FINANCIAL MODELING: OPTIONS, SWAPS AND DERIVATIVES

http://www.lifelong-learners.com/opt

André Jaun <jaun@kth,se>

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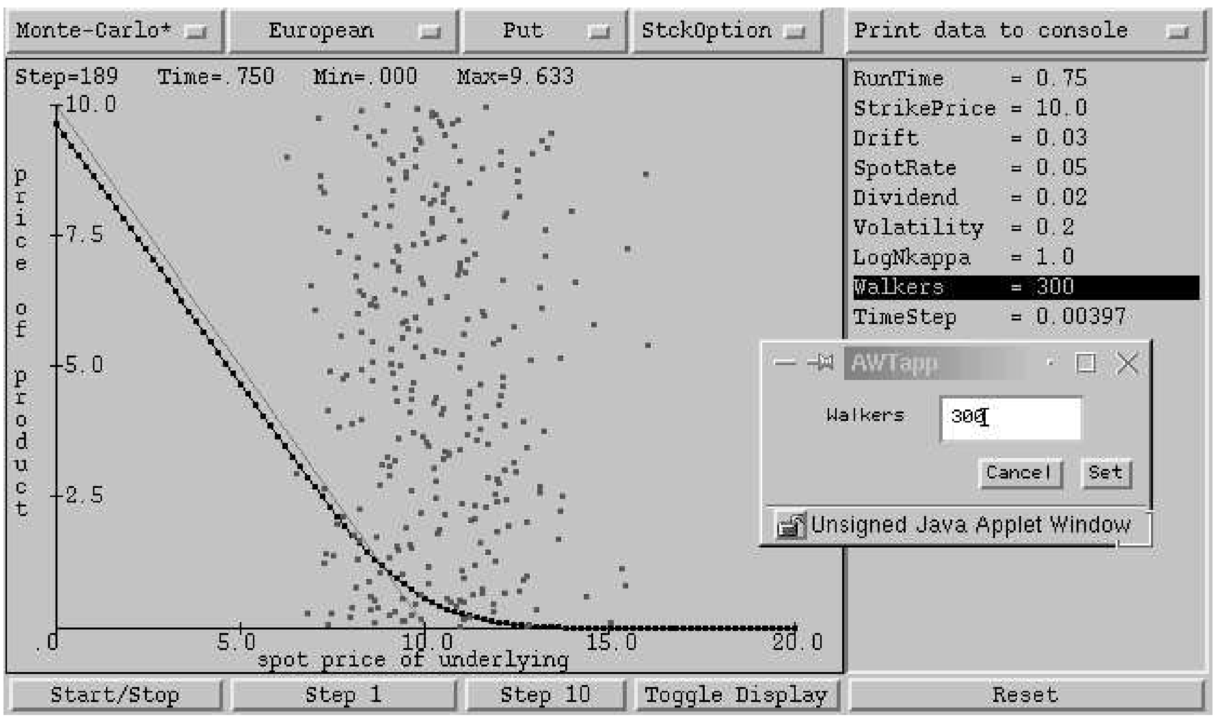

VMARKET applet: the Black-Scholes equation is here solved for a European vanilla put option using a Monte-Carlo method.

2004 edition for distance-learning students from the Swedish Netuniversity, the Master in Applied Finance programme at the University of Adelaide, the School for Financial Mathematics at the University of Gdansk, and students from the Royal Institute of Technology, Stockholm.

This document has been typeset with LaTeX 2ε[1] enriched with `latex2html` macros.

[1] http://www.tug.org/

Contents

1 INTRODUCTION **3**
1.1 How to study this course at 3 levels: the meaning of ♢,♠ 3
1.2 Capital and markets 6
1.3 The risk and return from conventional assets 8
1.4 Modern portfolio theory and basic risk management strategies ♢ 13
1.5 Historical data and modeling 15
1.5.1 Drift and volatility of market prices 15
1.5.2 Moving averages: UWMA, EWMA, GARCH ♢ 16
1.5.3 Maximum likelihood estimate of parameters ♠ 18
1.6 Computer quiz 21
1.7 Exercises 22
1.8 Further reading and links 24
1.9 Quick intermediate evaluation form 25

2 A VARIETY OF SECURITIES **27**
2.1 The stock market and its derivatives 27
2.1.1 Shares and market indices 27
2.1.2 Forward contract and futures markets 30
2.1.3 Plain vanilla options 31
2.1.4 Exotic options ♢ 33
2.1.5 LEAPS and warrants 35
2.2 The credit market and its derivatives 36
2.2.1 Interest rates: treasury note, LIBOR, credit spread 36
2.2.2 Underlying discount bonds and forward rates ♢ 37
2.2.3 Interest rate swaps and forward rate agreements ♢ 39
2.2.4 Bond options: caps, floors and swaptions ♢ 40
2.3 Convertible bonds 41
2.4 Hedging parameters, portfolio sensitivity ♢ 42
2.5 Computer quiz 43
2.6 Exercises 44
2.7 Further reading and links 46
2.8 Quick intermediate evaluation form 46

3 FORECASTING WITH UNCERTAINTY **47**
3.1 Option pricing for dummies 47
3.2 Simple valuation model using binomial trees ♢ 48
3.3 Improved model using stochastic calculus 51
3.3.1 Wiener process and martingales ♠ 51
3.3.2 Itô lemma ♠ 52
3.3.3 Evaluate an expectancy or eliminate the uncertainty ♠ 53
3.4 Hedging an option with the underlying (Black-Scholes) ♢ 54
3.5 Hedging a bond with another bond (Vasicek) ♠ 55
3.6 Computer quiz 56
3.7 Exercises 57
3.8 Further reading and links 58
3.9 Quick intermediate evaluation form 58

4 EUROPEAN OPTION PAYOFF DYNAMICS **59**
4.1 Plain vanilla stock options 59
4.1.1 The European Black-Scholes model for dummies 59
4.1.2 Parameters illustrated with VMARKET experiments 61
4.1.3 Application, time value and implied volatility 63
4.2 Exotic stock options 64
4.2.1 Binary options 64
4.2.2 Barrier options 65
4.3 Methods for European options: analytic formulation 66
4.3.1 Transformation to log-normal variables ♠ 66
4.3.2 Solution of the normalized diffusion equation ♠ 67
4.3.3 Black-Scholes formula ♢ 68
4.4 Methods for European options: finite differences (FD) 69
4.4.1 Naive implementation using financial variables ♢ 69
4.4.2 Improved scheme using log-normal variables ♠ 71
4.5 Methods for European options: Monte-Carlo sampling (MCS) 74
4.5.1 Forecast possible realizations of the underlying asset ♠ 74
4.5.2 Expected value of an option from sampled data ♠ 76
4.6 Computer quiz 78
4.7 Exercises 78
4.8 Further reading and links 80
4.9 Quick intermediate evaluation form 80

5 BONDS, SWAPS AND DERIVATIVES **81**
5.1 Discound bonds 81
5.1.1 Term structure models for dummies 81
5.1.2 Parameters illustrated with VMARKET experiments 83
5.2 Credit derivatives 84
5.2.1 Vanilla swaps 85
5.2.2 Cap-/floorlets 86
5.3 Methods for bonds and derivatives: finite elements (FEM) 87
5.3.1 The Vasicek model for a bond ♠ 87
5.3.2 Extensions for derivatives ♠ 91
5.4 Computer quiz 92
5.5 Exercises 93
5.6 Further reading and links 94
5.7 Quick intermediate evaluation form 94

