i \,\mathrm{d}W^{i+1} \\ &= -\sum_{j=i+1}^{n-1} \frac{\delta\gamma_i\gamma_j F_j}{(1+\delta F_j)}\,\mathrm{d}t + \gamma_i\,\mathrm{d}W^n. \end{aligned} \tag{21}$$

2.4 The BGM model

The BGM model [3], also called the Libor market model, is the evolution model for forward rates of the above type where the relative forward rate volatilities γ_i are deterministic functions. The forward rates are said to have a lognormal volatility structure as each of them are defined by a stochastic process of the form

$$\frac{\mathrm{d}F_i(t)}{F_i(t)} = \ldots dt + \gamma_i\,\mathrm{d}W, \tag{22}$$

where W is a standard d-dimensional Wiener process.

Under terminal measure $\mathbf{Q}_{P_N}$, using $P(t, T_N)$ as pricing numeraire, the forward rate processes are then governed by

$$\frac{\mathrm{d}F_i}{F_i} = -\sum_{j=i+1}^{N-1} \frac{\delta\gamma_i\gamma_j F_j}{(1+\delta F_j)}\,\mathrm{d}t + \gamma_i\,\mathrm{d}W^N. \tag{23}$$

These equations can be turned into equations in discrete time suitable for simulation. It is however not easy to do this without losing the arbitrage-free properties [4]. Instead, introduce $N-1$ state variables defined by

$$X_i(t) = F_i(t)\prod_{j=i+1}^{N-1}\left(1+\delta F_j(t)\right), \tag{24}$$

which, when found, can be returned to forward rates through

$$F_i(t) = \frac{X_i(t)}{1+\sum_{k=i+1}^{N-1}\delta X_k(t)}. \tag{25}$$

We state the following result from Glasserman [4], lemma 1.

Theorem 3 *Each X_i is a martingale that satisfies*

$$\frac{\mathrm{d}X_i(t)}{X_i(t)} = \left(\gamma_i(t) + \sum_{j=i+1}^{N-1}\frac{\delta\gamma_j(t)X_j(t)}{1+\sum_{k=j}^{N-1}\delta X_k(t)}\right)\mathrm{d}W^N. \tag{26}$$

The Euler discretisation of $\ln(X_i)$ with time step h is

$$\hat{X}_i((j+1)h) = \hat{X}_i(jh)\exp\left(-\frac{1}{2}\sigma_{\hat{X}_i}(jh)\sigma_{\hat{X}_i}(jh)h + \sqrt{h}\,\sigma_{\hat{X}_i}(jh)\Delta_{j+1}\right), \quad (27)$$

where

$$\sigma_{\hat{X}_i}(t) = \gamma_n(t) + \sum_{j=i+1}^{N-1} \frac{\delta \hat{X}_j(t)\gamma_j(t)}{1+\sum_{k=j}^{N-1}\delta \hat{X}_k(t)}. \quad (28)$$

and Δ_{i+1} is a d-dimensional vector of random samples from a standard normal distribution. These discrete variables were simulated, producing sufficient information for pricing derivatives based on forward rate.

3 Market instruments

3.1 Swaps

An interest rate swap is a contract between two parties where the buyer pays a contract specific fixed rate κ annually and receives a floating rate found in the interest rate market at specified dates. The size of each payment is based on the notional amount, the amount on which interest rate is calculated. The floating rate is almost always based on a 3- or 6-month interbank rate in the currency of the contract, paid at the end the period for which it is determined. For more information on swaps, see Hull [7] chapter 5.

The 3-month STIBOR (Stockholm Interbank Offered Rate) is used as reference for the floating rate side of Swedish swaps, paid at the end of the three months for which it is determined. The market considered will therefore be assumed to have a constant $\delta = \frac{1}{4}$ for a quarterly tenor structure. The fixed leg of the swap is changed from annual to quarterly payments in order to avoid adjustments for swaps that do not start with one year before first fixed payment.

A payer swap with a notional amount of one currency unit that starts at time T_s, has the first payment at time T_{s+1} and last payment at time T_e, has the time T_s value

$$SV_{s,e}(T_s) = \sum_{j=s+1}^{e} P_j(T_s)\mathbf{E}_{P_j}\left[\delta(F_{j-1}(T_s)-\kappa)|\mathcal{F}_{T_s}\right]. \quad (29)$$

The expectancy of the forward rates are simply

$$\mathbf{E}_{P_j}\left[F_{j-1}(T_s)|\mathcal{F}_{T_s}\right] = \frac{1}{\delta}\left(\frac{P_{j-1}(T_s)}{P_j(T_s)} - 1\right), \quad (30)$$

which inserted into (29) and using that $P(t,t)=1$ leads to the swap value given by

$$SV_{s,e}(T_s) = 1 - P_e(T_s) - \sum_{i=s+1}^{e} P_i(T_s)\delta\kappa. \quad (31)$$