6 AMERICAN OPTION PAYOFF DYNAMICS **95**
6.1 American stock options 95
6.1.1 The American Black-Scholes model for dummies 95
6.1.2 Parameters illustrated with VMARKET experiments 96
6.1.3 Application 96
6.2 Methods for American options: finite elements (FEM) ♠ 97
6.2.1 The Black-Scholes equation for American options 97
6.2.2 Solution of the variational inequality using finite elements 98
6.3 Computer quiz 101
6.4 Exercises 101

6.5 Further reading and links 102
6.6 Quick intermediate evaluation form 102

7 EXTREMAL EVENTS **103**
7.1 Basel accord on banking 103
7.2 Value at risk (VaR) 103
7.3 Copulas for risk management 103

8 MULTI-FACTOR MODELS **105**
8.1 Principal components analysis 105
8.2 Martingales and measures 105
8.3 Two factor models 105

9 LEARNING LABORATORY ENVIRONEMENT **107**
9.1 Typesetting with TEX 107
9.2 Programming in JAVA 109
9.3 VMarket parameters and preset in HTML 111
9.4 Quick intermediate evaluation form 113

10 APPENDIX **115**
10.1 Glossary of keywords 115
10.2 Notation and symbols 120
10.3 Final interactive evaluation form 120

PREFACE

This is the syllabus of the course taught at the Royal Institute of Technology in Stockholm (KTH, course 2D5244, 4 points), the University of Adelaide (Master of Applied Finance programme), the Gdansk University of Technology and the Swedish Netuniversity (course 2D4282, 6 ECTS) and is also shared with lifelong-learners on the Internet. The goal is to familiarize students with tools that are commonly used in the modeling of financial products, using real-time data and numerical models to test hedging strategies in a problem-based learning (PBL) environment.

The material has been designed so that it can be studied at three different levels depending on the mathematical background and the ambition of each participant. At the basic level, the concepts are explained using numerical simulation instead of formulae: virtual market experiments, such as the VMARKET applet on-line,[2] translate financial arguments into an intuition for the subject without any mathematics. At an intermediate level, simple algebra is used to complete the basic picture: sections that are marked with a diamond suit$^\diamondsuit$ in the printed edition appear in grey or black in the document on-line depending on the profile that is associated with every user at login. Such content can be skipped without fear that this will later preclude the understanding of the material at a basic level. In the same manner, advanced sections that are identified with a spade suit$^\spadesuit$ enable graduates from quantitative fields to use stochastic calculus, formulate their own models and implement them numerically to calculate the price of exotic contracts. Even if the applets have been specifically designed for this syllabus, virtual experiments also provide useful complements to more classical textbooks, such as the book from *J. Hull* [11].

Over the years, M. Bungener, P. Cotton and J. Paget have been an important source of inspiration and it is a pleasure to acknowledge them here. A considerable amount of work was necessary to produce a complete problem-based learning environment, with a richness and complexity that can only multiply mistakes ranging from typos, inconsistencies to programming errors. An large effort has been made to guarantee the highest standards and I would like to thank T. Hurtig and S. González for their contributions. Needless to say that the responsibility of the remaining mistakes is all mine and I will be grateful for criticism and encouragement from the learners directly in the user forum on-line.

André JAUN, Stockholm, July 2004

[2] accessible, after login, using a Java-powered browser from `http://www.lifelong-learners.com/opt`

1 INTRODUCTION

1.1 How to study this course at 3 levels: the meaning of ♢, ♠

Studying is fundamentally an individual process, where every student has to find out himself what is the most efficient method to understand and assimilate new concepts. Experience however shows that major steps are taken when a theory is first exposed by the teacher (in a regular classroom, a video-lecture or a syllabus), later reviewed and discussed with peers (best accross the table during coffee break, or if this is not possible, in a video-conference, during user forum discussions or even a computer quiz) and finally applied to solve practical problems.

The educational tools, which have been developed to study the course on-line[3] reflect this pedagogical understanding. They can be combined in different manners, using technology to provide flexibility to study at the place, time, pace and level that best suits every learner. The progress made by every learner is continuously monitored with a system of *bonus points:* they reward original contributions from different activities, including user forum discussions and assignments that are performed with the help an corrections from a human teacher. Particularly nice solutions will be selected for reference and shared for discussions with the rest of the class.

An example showing how you can study the material during a typical day of an intensive course involves three distinct phases; those marked with an asterisk* require that you login to enjoy the full pedagogical support.

Passive learning (1h). This is when new concepts are first brought to you and you only have to carefully follow the teacher's line of thought. In this phase, you may combine

- **Video-Lecture.*** From the course main page, select *COURSE: video-lecture* and download the video file once for all to your local disk (press SHIFT + select link). This enables you to scroll back and forth, stop and replay different arguments in the lecture. If they are present, you can use the under-titles for synchronization with the syllabus on-line and perform the experiments directly as they are discussed. Open your video player next to, or on the top of your web browser to work with both tools simultaneously–Windows users can select *Always on top when playing*).
- **Syllabus.** Select the *COURSE: syllabus* to access the Java-powered document where you can execute all the virtual experiments on-line. As an alternative, you can also download and print the equivalent paper edition in *PDF* or *Postsript* format, using with the browser only to perform the experiments and to follow the links that appear in italic in the printed edition. Depending on your level and ambition, you can choose either to read or skip the material that is labelled on paper with symbols for intermediate♢ and more advanced♣ students. After login, the same material on-line will be displayed with grey fonts if a particular section is more advanced than the level that has been defined for you at the time of the registration.

Active learning (2h). Following the passive phase, you are meant to question the validity of the new concepts, verify the calculations and test parametric dependencies.

[3] http://www.lifelong-learners.com/opt

- **Syllabus.** Repeat the analytical derivations that are on purpose left scarce to force you to fill-in the intermediate steps by yourself.
- **Applets.*** Perform the numerical experiments that are suggested and modify the parameters to challenge your understanding. The original values can always be recovered with a partial reload of the webpage–simply by pressing the F5 key with Microsoft Explorer or selecting *Reload* with Netscape, Mozilla and Firefox.
- **Quiz.*** Answer the review questions making sure that you properly understood all the material. Reading the syllabus on-line, you can verify your answer and follow a correction link directly back into the syllabus.
- **Tutorials / Video-Conference.*** With a sufficient number of participants, tutorials (locally) or video-conferences (at a distance) are sometimes organized to discuss and refine the understanding that has PREVIOUSLY BEEN ACQUIRED in the passive phase. This is an opportunity for everyone not only to ask, but also to answer and comment the questions from peers.
- **User Forum.*** Regular students choose the *classroom* (others the *world*) forum both to obtain and provide help and also to improve the general understanding of the material. You are strongly encouraged to discuss related topics and share your views with answers to your classmates. Remember that this virtual classroom activity is mandatory and rewarded with 1-5 bonus points depending on the effort made for every contribution. Note that it does not really matter whether your arguments are correct or not: it is the teachers' duty to correct potential errors. Consult the *Forum: rules* and take a minute to think how you can make your contributions beneficial for everyone in the course (exercise 1.00).