The swap rate can be determined from this relation, as the swap rate is defined as the fixed rate $S_{s,e}(t)$ that makes the swap have zero value. Replacing κ with $S_{s,e}(t)$, equation (31) becomes

$$0 = 1 - P_e(T_s) - \sum_{i=s+1}^{e} P_i(T_s)\delta S_{s,e}(t),$$

so that

$$S_{s,e}(t) = \frac{1 - P_e(T_s)}{\delta \sum_{i=s+1}^{e} P_i(T_s)}. \tag{32}$$

3.2 Caps

An interest rate cap is a simple series of interest rate options, caplets, each paying the difference between the prevailing floating rate and a contract specific cap rate when the floating rate is higher. The size of each payment is based on the principal amount, the amount on which interest rate is calculated. As for swaps, the cashflow occurs at the end of the period for which the floating rate is determined. After the first period, however, most caps do not have any cashflow. The 3-month STIBOR is used as reference for the floating rate side of Swedish caps. For more information on caps, see Hull [7] pages 537-543.

A cap with a principal amount of one currency unit that starts at time T_s, has the first payment at time T_{s+2} and last payment at time T_e, has the time T_s value

$$CV_{s,e}(T_s) = \sum_{j=s+2}^{e} P_j(T_s)\mathbf{E}_{P_j}\left[max(\delta(F_{j-1}(T_s) - \kappa), 0)|\mathcal{F}_{T_s}\right]. \tag{33}$$

Assume that each forward rate is lognormal with a flat volatility σ_C, then the price of the cap can be determined through the Black-Scholes type formula

$$CV_{s,e}(T_s) = \sum_{j=s+2}^{e} P_j(T_s)\left[\delta(F_{j-1}(T_s)N(d_1) - \kappa N(d_2))\right], \tag{34}$$

where

$$d_1 = \frac{\ln(F_{j-1}(T_s)/\kappa) + \sigma_C^2(T_j - T_s)/2}{\sigma_C\sqrt{(T_j - T_s)}}, \tag{35}$$

$$d_2 = \frac{\ln(F_{j-1}(T_s)/\kappa) - \sigma_C^2(T_j - T_s)/2}{\sigma_C\sqrt{(T_j - T_s)}}, \tag{36}$$

Caps are quoted for indicative prices by the volatility σ_C for a κ equal to the swap rate at the time T_s for a swap that excludes the first cashflow. The formula (34) is used for calculating the corresponding price of the cap.

The lognormal assumption for the forward rates is identical to the lognormal volatility structure used for the BGM model, which makes caps ideal for calibrating the model.

3.3 European swap options

A European swap option is a standardised option on a swap. For more information on swap options, see Hull [7] pages 543-547.

A European call option exercisable at T_s on a payer swap with a notional amount of one currency unit, starting at time T_s with last payment at time T_e, has the time $t \leq T_s$ value

$$SOV_{s,e}(t) = P_s(t)\mathbf{E}_{P_s}\left[\left(\sum_{j=s+1}^{e} P_j(T_s)\mathbf{E}_{P_j}\left[\delta(F_{j-1}(T_s) - \kappa)|\mathcal{F}_{T_s}\right]\right)^+ |\mathcal{F}_t\right], \tag{37}$$

where κ is the strike swap rate and $(\cdot)^+$ is $\max(\cdot, 0)$.

Using the martingale property of the above expression, it can be simplified to

$$SOV_{s,e}(t) = P_s(t)\mathbf{E}_{P_s}\left[\left(\sum_{j=s+1}^{e} P_j(T_s)\delta(F_{j-1}(T_s) - \kappa)\right)^+ |\mathcal{F}_t\right]. \qquad (38)$$

Although equation (38) cannot be expressed through a Black-Scholes type formula under the multi-factor BGM model, several single factor approximations exist. The approximation suggested by Andersen [1] results in the formula

$$SOV_{s,e}(t) = \sum_{k=s}^{e-1} \delta P_k(t)\left[S_{s,e}(t)N(d_1) - \kappa N(d_2)\right], \qquad (39)$$

where

$$d_1 = \frac{\ln(S_{s,e}(t)/\kappa) + v_s/2}{\sqrt{v_s}}, \qquad (40)$$

$$d_2 = \frac{\ln(S_{s,e}(t)/\kappa) - v_s/2}{\sqrt{v_s}}, \qquad (41)$$

$$v_s(t, T_s) = \int_t^{T_s} \sum_{j=s}^{e-1} \left\|w_j(t)\gamma_j(u)\right\|^2 \,\mathrm{d}u, \qquad (42)$$

and

$$w_j(t) = \frac{F_j}{S_{s,e}(t)}\left[\frac{\delta S(t)}{1+\delta F_j}\left(\frac{P_e(t)}{P_s(t) - P_e(t)} + \frac{\sum_{k=j}^{e-1}\delta P_{k+1}(t)}{\sum_{k=s}^{e-1}\delta P_j(t)}\right)\right]. \qquad (43)$$

This approximation is used during the Monte Carlo simulation for evaluating whether option exercise is optimal. Note that equation (42) is slightly modified from the equation in Andersen's article as the original version gave quite strange results.