Problem based learning (5h). Having understood the principles, a new skill is finally acquired by solving practical problems. Select *USER: login* to open your personal account and list your problem set under *WORK: assignments*. Each exercise can be edited in your browser by clicking on the identification number (e.g. *1.00*): below the handout, different windows invite you to edit (alt. cut-paste from an editor) and then submit your solution to different compilers:

- **TeX.*** The first window can handle both regular text (ASCII) and LaTeX input♠, allowing advanced students to enrich their solutions with mathematical derivations (symbols inserted between two dollar signs, such as `$c=\sqrt{a^2+b^2}$`, will appear as regular algebra $c = \sqrt{a^2 + b^2}$ in your web browser). In this TeX window, you should explain how you derive your solution, how you implement it and discuss the numerical values or plots you have observed in your experiments. Users who are not familiar with LaTeX generally find it easy to perform only small modifications of the templates that are provided for every new assignment. For documentation, consult the list of *symbols* in sect.9.1, which is most conveniently accessed using the link directly on the top of the TeX input window.
- **JAVA.*** The content of the JAVA window will be inserted and compiled into an actual applet, allowing advanced students to develop and execute their own numerical schemes directly on-line. It is not necessary to know any Java programming to follow this course: most of the tasks involve small modifications of templates that are given and part of the syntax will automatically be acquired

through the context. Be careful, however, to always correct all the compiler errors before you switch to another exercise... or the applet will stop working in all your assignments. For documentation concerning *Java* consult chapter 9.2, which can again be accessed directly from the top of the Java input window.

Important for advanced students who will perform modifications to their Java code: most browsers store the applet once for all in a local cache directory. To access the newly compiled version of your own applet, you have to force your browser to COMPLETELY reload the solution web page where the applet appears (check the frequently asked questions FAQ to find out how you can do this by clicking RIGHT in the white area and then press CTRL-F5 with Microsoft Explorer or press SHIFT + select *Reload* using Netscape, Mozilla and Firefox. Finally, if you don't get immediate programming advice from the *User Forum*, you may temporarily deactivate a problematic scheme using the `/* Java comment delimiters */`.

- **Parameters.*** The *tags* window allows you to preset parameter values in your applet that are different in every exercise: choose them so as to highlight the phenomenon you want to illustrate. Only the parameters that appear in the tags will also be displayed in the applet.
- **Figures.** Screen copies produced with external software can be submitted as figures in bitmat format: png, gif, jpeg in decreasing order of preference.

Finally, be sure to submit only one input window at any time and always compile your work before you navigate further in the syllabus or in the forum. Sometimes the *Back* button of your browser may restore data that has been lost... but don't count on it! As soon as your solution is ready or when you need a specific piece of advice that only the teacher can provide, click on the *CheckMe* button (appearing on the left of every *WORK: assignment* after the first compilation) and press *Submit Check* (at the bottom of the table) to send your solution for correction to the teacher. Take into account the corrections that will be returned after a couple of days until the solution is accepted and your exercise is signalled as passed.

- **Evaluation.*** The last section of every chapter consists of a short anonymous *evaluation form* where you are kindly requested to communicate your impressions each time before you start a new chapter. By sharing your impressions as you work yourself through the material, we will try to maximise your satifaction not only at the end of the course, but also optimize the path leading there.

The amount of work in each module is sufficiently large that it is usually not possible to complete all the course requirements within the short duration of an intensive course; rather than proceeding sequentially, it is then important that you start at least one assignment before every topic is discussed in a tutorial / video-conference. Remember that these are not lectures and tend to be useless if you are not at all familiar with the course material.

Project (1 week). Regular students are given an opportunity to apply their newly acquired skills in a topic that could be of interest for their own research. The intention is to reward taking a risk (stricktly limited to one week), to assess whether some tools could potentially result in an useful development in the frame of their PhD thesis. A small report with no more than six A4 pages will be published under the course main web page.

1.2 Capital and markets

Most of the ideas discussed in this course derive from one particular model of the society called *capitalism*. At the core lies an idea that *capital* (indeed any kind of *asset* such as money, raw material, even patents) owned by an individual (the *investor*) can be lent to another (the *entrepreneur*) to produce a certain number of goods or services. The separation of roles played by the owner and producer is not granted for example in feudal, communist or family based societies, where the *suzerain*, the state or the father respectively are as much the owners as the chief producers of goods. With no implied judgment for choosing one particular model, this separation of interests does however lead to a number of interesting characteristics:

1. Entrepreneurs with little resources but good ideas can realize projects for the larger benefit of the society and are rewarded for their work with a regular income.

2. Investors have an independent judgment of what they consider good ideas, which reduces the likelihood that powerful individuals with bad ideas allocate large resources to realize projects that have little but self-interest.

3. Investors have an interest in putting their wealth to work for the larger benefit of the society and will sometimes make a profit.

4. The mutual interest and also the competition between investors and entrepreneurs can, via regulations, be used to maximize the efficiency of reaching certain goals the society wants to pursue – such as the growth in the *gross domestic product (GDP)* that measures the total amount of goods produced in a country.

By helping entrepreneurs to realize their ideas, investors take a certain *risk* that their initial assets (the *investment*) will be consumed without producing the *expected return:* to statistically compensate for more frequent losses, investors demand a larger return from a risky investment. This is apparent in all the assets that constitute the savings of an individual, which are commonly called *portfolio*.

An important feature of capitalism is the *markets*, where investors exchange standardized assets in the form of *securities*, for a market price (the *spot price*) that is openly disclosed to all the participants in the market. Examples include the well known stock markets (such as the New York Stock Exchange NYSE, the European Virtual Exchange VTX) and less well know exchanges (such as the New York Mercantile Exchange NYMEX, the New York Commodity Exchange COMEX, or the Chicago Board of Trade CBOT) where raw material are traded (such as cattle, oil, gold).

The spot price of a security depends on the consensus reached via offer and demand from the sellers and the buyers: if everything goes well for the investors, it slowly *drifts* in time at a rate that reflects the growing value of this security. Uncertainties in the valuation lead to different opinions and are the source of price fluctuations: quantified as the standard deviation of normalized increments measured over a period of time, the fluctuations are called *volatility* and play a central role in the description of any security. Combining the effects from drift and volatility, the spot prices are said to evolve in a *stochastic* manner, i.e. they never follow any quite predictable pattern: rather, they look like the *random walk* that was first described in biology, when Brown observed the motion of small particles under a microscope and is illustrated with horizontal motions in the VMARKET applet on-line.

Virtual market experiments: the random evolution of market prices

1. Reduce the value of the `Volatility` parameter and test how you can affect the "amount of randomness" in the price increments. Taking one step at a time, verify that you cannot predict with any certainty whether the next movement of the price will go up or down.

2. Increase the value of the `Drift` parameter to add a small, positive or negative but systematic and predictable drift to the price increments.

3. Raise the number of independent `Walkers` to 100 and higher to verify how a *Monte-Carlo simulation* can be used to compute a large number of possible realizations of the market that all start from the same present value.

Masters: probability of an outcome. $\diamondsuit$
Even if one cannot predict with certainty the evolution of a random variable such as a spot price, it is often possible to say at least what are the *possible realizations* and to attribute a *probability* to a certain *outcome*. Assuming that the drift and the volatility of an asset are known over a period of time, the experiments above suggest that a computer can simulate possible realizations with random walkers, by adding small increments to an initial value that is known. Monitoring the evolution of a large number of walkers N, the probability of the chosen outcome can then be estimated by dividing the number of realizations n that satisfy this outcome by the total number of walkers, P=n/N; the relative precision of the estimate is $\epsilon \approx 1/\sqrt{N}$. This procedure can be used to estimate the probability of winning in a market (exercise 1.04) and, more generally, of expecting a price in an interval [a,b]: quants view this as an approximate integral over the probability distribution P=$\int_a^b p(S)dS$.