3.4 Bermudan swap options

A Bermudan swap option is not a standardised option, but is here defined as follows.

Definition 9 (Bermudan swap option) *A Bermudan swap option is an option on a swap with ending payment at a set date T_e and a variable starting date T_s. The starting date is the date when the holder chooses to exercise the option. Possible exercise dates range from a lock-out date, T_l, until one period before the last payment of the underlying swap, T_{e-1}.*

The Bermudan swap option can be viewed as a max option, taking the best value of European options on swaps with decreasing length. As soon as an exercise strategy is devised, the BGM model can be used for pricing Bermudan swap options.

4 Optimal exercise

When exercised, the Bermudan and the European swap options are both worth the difference between the values of a swap at strike rate and a swap at market rate. The difficulty of the Bemudan swap option is determining the optimal exercise time $\tilde{T}$ from the available possible exercise times $T_l \leq \tilde{T} \leq T_{e-1}$.

Let $I(t) = \{0, 1\}$ be a $\mathcal{F}_t$-adapted exercise indicator function, taking the value 1 when exercise is optimal, otherwise 0. How should such a function be designed?

4.1 Superoptimal exercise

In order to establish an upper bound for the premium of a Bermudan option compared to a European option, it is possible to locate, for each simulation, when the option had the highest value, or

$$BerSOV_{l,e}(t) = \max\left(0, P_l(t)SOV_{l,e}(T_l), \cdots, P_{e-1}(t)SOV_{e-1,e}(T_{e-1})\right). \quad (44)$$

This is defined as the superoptimal exercise strategy, as the exercise decision "looks into the future" and is clearly not adapted to the filtration of the Wiener processes. Instead, with this strategy the option is more of a look-back option, always taking the maximum value possible for the Bermudan swap option.

4.2 Suboptimal exercise

More important than finding the optimal exercise strategy is to find a simple enough exercise strategy that can be used during simulation with low cost of time. Two suboptimal strategies will be evaluated, one absolute and one relative exercise strategy, corresponding to strategies I and III in Andersen [2]

Definition 10 (absolute exercise strategy) *The absolute exercise strategy is defined by the exercise indicator*

$$I(T_i) = \begin{cases} 1 & \text{if } SV_{i,e}(T_i) > H(T_i), \\ 0 & \text{otherwise}, \end{cases} \quad (45)$$

where $H(t)$ is an absolute exercise boundary function.

Definition 11 (relative exercise strategy) *The relative exercise strategy is defined by the exercise indicator*

$$I(T_i) = \begin{cases} 1 & \text{if } SV_{i,e}(T_i) > H(T_i) + \max(SOV_{j,e}(T_i)) \ \forall j > i, \\ 0 & \text{otherwise}, \end{cases} \quad (46)$$

where $H(t)$ is a relative exercise boundary function.

In both suboptimal exercise strategies the boundary function $H(t)$ is calculated by running a set of simulations, called calibration simulations, and assigning the value to each $H(T_i)$ that maximises the option value. As the last possible exercise date is T_{e-1} the boundary function always ends with $H(T_{e-1}) = 0$. Looking backwards at the preceding exercise date, and assuming that the option has not yet been exercised, the boundary is found by linear search of a range starting at zero and ending above the largest swap value attained during the simulations. This procedure is repeated until all exercise dates has a corresponding value of the boundary function.

When $H(t)$ has been determined, the calibration simulations are discarded in order to avoid a perfect forsight bias [2]. A separate set of simulations are run, pricing the option by use of the exercise strategy with the recently determined boundary function.

5 Model calibration

The BGM model has several variables that must be calibrated before any simulation can take place. As the market quotations used for calibration are always divided into bid and ask, a mid value, calculated as halfway between the bid and ask quotations, is used for each necessary value.

5.1 Forward rates

Initial values of the X martingale are calculated through equation (25) using initial values of the forward rates. These in turn are calculated from an interest rate curve based on instruments comparable in risk to STIBOR loans.

The instruments used here are deposits and swaps. Deposits are simple instruments with only one cashflow each, and need no further treatment. Swaps have fixed annual cashflows, so all but one cashflow must be removed for calculating the interest rate for the date of that cashflow. The Bootstrap method is used for this, using the interest rate of a one year instrument to discount the first year's cashflow of a two year instrument, and so on. The resulting interest rates, called zero coupon rates, reflect the market price of single cashflows on given dates. Yearly effective interpolation of these zero coupon rates completes the interest rate curve for all dates.

5.2 Volatility and correlation

The relative forward rate volatilities γ in equation (23) are calibrated in order for the simulator to price caps correctly and to produce a model correlation structure between forward rates that is close to the correlations in the interest rate market. Using the method of Rebonato [8], these tasks can be performed separately.

The relative forward rate volatility matrix is separated into two components, $\gamma_{ik} = b_{ik}s_i$, where s_i is a vector of volatilities for each forward rate F_i and b_{ik} is a matrix designed to catch the correlation structure.

As there are quotatations for caps starting with one year, the first year's forward rate volatilities are set to the flat volatility of a one year cap. For subsequent years, the volatility of forward rates are found by inverting formula (34) for the last part of a cap, under the no arbitrage assumption that the first part of the cap has the same value as the previous cap. The formula (34) has no analytical inversion, so finding the volatility from the price is performed through iteration.

As for the market correlation, there are no traded instruments that produce an implicit correlation structure. Instead, historical correlation is calculated through basic time series analysis of the forward rate movements the last months. This produces an NxN market correlation matrix $Corr^{market}$ used for estimating the future correlation between forward rates.

A general description of the (b_{ik}) matrix is, according to Rebonato [8]

$$
\begin{aligned}
b_{ik} &= \cos(\theta_{ik}) \prod_{j=1}^{k-1} \sin(\theta_{ij}), \;\; k = 1, s-1, \\
b_{ik} &= \prod_{j=1}^{k-1} \sin(\theta_{ij}), \;\; k = s,
\end{aligned} \tag{47}
$$

where θ_{ij} are angles to be determined. The aim is to minimise the difference between the model correlation,

$$Corr^{mod} = BB^t,$$

and the market correlation matrix. Equations (47) ensures that

$$\sum_{k=1}^{s} b_{ik}^2 = 1, \;\; \forall i,$$

producing the vital property of a diagonal of ones in the model correlation matrix. For determining the other matrix elements, a fitting error is defined as the sum of square discrepancies between the model and market correlation matrices. The Levenberg-Marquardt optimising algorithm is used to find optimal angles.

6 Numerical results

Interest rate market data was sampled from values of Swedish deposits, swaps and caps from 2002-04-08 to 2003-04-08. Specifically, the options pricing is performed for 2003-04-08. The results obtained are here graphically presented. For actual values, see appendix.

The Mersenne Twister is used for generating pseudorandom numbers used for the Monte Carlo simulation.

6.1 Calibration tests

The parameters of the evolution equation (27) are calibrated to market forward rates and the caplet prices calculated from market date. A perfect model would recover these data after simulation.

Comparing the market forward rates with average forward rates of 10000, 100000, 1000000 and 5000000 simulation trials, figure 1 shows a systematic error of up to 5 basis points (0.05 percentage points).

This phenomenon has been observed in other studies [4] and and can be altered but not removed with a different choice of measure for relating forward rates in the model. The terminal measure used here imposes an error free final forward rate. Due to the size of the systematic errors a compensation sceme must be devised, see the discussion. More important at this stage, practically no additional information is added after 100000 simulations.

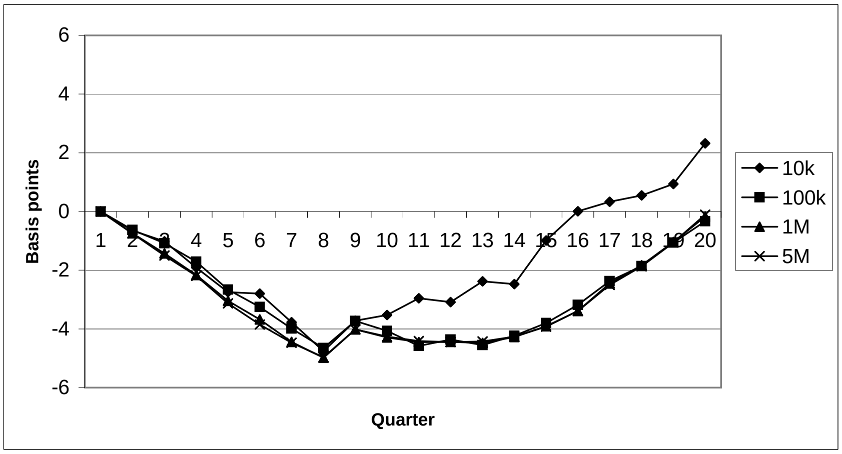

Figure 1: Absolute difference between forward rates up to 5 years

Comparing the caplet prices calculated from market data with average caplet prices of 10000, 100000, 1000000 and 5000000 simulation trials, figure 2 shows a systematic error of up to 6% for the BGM model under terminal measure. The error is quite large, but seems heavily correlated with the systematic error in simulated forward rates. Again, practically no additional information is added after 100000 simulations.

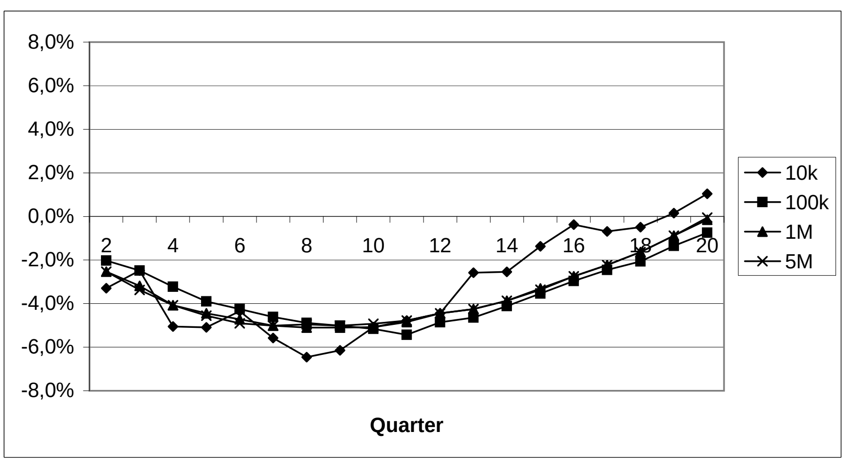

Figure 2: Relative price difference of caplets up to 5 years

The conclusion of the calibration tests is that at least 100000 simulation trials must be performed for pricing a claim with sufficiently small non-systematic errors.