Not all the trades are openly disclosed in exchanges: non-standard deals are generally carried out *over-the-counter (OTC)* by a *broker*, who's job as a *market maker* is to determine a fair price that will match buyers with sellers, while keeping a small fraction of the money for himself in *transaction costs*. Neither are the trades always for investment purposes: markets are inhabited by *speculators* who bet on the price evolution, *hedgers* who seek protection to reduce the investment risk and *arbitrageurs* who try to exploit small price differences to make immediate and risk free profits.

Financial regulations try to guarantee a fair treatment for all the participants in an open market. Clearinghouses, via a deposit in cash, ensure that the deals are carried out according to the contracts: *clearing margins* are particularly important when a party enters an obligation toward another some time in the future: instead of buying (i.e. *go long*) a security in the hope that the price will rise, this allows members of a clearinghouse to *sell short* a security, i.e. sell something for future delivery that they do not currently own, in the hope that they will be able to buy it more cheaply later.

Private investors generally have access to the markets through a bank or a Internet broker who will carry out market operations on their behalf, generally charging a *fixed fee* plus a *commission* around 1-2% of the value of the deal, which have both to be added to the total *transaction costs*. Because of the risk of defaulting on a deal, securities that carry an obligation are often not accessible to the private investors; chapter 2 will show how a put option can be used instead to earn money in falling markets.

1.3 The risk and return from conventional assets

Before we look into more advanced securities called *derivatives* because their value can be derived from others, it is useful to review some of the conventional assets held in a portfolio.

Bank savings account. Investors who want keep the possibility to quickly withdraw a limited amount of cash usually make a deposit in a bank savings account (for example, check UBS, Handelsbanken, Deutsche Bank). Depending on the total amount invested and the seasonal variations in the interest rates, deposits are rewarded with 0-2% interest excluding fees and taxes. The bank will of course invest the money further for its own profit, but tough regulations ensure that the risk of a bank defaulting on savings account is tiny and the governments often protect deposits to an upper limit around EUR 50,000.

Bank certificates of deposit. Investors willing to lock up their money for a couple of years until a certificate of deposit reaches the *maturity date* can expect larger returns around 3-5% (for example, check UBS). For this first type of longer term investment, it becomes important to distinguish the simply compounded *annual percentage rate (APR)* from the discretely compounded *annual percentage yield (APY)* that includes the interest on interest rates.

Masters: simple, discrete and continuous compounding of interest rates ♢
Consider an amount A invested for an annual interest rate R during n years. If the money earned once a year is not reinvested, the terminal value from a *simply compounded* calculation leads to

$$W = \underbrace{AR_s}_{\text{1st year}} + \underbrace{AR_s}_{\text{2nd year}} + \ldots + \underbrace{AR_s}_{\text{n-th year}} + A = A(1 + nR_s) \qquad (1.3\#\text{eq}.1)$$

If the money is compounded m times a year and immediately re-invested at the same rate, the terminal value from a *discretely compounded* calculation becomes

$$W = A\left(1 + \frac{R_m}{m}\right) \times \left(1 + \frac{R_m}{m}\right) \times \ldots \times \left(1 + \frac{R_m}{m}\right) = A\left(1 + \frac{R_m}{m}\right)^{mn} \qquad (1.3\#\text{eq}.2)$$

Increasing the compounding frequency to infinity $m \to \infty$, the terminal value from the *continuously compounded* calculation tends to

$$W = A\exp(Yn) \qquad (1.3\#\text{eq}.3)$$

where the yield Y can be understood as the annual growth rate of the investment. Continuous compounding is often used for simplicity instead of the more realistic discrete compounding; both are in any case simply connected via

$$R_m = m(\exp[Y/m] - 1), \qquad R = \exp[Y] - 1 \qquad (1.3\#\text{eq}.4)$$

where R_m is the discretely and R the continuously compounded annual rate.

For example, take a 1% monthly rate of a unit investment: depending on the compounding, this translates into $W_{APR} = 1.01 \times 12 = 1.12$, $W_{APY} = 1.01^{12} = 1.126825$, which approaches the continuous annual value of $W = \exp(0.01 \times 12) = 1.1275$. In other words, the return has an APR of 12% and an APY of 12.68%. Brokerage houses often insure a single certificate of deposit for up to EUR 500,000. The main disadvantage of a certificate of deposit is that it locks up the money for a long time unless a steep penalty is payed... up to half of the return from the interest rate!

Money market funds. With similar interest rates, around 3-5%, and no maturity constraints, *money market funds* are only slightly more risky (for example, check CS First Boston, SEB, Credit Lyonnais). A fund manager collects money from a pool of investors and distributes it into a large number of bonds to spread out the risk of defaulting on a debt. Good managers will pick bonds with high return to risk ratios, for a managing commission of up to one fifth of the fund's return, which is directly deducted from the investors profit. Specialists argue that no money market fund has "broken the buck" (i.e. returned less than the original investment) in the last 15 years; it is not easy to verify even such a strong statement, but our readers from around the world may want to comment in our World Forum.

Mutual and hedge funds. By combining holdings in cash, bonds and stock, fund managers can produce larger returns with an increased amount of risk. Every day, the manager counts up the value of all the fund's holdings and, by dividing by the number of shares that have been purchased by the investors, calculates the *Net Asset Value (NAV)* per share of that fund. New investors send their money to the fund manager, who will issue new shares from that fund for the latest quoted value. Holdings are continuously sold and reinvested, which is why mutual funds are sometimes called *open-end funds*. If the fund manager is doing a good job, the net asset value increases and the investors make a profit when they eventually sell their shares. Nevertheless, management commission of around 1-2% of the NAV eat away a considerable fraction of the average 5-10% growth that can be expected in the long term.

A large variety of funds pursue different investment strategies (countries, industries, risk levels, ethical factors): on Feb 23, 2002, the Financial Times newspaper listed more than seven pages with funds... more than shares! In this context, it is good to remember that market indices (such as FTSE-100, Russel 1000, NASDAQ-100) are by definition an arithmetic average; since funds now represent a larger fraction of the market, it is clear that roughly half of the funds under-perform that index. Pursuing a variety of often contradicting strategies can nevertheless be exploited by the marketing departments of the management firms, who simply highlight even few funds that outperformed the index to advertise the skills of all the managers.

Bonds. Investors aiming for the 5.2% long term average return produced by the US bond market have to minimize the fees and commissions payed in transaction coses to the managers... but then have to manage the investment risk by themselves. As a matter of fact, individuals can lend money both to the government and to corporations: both borrow capital from the public by issuing bonds and other *fixed income instruments* that are traded on the *bond markets*. The price of a bond evolves from its initial *nominal principal* or *face value* and the issuer pays a regular predetermined amount of cash called *coupon* until the bond reaches the *maturity* or *redemption date*, when the principal is finally payed back to the investor.