6.2 Approximation test

Figure 3 compares the time zero European swap option prices of different maturities on a swap ending after five years, calculated using the approximation (39) or from 100000 simulations. It shows that in general the approximation yields a lower price than the simulation. This is good news, as the boundary function with its current design only adds to the maximum option price of future European swap options.

The approximation error is defined as the difference between the simulated price and the approximation.

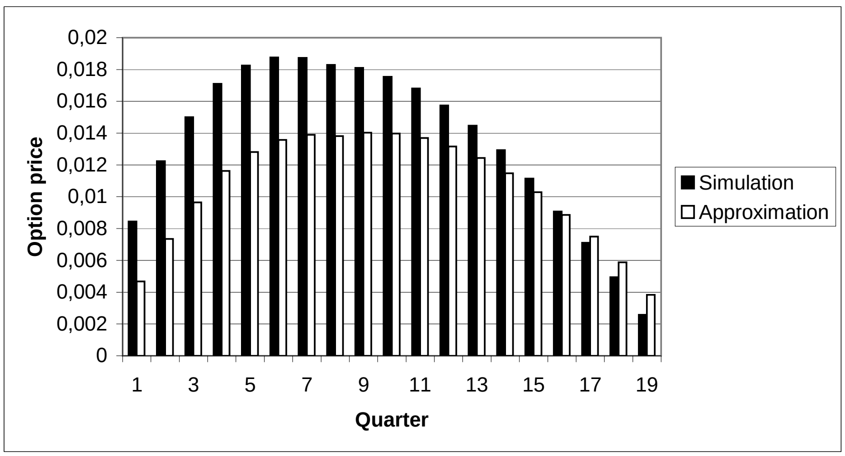

Figure 3: Prices of European options on a swap ending after five years

6.3 Determining the boundary function

Figure 4 shows the boundary functions for a five year Bermudan swap option using the two exercise strategies (45) and (46), found through 10000 calibration simulations. As it is rarely optimal to exercise this Bermudan swap option until after at least a year, the first part of both exercise boundaries is slightly erratic while the latter part behaves nicely. The relative boundary function only contains small values. The strategy almost entirely relies on the approximate values of future European options.

Comparing the time zero approximation error with the relative boundary function, figure 5 shows that the boundary function partly compensates for the error. Note that this comparison is rather rough as the approximation is calculated at time zero while the relative boundary is optimised to yield the highest option price at time of exercise. The aim of this comparison is merely to highlight the sign of the approximation error.

6.4 Option pricing

Options on swaps ending after two, tree, four and five years were priced, each in a series of ten independent runs. Each option pricing started with 10000