Masters: fixed stream of payments of a bond / discount factor. $\diamondsuit$
Throughout its life, a bond generates a predetermined stream of payments

$$A + \left(\sum_{i=1}^{n} A \tau_i X_i \right) \qquad (1.3\#\text{eq.}5)$$

where the amount A is the principal value outstanding at maturity (usually normalized to 1 or 100), τ_i is the *tenor* or the frequency (in fractions of years, e.g. 30/360 for a monthly coupon in the *LIBOR* convention) and X_i the *fixed annual interest rate* used to calculate the coupon per unit investment (e.g. 0.05 for a 5% coupon).
As the name suggests, a *zero-coupon bond* does not pay any coupon and simply returns the contractual value AP(T,T)=A on the maturity date T. Its present value AP(t,T) measured at time $t < T$ can be calculated from a *no-arbitrage argument*, provided that the interest rates are fixed and that the issuer is certain to pay the loan back on time. Indeed, investing an equivalent amount of cash on the money market for a yield Y should result in the same final value as when the bond matures; if this were not true and the price lower (alt. higher), it would be possible to buy (alt. sell) bonds in exchange of cash on the money market and generate a risk-free profit at the maturity date. Arbitrageurs would immediately take advantage of such opportunities until the demand (alt. offer) moves the price back to the equilibrium value

$$\begin{aligned} P(t,T)\exp(Y[T-t]) &= P(T,T) = 1 \\ &\Rightarrow P(t,T) = e^{-Y[T-t]} \end{aligned} \qquad (1.3\#\text{eq.}6)$$

In a risk-free economy, the present value of an asset can always be calculated from a price known in the future by multiplication by the *discount factor* $\exp(-Y[T-t])$. This is also true for coupons payed at $t_1 = t + \tau_1, t_2 = t_1 + \tau_2, \ldots, t_n = t_{n-1} + \tau_n \equiv T$, which can be discounted back in time as

$$\begin{aligned} Bnd(t, \{t_i\}, T) &= Ae^{-Y[t_n - t]} + \left(\sum_{i=1}^{n} AX_i \tau_i e^{-Y[t_i - t]} \right) \\ &= AP(t,T) + \sum_{i=1}^{n} X_i \tau_i AP(t, t_i) \end{aligned} \qquad (1.3\#\text{eq.}7)$$

showing that the present value of a coupon-bearing instrument can always be reduced to a linear combination of zero-coupon bonds (exercise 1.07).

In the real world, the spot price of a bond is determined by the offer and demand from investors and depends also on the credit worthiness of the issuer. Rating agencies such as Standard & Poor, Moody's or KMV use different criteria to judge issuers who are labeled from the safest "investment grade" (AAA, AA, A, BBB, of which 2.95% American corporate bonds defaulted in 2002) down to "speculative" (BB, B, CCC, CC), "junk" or "default" (C,D). The price of a bond drops sharply when the risk of defaulting on a debt rises: check the historical value of the Argentinian government bonds as its credit worthiness was finally downgraded from C to D in December 2001.

The spot price quoted for a variety of bonds can be accessed with a dozen minutes delay free of charge over the Internet (take e.g. Yahoo, Bloomberg, etc) and the closing prices are reported one day later in the press: for example, on Feb 23, 2002, the Financial Times printed the values in 1.3#tab.1.

Issuer	Red date	Coupon	S&P rating	Bid Price	Bid Yield
Sweden	01/09	5.000	AA+	100.154	4.97
Ford	06/10	7.875	BBB	102.241	7.50
Marconi	03/10	6.375	B+	35.000	26.73

Table 1.3#tab.1: Bonds traded in London quoted on Feb 23, 2002 by the press

The first row shows a Swedish government bond that matures Jan 2009 and pays a 5% annual coupon: with a good investment grade AA+ and a yield in line with the market's expectations, the price (given as a percentage of the principal value of EUR 1,000) is 0.154% higher than the principal. If you bought this bond on Feb 22, you would now earn 0.03% less than the original coupon.

Marconi's corporate bond expires March 2010 and pays a coupon of 6.375%; after the downgrade of telecom operators and speculations about the company's financial fitness, the coupon is now well below what the market expects for the speculative B+ rating. This explains why the bond lost 65% of its principal value and was now only worth 35.00. If you bought this bond on Feb 22, you could earn a very high yield of 26.73% during the next 10 years, provided that the company does not go bankrupt in the mean time.

Small systematic costs have a large impact on the long term return of a portfolio: investors should never neglect the possibility of tax reductions or outright exemptions when buying municipal, state or government bonds.

Stock. Encouraged by the average 7-11% long term average growth of the stock market, investors often add company shares to their portfolio. By doing so, they become co-owners and link the fate of their investment to the future earnings of these companies. Every quarter of a year, the management appointed by the shareholders assembly reports on the profits or losses and sometimes distributes a fixed *dividend* for every share to reward the investors.

In many countries, the tax on dividend income is higher than the tax on the gain in capital – although recent modifications of the taxation in the US may revert this trend. Shareholders therefore prefer to keep the dividend yield low and let the value of shares grow with the company as long as growth remains possible. The valuation of the company's assets, together with the latest results and the expectation of future earnings directly impact on the offer and demand from investors on the stock market (such as NYSE, NASDAQ) which ultimately determines the price of shares.

The spot prices quoted for every share can be read free of charge on the Internet after only a dozen minutes delay (take e.g. Yahoo, Bloomberg, etc) and the closing prices are reported one day later in the press: for example, on Feb 23, 2002, the Financial Times printed the values in 1.3#tab.2 below. The first row shows that the share from Hilton hotels fell GBP 4.75 to 215.50, in a liquid market with more than 8 million shares exchanged during the trading day. This price is somewhere in the middle of

Company	Price	+/-	High	Low	Volume	Yield	P/E
Hilton	215 1/2	-4 3/4	259 3/4	152	8,081	4.0	11.9
AstraZeneca	3519	+64	3564	2724	3,874	1.4	29.9
Marconi	16 3/4	-2 3/4	800	12 1/4	103,986	-	-

Table 1.3#tab.2: Stocks traded in London quoted on Feb 23, 2002 by the press

the range over which the share was trading during the last 12 months, as indicated by the high and low ends of the price interval. The price-to-earning ratio shows that it approximatively takes P/E=11.9 years for the company earnings to add up to the original purchase price ("paying back your investment") if the earnings remain fixed. Assuming a small one percent growth G=0.01 for a mature industry, you can show (exercise 1.06) that this corresponds to an expected return on investment of $G + E/P = 0.01 + 1/11.9 = 0.094$ or 9.4%, which is indeed much larger than the 4% dividend yield payed in cash to the shareholders; investors should therefore expect Hilton's share price to rise by an annual 5.4%.

AstraZenca pharmaceuticals have been growing very fast during the last years and, expecting that this will continue into the future, investors are willing to pay a much larger price-to-earning ratio of 29.9. For the sake of simplicity, assume that the exponential growth reached from the Low to the quoted Price during exactly one year, so that the growth rate can be estimated from $G = \ln(P_2/P_1)/(t_2 - t_1) = \ln(3519/2724) = 0.25$.This translates into astronomical returns $G + E/P = 0.25 + 0.033 = 0.28$ which exceed by far the 3.3% justified by the earnings and the 1.4% payed as dividends.

Finally, note the near collapse of Marconi's share from GBP 800 to 16.75, which shows that the company has large financial difficulties and may go bankrupt, i.e. the share value drop to zero forever. This is consistent with the low credit worthiness perceived for its corporate bond (1.3#tab.1) and underlines the fact that a high investment risk can lead to large losses.