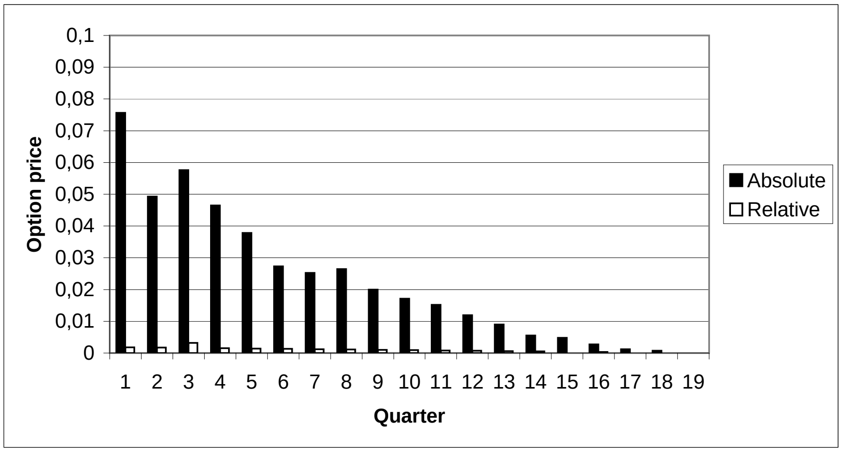

Figure 4: Boundary functions for different strategies

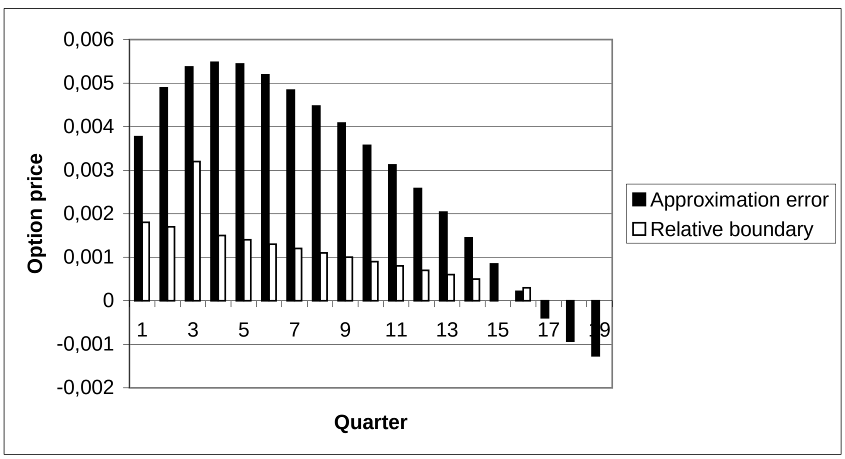

Figure 5: Comparison between the relative boundary function and the approximation error

calibration simulations that were used for finding the optimal boundary function (either absolute or relative). The calibration simulations were discarded and the boundary function kept. The option price was determined through 100000 simulations, using the exercise strategy determined by the boundary function.

Figure 6 shows the average value of the best European option, and then the Bermudan option value using the three different exercise strategies: suboptimal absolute exercise (45), suboptimal relative exercise (46) and superoptimal exercise (44). Figure 7 shows the corresponding standard deviation.

The first notable result in figure 6 is the significant difference between the superoptimal strategy and the suboptimal strategies. As the suboptimal strategies represent a more realistic valuation, we conclude that the superoptimal strategy was greatly overpricing the Bermudan swap option. A better superoptimal

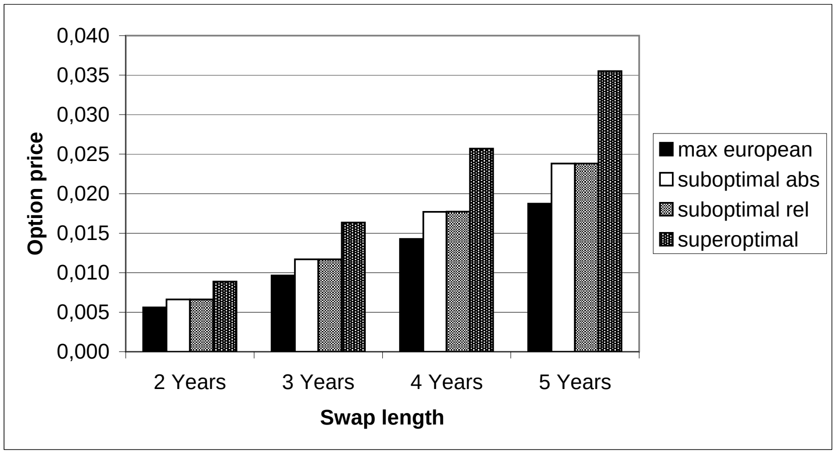

Figure 6: Option prices for different exercise strategies

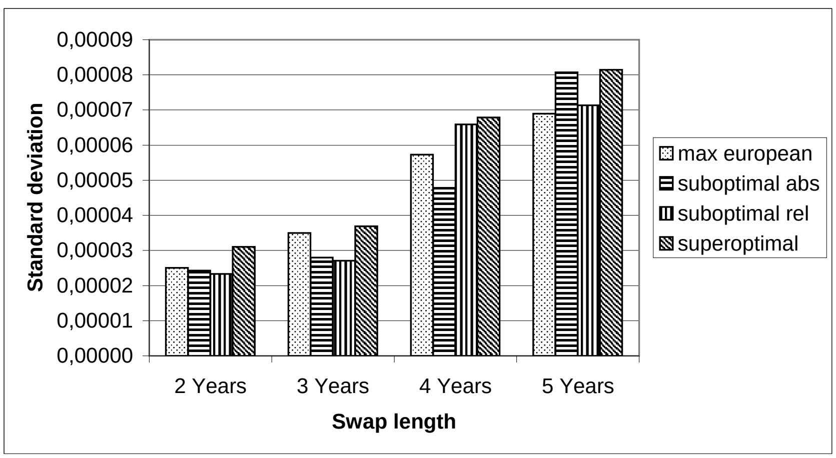

Figure 7: Standard deviation of the option prices

strategy is needed.

Looking at the prices of figure 6 and the standard deviations of figure 7, both the price difference between the best European option and the Bermudan options and the price difference between any of the suboptimal strategies and the superoptimal strategy can be considered statistically significant. The only prices that cannot be separated are the prices using the two different suboptimal exercise strategies on a swap with the same length. None of the two strategies can therefore be said to outperform the other.

Each pricing run of an option on a five year swap in my calculations on an ordinary PC (1 GHz processor) took about half an hour, the time roughly increasing with the square of the swap length.