Intra-day prices can in general not be accessed free of charge; daily values adjusted for occasional *splits* can, however, be downloaded using the MKTSolution applet on-line.

Market data: historical values from closing prices

1. Study the price history and the market volume for shares in General Motors traded during one year. How large is the drop that can be associated with the WTC attack on Sep 11, 2001?

2. Have a closer look at the daily price increments and compare them with the random walk described in the previous section.

3. Follow the link to the *Market data* applet and identify the market *symbol* for an update of your favorite companies.

From this overview, it should be clear that a higher return can be expected if the investor accepts a larger risk. The comming sections describe simple methods to maximize the return from a portfolio and determine the risk from historical data. But how much risk should an investor take anyway? A mountaineer says this a matter of taste, while common sense tells you not to wake up in the middle of the night to worry about a portfolio!

1.4 Modern portfolio theory and basic risk management strategies ♢

With his conjecture that *investment risk* can be quantified as *volatility* from the *standard deviation* of the expected return, H. Markowitz started in 1952 at the University of Chicago what has become the *Modern Portfolio Theory (MPT)* and awarded him in 1990, the counterpart of the Nobel Prize in Economics [16].
Rather than looking at the risk from a single investment, he examined the stochastic evolution from different asset classes and found that seemingly random prices are sometimes correlated, for example when two industries compete or complement each other in the same market. This led him to distinguish *specific risk* associated with small groups of interdependent assets (e.g. bad weather affects the harvest of coffee and the valuation of all the shares in the food industry) from the *non-specific risk* that affects the market as a whole (e.g. a stock market crash). Using a judicious choice of anti- and uncorrelated securities, Markowitz showed that the total volatility of a portfolio can be minimized without reducing the expected return, by *diversifying* out the specific risks.
For a simple example, imagine a portfolio composed of two assets that are perfectly anticorrelated, but have the same expected return: by canceling each other's price fluctuations, the total volatility can be reduced to zero without changing the total expected return.
A second example illustrated in (1.4#fig.1) suggests how the combined risk from two assets can be minimized ($\sigma_b < \sigma_a$) at a constant expected return ($E_a = E_b$) provided that the prices are partly de-correlated. By varying the proportions invested in each asset, an *efficient frontier* can be calculated where the highest return is expected for a given volatility (continuous line in blue).

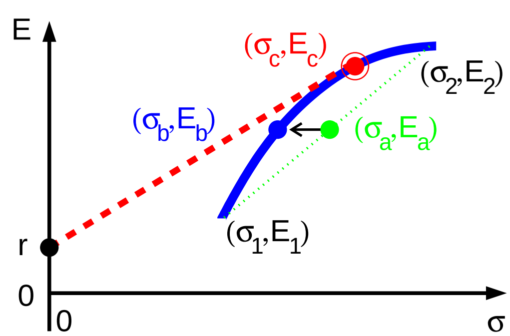

Figure 1.4#fig.1: Expected return E from a mix of two volatile assets (σ_1, E_1) and (σ_2, E_2) that are partly correlated. The plot shows how, for the same expected return $E_a = E_b$, the combined risk from two assets is lower than the average that is calculated when the risks of the assets are evaluated separately $\sigma_b < \sigma_a$. The tangent (broken line in red) drawn from the risk-free rate $(r, 0)$ intersects the efficient frontier (line in blue) at (σ_c, E_c) where the best portfolio is located.

The analysis is only slightly more complicated when more than two assets are involved: the applet on-line calculates the efficient frontier based on monthly variations of the price of raw materials, bonds and stock as they have been observed in the markets during the years 1986–1996.

Market history experiments: efficient frontier

1. Select "`US LT Govt Bnd`" and "`US S&P500`" to plot the joint expected return from a variable amount of money invested in bonds (10 years government bonds) and shares (stock market index) in the US.
2. Add the "`US 30Day TBill`" and explain the similarity with (1.4#fig.1).
3. Explain what happens with an investment in "`Gold`". You may check the glossary on-line for a complete list of other symbols appearing in the applet.

To find out which combination is most risk efficient under the present market conditions, another Nobel laureate, W. Sharpe, looked at the return from a variable amount invested partly for a risk-free rate r and partly in assets located on the efficient frontier (1.4#fig.1). He found that the "best" portfolios, located on the broken line in red, intersect the efficient frontier where the highest return E_c is achieved if the money has to be borrowed on the market. This led him to the definition of what is now known as the *Sharpe ratio*, where the reward-to-risk from a portfolio x is measured as the excess return over the risk free rate divided by the total volatility

$$S_x = \frac{E_x - r}{\sigma_x} \qquad (1.4\#eq.1)$$

To compare the relative performance from a portfolio (σ_x, E_x) with the average performance from a market index $(\bar{\sigma}, \bar{E})$ over different periods of time, Sharpe later developed the *capital asset pricing model (CAPM)*

$$\alpha + \beta(\bar{E} - r) = E_x - r \qquad (1.4\#eq.2)$$

where a linear least-square fit is calculated to obtain the slope beta (β relative performance from taking risks) and the offset alpha (α relative performance from arbitrage and costs). To "beat the market" alpha should always be positive and beta larger than unity. Unfortunately for most of the investors, commercial funds generally have a negative alpha because of the 1-2% management costs that are payed in the form of commissions. Some say that expertise justifies the costs because of additional earnings made from arbitrage: this is in general not true, because arbitrage is a *zero-sum game*, so that whatever makes one manager look better only make another look worse! The value added by the fund manager therefore resides mainly in beta, i.e. in the management of risk.
All together, the modern portfolio theory largely justifies investments in funds, provided that a large number of weakly correlated assets are managed at a very low cost. It also explains how different funds can be classified according to their growth (drift), standard deviation (volatility), reward-to-risk (or Sharpe ratio 1.4#eq.1) and every fund can always be compared with a market index using the CAPM parameters (alpha and beta). The correlation between individual assets and the portfolio as a whole provides a more detailed description (execise 1.03). For the layman, the theory leads to the simplest and best known risk management strategy: **diversify your portfolio by investing in a variety of weakly or anti-correlated securities**. To maximize the reward, it is better to **blend different types of investments**, for example by selecting assets according to the time that it will take to average out fluctuations. Some investors subtract their age from 100 to determine the percentage to invest in stocks and put the rest in bonds: the younger the an investor is, the more risk he can afford to take.

In the sixties, E. Fama [8] proposed another important conjecture following market observations, called the *Efficient Market Hypothesis:* at any time, the price of a security fully reflect all the information that is available about this security. The reason is that the market is inhabited by *arbitrageurs*, whose highly paid job is to seek out and exploit possible mis-pricings. Under the efficient market hypothesis, no-arbitrage arguments state that **it is not possible to find a self-financing trading strategy leading to an immediate risk-less profit**. This means that there is no way for investors to buy securities at a bargain price: even if the prices just fell, there are equal chances for them to move back up or fall down even further. There is no way to make a statement such as "the market is too high now".

Of course, not all the markets are efficient and human psychology is such that investors tend to buy more in rising than falling markets: buying stocks in a falling stock market sounds easy, but very few people have the stomach to do it! To avoid arbitrageurs taking advantage of the psychology, portfolio managers sometimes perform a cost averaging by regularly buying a fraction of the security they want to buy or sell – independently of the short time fluctuations of the market (exercise 1.02). This strategy has, however, also its limits since investors should imperatively minimize the transaction costs that are associated with every operation.

In conclusion, simple management strategies can be used to reduce the investment risk in a portfolio: **ignoring the advice to diversify and regularly pay large commissions and transaction costs have the worst long term effects**. For a more quantitative and a flexible approach of managing investment risk, the next chapter will examine a new class of securities: so-called *options*, which can be combined with other assets to hedge a portfolio to any level and type of risk chosen by the investor.

1.5 Historical data and modeling

Having loosely introduced the *volatility* σ as a measure of the *investment risk*, it is time now to develop an intuition for this important quantity and show how the volatility and the drift of a spot price can be calculated as averages from historical data from the markets.

1.5.1 Drift and volatility of market prices

Have a look first at (1.5.1#fig.1, top), which shows the price of the Cisco share quoted on NASDAQ every trading day between 1994 and 2004. After a prolonged period of exponential growth from USD 1 in Jul 94 to USD 80 in Dec 1999, the price drops by $\sim$70% following a sector-wide correction of technology shares during the year 2000. The repercussions from the attack Sep 11, 2001 on the world trade center (WTC) are also visible, but led only to a temporary $\sim$25% drop in the share price.

How does the volatility in (1.5.1#fig.1, bottom), updated after every trading day using only information from the past, reflect the financial risk that can be judged a posteriori? To answer this question, note first that the long term average volatility of around 40% per annum does not really depend on the actual price of the share: the volatility only shows that typical gains or losses of at least 40% can be expected during any year under consideration. The volatility jumps to even higher values immediately AFTER every significant change in the share price, both on the way up and on the way down: a large movement of the price reflects the uncertainty of the investors, who are unsure if the amplitude of the change is exaggerated or if it should be even larger. For example, the volatility was large (∼100%) at the end of the year 2000 during the whole period when the price kept falling, but it was also large after the WTC attack when the prices recovered within only a couple of weeks. This illustrates that the volatility cannot be used to forecast whether a spot price will rise

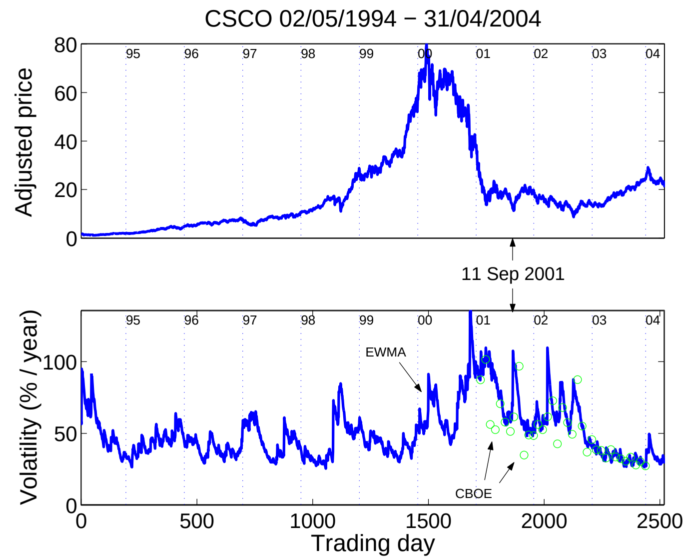

Figure 1.5.1#fig.1: The upper plot shows the historical price (adjusted for splits) of the Cisco share during 10 years as a function of the trading day. On the bottom, the corresponding volatility calculated using the EWMA/$\lambda = 0.94$ model (blue line) is displayed in a comparison with the volatility measured by the Chicago Board of Exchange (green circles).

or fall, but it gives a good idea by how much the price may move in either direction: this is indeed the measure of risk we are seeking. In a word of caution, however, note that the volatility of a spot price is not a value that can directly be observed: different models yield different values, suggesting that it can be misleading to use values from the Internet without knowing exactly how they have been calculated.

1.5.2 Moving averages: UWMA, EWMA, GARCH ♢

Consider a sequence of spot prices $\{S_1, S_2, ...\}$obtained from the market at regular time intervals labelled $i = 1, 2, 3, ...$ Introduce the normalized increments $s_i = \ln(S_i/S_{i-1})$which, it will be shown in the next chapter, are typical of a log-normal distribution of the price increments observed on the stock market. Following Markowitz' definition of *volatility* as *standard deviation* of the expected return, it is useful first to estimate the mean (drift) and the variance (square of volatility) per unit time Δt using the m most recent observations

$$\bar{\mu}_n = \frac{1}{m\Delta t}\sum_{i=1}^{m} s_{n-i}, \qquad \bar{\sigma}_n^2 = \frac{1}{(m-1)\Delta t}\sum_{i=1}^{m}(s_{n-i} - \bar{\mu}\Delta t)^2 \qquad (1.5.2\#\text{eq.1})$$

This formula provides the basis for the so-called uniformly weighted moving avergage (UWMA) and has been implemented in the `MKTSolution` applet,♠ using a window with m=126 values, corresponding to half of the 252 trading days during a year. The estimated drift and the volatility expressed on an annual basis is finally obtained from the scaling

$$\mu_{\text{yr}} = \mu \times 252, \qquad \sigma_{\text{yr}} = \sigma \times \sqrt{252}. \tag{1.5.2\#eq.2}$$

For small changes $\ln(S_i/S_{i-1}) = \ln([S_{i-1} + \Delta S_i]/S_{i-1}) = \ln(1 + \Delta S_i/S_{i-1}) \approx \Delta S_i/S_{i-1}$ the normalized increments are generally approximated with the ratio

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}} \tag{1.5.2\#eq.3}$$

and the small drift associated with the mean is generally neglected in comparison to the much larger fluctuating component. For a large number $m \approx m - 1$ the formula for the variance can therefore be simplified to

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2 \qquad \text{or, the equivalent} \qquad \sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \quad , \quad \alpha_i = \frac{1}{m}. \tag{1.5.2\#eq.4}$$

The UWMA of drift, volatility and other quantities that can be estimated from time series, such as correlations, suffers from two main short-commings: the most recent events that are most significant, only carry the same uniform weight α_i as all the others in the averaging window, this until the information is abruptly lost after m days. In addition, the UWMA is independent of a long term average towards which temporary deviations tend to revert to.

To tackle the first problem, the average window can be dropped in favour of a recursive or auto-regressive definition, producing an exponentially weighted moving average (EWMA) where the last known quantity is constantly updated with the most recent market increment

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2. \tag{1.5.2\#eq.5}$$

Insert (1.5.2#eq.5) back in itself and work through the recursion a few times to convince yourself that the weights, which were uniform in (1.5.2#eq.4), now are exponentially decaying with a "forgetting rate" $\alpha_i = (1-\lambda)/\lambda^{i-1}$ that accelerates as $\lambda \in [0;1]$ gets smaller (exercise 1.09). In its RiskMetrics database, J.P.Morgan for example uses an EWMA model with $\lambda = 0.94$, and has also been implemented in the `MKTSolution`♠ applet. Alternatively, a maximum likelyhood estimate can be calculated for every spot price using the method described in the next section (exercise 1.05).