Analysing the exercise pattern of the studied Bermudan options, figure 8 shows the price of an option on a five year swap divided according to exercise quarter. More than 60% of the option price stems from the five quarters centred on the quarter with the best European option, quarter 6.

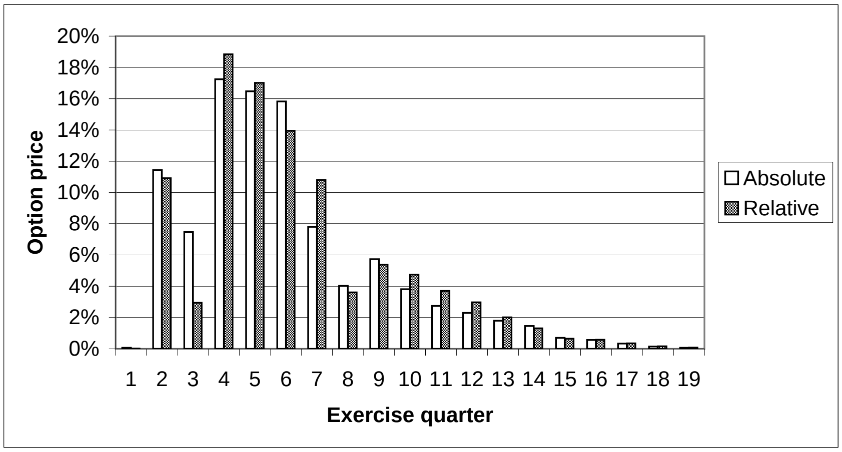

Figure 8: Option price from different exercise times on a five year swap

7 Discussion

This study has found that practical pricing of Bermudan swap options is feasible. Already at 100000 pricing simulations interesting results can be obtained. The half an hour of simulation time needed for pricing a five year claim, can be reduced to a few minutes through optimised code, variance reduction techniques and parallel computing if the need arises. This time is not good enough for real-time trading but enough for selling options over-the-counter.

In line with the conclusion of Andersen [2], the relative exercise strategy, using the approximate future European swap option values, is found not to be better than the absolute exercise strategy. This, together with the higher complexity of the relative strategy, makes the absolute strategy the exercise strategy of choice. This does not, however, rule out the possibility that the relative strategy can be improved. The relative strategy is highly dependant on the precision of the approximation, a precision that unfortunately is not very high (figure 3). It also cannot in its current form compensate for instances when the approximation yields a higher price than the simulation (figure 5).

In order to find a superoptimal exercise value that is closer to the theoretical optimal value of the Bermudan option a strategy using a more realistic forecast than the whole swap duration must be devised. A suggested strategy for further analysis is a strategy based on a single period perfect forecast.

The systematic error in the forward rates due to the choice of the terminal measure (figure 1) leads to an unknown systematic error in the swap option pricing. As the market price of European swap options are often available to traders, these can be used as control variates to improve the reliability of the Bermudan option prices and remove some of the systematic errors. This is accomplished by taking the simulation price difference between, the best European and the Bermudan swap option and adding this to the market price of the European swap option. Figure 8 supports this view as the exercised Bermudan options are more often than not priced using nearly the same forward rates as the best European option. Another alternative could be using a basket of known European options for comparison.

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A Appendix: Numerical values

Quarter	*Forward rate*	*Simulation rate difference (bp)*			
		10k	*100k*	*1M*	*5M*
1	4.1450%	0	0	0	0
2	4.7557%	-0.65	-0.63	-0.74	-0.75
3	5.0520%	-1.02	-1.08	-1.43	-1.50
4	5.3946%	-1.91	-1.71	-2.16	-2.19
5	5.5305%	-2.75	-2.66	-3.04	-3.13
6	5.8073%	-2.80	-3.25	-3.68	-3.85
7	6.0837%	-3.77	-3.99	-4.44	-4.47
8	6.3597%	-4.76	-4.65	-4.98	-4.97
9	5.9528%	-3.72	-3.73	-4.02	-4.02
10	6.0775%	-3.53	-4.07	-4.29	-4.26
11	6.2021%	-2.96	-4.58	-4.45	-4.41
12	6.3266%	-3.09	-4.36	-4.44	-4.45
13	6.1671%	-2.38	-4.55	-4.48	-4.43
14	6.2480%	-2.47	-4.23	-4.28	-4.26
15	6.3289%	-0.99	-3.80	-3.91	-3.92
16	6.4098%	0.01	-3.18	-3.39	-3.39
17	6.3283%	0.33	-2.37	-2.45	-2.50
18	6.3900%	0.55	-1.86	-1.83	-1.87
19	6.4518%	0.94	-1.06	-1.04	-1.05
20	6.5135%	2.32	-0.33	-0.17	-0.11

Table 1: Comparison between true and simulated forward rates in figure 1

Quarter	*Caplet price*	*Simulation rate difference (bp)*			
		10k	*100k*	*1M*	*5M*
2	0.000427	-3.2911%	-2.0253%	-2.5316%	-2.5316%
3	0.000641	-2.4823%	-2.4823%	-3.1915%	-3.3688%
4	0.000838	-5.0562%	-3.2303%	-4.0730%	-4.0730%
5	0.001165	-5.0921%	-3.9003%	-4.4420%	-4.5504%
6	0.001367	-4.3478%	-4.2553%	-4.7179%	-4.9029%
7	0.001567	-5.5780%	-4.6079%	-5.0121%	-5.0121%
8	0.001769	-6.4609%	-4.8816%	-5.0969%	-4.9533%
9	0.001611	-6.1419%	-5.0173%	-5.1038%	-5.0173%
10	0.001744	-5.0834%	-5.1628%	-5.0834%	-4.9245%
11	0.001875	-4.7759%	-5.4372%	-4.8494%	-4.7759%
12	0.002006	-4.4460%	-4.8564%	-4.4460%	-4.4460%
13	0.002083	-2.5773%	-4.6392%	-4.2526%	-4.2526%
14	0.002196	-2.5408%	-4.1137%	-3.8717%	-3.8717%
15	0.002307	-1.3683%	-3.5348%	-3.3067%	-3.3637%
16	0.002418	-0.3772%	-2.9634%	-2.7478%	-2.7478%
17	0.002274	-0.6857%	-2.4571%	-2.2286%	-2.2286%
18	0.002366	-0.4899%	-2.0686%	-1.6331%	-1.6331%
19	0.002457	0.1559%	-1.3514%	-0.8836%	-0.8836%
20	0.002548	1.0443%	-0.7459%	-0.1492%	-0.0497%

Table 2: Comparison between true and simulated caplet prices in figure 2

Quarter	*Simulation*	*Approximation*
1	0.008441	0.004671
2	0.012238	0.007347
3	0.015009	0.009641
4	0.017100	0.011621
5	0.018248	0.012811
6	0.018756	0.013568
7	0.018737	0.013901
8	0.018287	0.013817
9	0.018102	0.014022
10	0.017546	0.013978
11	0.016812	0.013691
12	0.015744	0.013166
13	0.014475	0.012438
14	0.012929	0.011481
15	0.011137	0.010293
16	0.009074	0.008859
17	0.007108	0.007499
18	0.004948	0.005873
19	0.002573	0.003837

Table 3: Comparison between simulated and approximated European option values in figure 3

Quarter	*Absolute*	*Relative*	*Approx. error*
1	0.0756	0.0018	0.003770.
2	0.0493	0.0017	0.004891.
3	0.0576	0.0032	0.005368.
4	0.0465	0.0015	0.005479.
5	0.0378	0.0014	0.005437.
6	0.0273	0.0013	0.005188.
7	0.0252	0.0012	0.004836.
8	0.0264	0.0011	0.004470.
9	0.0200	0.0010	0.004080.
10	0.0171	0.0009	0.003568.
11	0.0152	0.0008	0.003121.
12	0.0119	0.0007	0.002578.
13	0.0090	0.0006	0.002037.
14	0.0055	0.0005	0.001448.
15	0.0048	0.0000	0.000844.
16	0.0027	0.0003	0.000215.
17	0.0012	0.0000	-0.000391.
18	0.0007	0.0000	-0.000925.
19	0.0000	0.0000	-0.001264.

Table 4: Boundary functions for different strategies and approximation error in figure 4 (columns 2 and 3) and figure 5 (columns 3 and 4)

2 years swap

	max European	*suboptimal (abs)*	*suboptimal (rel)*	*superoptimal*
Average	0.005603	0.006625	0.006629	0.008869
Std dev	0.000025	0.000024	0.000023	0.000031

3 years swap

	max European	*suboptimal (abs)*	*suboptimal (rel)*	*superoptimal*
Average	0.009657	0.011692	0.011690	0.016357
Std dev	0.000035	0.000028	0.000027	0.000037

4 years swap

	max European	*suboptimal (abs)*	*suboptimal (rel)*	*superoptimal*
Average	0.014264	0.017708	0.017714	0.025698
Std dev	0.000057	0.000048	0.000066	0.000068

5 years swap

	max European	*suboptimal (abs)*	*suboptimal (rel)*	*superoptimal*
Average	0.018732	0.023823	0.023817	0.035501
Std dev	0.000069	0.000081	0.000071	0.000081

Table 5: Comparison between different exercise strategies in figure 6 and figure 7

Quarter	*Absolute*	*Relative*
1	0.1%	0.0%
2	11.4%	10.9%
3	7.5%	2.9%
4	17.2%	18.8%
5	16.5%	17.0%
6	15.8%	13.9%
7	7.8%	10.8%
8	4.0%	3.6%
9	5.7%	5.4%
10	3.8%	4.7%
11	2.7%	3.7%
12	2.3%	3.0%
13	1.8%	2.0%
14	1.5%	1.3%
15	0.7%	0.6%
16	0.6%	0.6%
17	0.3%	0.4%
18	0.1%	0.2%
19	0.1%	0.1%

Table 6: Exercise price weights in figure 8