The second issue is generally solved by writing the long term average as $V = \omega/(1-\alpha-\beta)$ and introducing a reversion term in a so-called generalized auto-regressive conditional heteroscedasticity model, using the p most recent increments and the q most recent volatility estimates in GARCH(p,q). The most commonly used is GARCH(1,1)

$$\sigma_n^2 = \omega + \beta\sigma_{n-1}^2 + \alpha u_{n-1}^2 \tag{1.5.2\#eq.6}$$

where α controls the sensitivity to most recent increments, β the forgetting rate and $\omega = \gamma V$ is linked with long term average. For consistency, the parameters must satisfy $\alpha+\beta+\gamma = 1$ and to prevent negative long term average volatility, it is important that $\alpha+\beta < 1$. Clearly, the EWMA model is a particular case of GARCH(1,1), where $\omega = 0,\ \alpha = 1-\lambda,\ \beta = \lambda$.

Market data: volatility models

1. Plot the price history and the market volume for shares in General Motors during one trading year and examine how they affect the volatility.
2. Compare different volatility models and try to summarize their response to a market event such as the WTC attack on Sep 11, 2001.
3. Follow the link to the *Market solution* applet and compare the model volatilities with the values quoted elsewhere on the Internet.

Qualitative arguments support models with features such as the exponential weighting ("forgetting"), a reversion mechanism ("long term average") and the tendency to reproduce the auto-correlation of the market ("clustering", i.e. large large u_i^2 tends to produce large u_{i+1}^2, u_{i+2}^2, etc). Since the volatility is not a quantity that can be directly measured on the market, it is not easy to judge which model is better or worse. Nevertheless, an independent test could compare the *implied volatility* of options (later defined in sect.4.1.3) with the value calculated here using the underlying share. The plot in (1.5.1#fig.1,bottom) shows such a comparison for Cisco during the period 2001-2004: the EWMA calculated from the stock market history assuming the parameter $\lambda = 0.94$ does indeed accurately reproduce the implied volatility calculated from the option market, except in 2001 during the period of high volatility when the EWMA appears to predict larger values.

Apart from following the advice from financial institutions, is there an independent way to calibrate the parameters $\alpha, \beta, \lambda, \omega$, in a manner that achieves the best possible fit between a model and the data? Yes, this will be the last topic of this introduction.

1.5.3 Maximum likelihood estimate of parameters ♠

Even if a model provides an accurate description of statistical data, it is important that occasional outliners be discarded from the fit. Instead of minimizing a residual between the model and all the data points, a *maximum likelyhood estimation* therefore aims at maximizing the probability that the model reproduces most, but not all the data points. For example, imagine a coin thrown five times into the air with, as an outcome 1 heads and 4 tails. The maximum likelyhood estimate of observing the sequence in that order can be calculated by maximizing the probability of the observation $\max[p(1-p)^4]$: setting the derivative equal to zero $(1-p)^4 - 4p(1-p)^3 = 0$, this yields $p = 0.2$ as expected.

The same method can be applied when the market increments are normally distributed with a variance $\nu_i = \sigma_i^2$ that is allowed to change over time. The maximum likelyhood of reproducing the market data in that order can then be calculated from the optimum

$$\max \prod_{i=1}^{m} \left[\frac{1}{\sqrt{2\pi\nu_i}} \exp\left(-\frac{u_i^2}{2\nu_i} \right) \right] \quad \Leftrightarrow \quad \min(\chi), \quad \chi = \sum_{i=1}^{m} \left[\ln(\nu_i) + \frac{u_i^2}{\nu_i} \right] \quad (1.5.3\#eq.1)$$

where the second expression has been obtained after realizing that the maximum of a quantity coincides with the maximum of its logarithm and the minimum of the opposite. Model parameters, such as λ in $\sigma_i^2(\lambda)$ for the EWMA model of the volatility (1.5.2#eq.5) are then calculated to maximize the likelihood of reproducing the data by setting the first derivative equal to zero $d\chi/d\lambda = 0$ and keeping the second derivative positive $d^2\chi/d\lambda^2 > 0$.

Quants: statistics. ♠

Remember that the correlation coefficient between two random variables X,Z is given by

$$\mathrm{Corr}[X,Z] = \frac{\mathrm{Cov}[X,Z]}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Z]}} \qquad \in [-1;1] \qquad (1.5.3\#eq.2)$$

the covariance, the variance and the expectancy operators are defined by

$$\begin{array}{rl} \mathrm{Cov}[X,Z] & = E[XZ] - E[X]E[Z] = E[(X-\mu_x)(Z-\mu_z)] \\ \mathrm{Var}[X] & = E[X^2] - (E[X])^2 = E[(X-\mu)^2] \equiv \sigma^2 \\ E[X] & = \int_\Omega x f(x) \equiv \mu \end{array} \qquad (1.5.3\#eq.3)$$

where μ is the mean, σ^2 the variance and σ the standard deviation. Higher order central moments are defined from $\mu_k = E[(X-\mu)^k]$, such as the skewness $\gamma_1 = \mu_3/\sigma^3$ and the curtosis $\gamma_2 = (\mu_4/\sigma^4) - 3$. Under general conditions, the sum of a large number of random variables is approximatively normally distributed $f \approx \mathcal{N}[\mu,\sigma^2]$

$$\mathcal{N}[\mu,\sigma^2](x) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right), \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2) \qquad (1.5.3\#eq.4)$$

and the normalized probability $P(-a \le (X-\mu)/\sigma \le a) = \int_{-a}^{+a}\varphi(x)dx = \mathrm{erf}(a/\sqrt{2})$. Unbiased estimates for the mean and variance of n data points $\{x_1, x_2, \ldots, x_n\}$ generated by a normally distributed process can be calculated from

$$\bar{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \bar{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{\mu})^2 \qquad (1.5.3\#eq.5)$$

Finally, a least-square fit to a linear model $\alpha + \beta x = y$ is obtained by solving the system of normal equations

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ . & \ldots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \approx \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{pmatrix} \qquad \Leftrightarrow \qquad \mathbf{Xc} \approx \mathbf{y} \qquad \Rightarrow \qquad \mathbf{X^T X c} = \mathbf{X^T y}$$

(1.5.3#eq.6)

Even if several parameters have to be determined $p_i, p_j \in \{\alpha, \beta, ...\}$ the gradient and the Hessian can be conveniently calculated from previous values

$$\frac{\partial \chi_n}{\partial p_i} = \frac{\partial \chi_{n-1}}{\partial p_i} + \frac{1}{\nu_n}\left(1 - \frac{u_n^2}{\nu_n}\right)\frac{\partial \nu_n}{\partial p_i} \qquad (1.5.3\#eq.7a)$$

$$\begin{aligned} \frac{\partial^2 \chi_n}{\partial p_i \partial p_j} &= \frac{\partial^2 \chi_{n-1}}{\partial p_i \partial p_j} + \frac{1}{\nu_n^2}\left(\frac{2u_n^2}{\nu_n} - 1\right)\left(\frac{\partial \nu_n}{\partial p_i}\right)\left(\frac{\partial \nu_n}{\partial p_j}\right) + \frac{1}{\nu_n}\left(1 - \frac{u_n^2}{\nu_n}\right)\frac{\partial^2 \nu_n}{\partial p_i \partial p_j} \\ &\approx \frac{\partial^2 \chi_{n-1}}{\partial p_i \partial p_j} + \left(\frac{2u_n^2}{\nu_n^3}\right)\left(\frac{\partial \nu_n}{\partial p_i}\right)\left(\frac{\partial \nu_n}{\partial p_j}\right) \end{aligned} \qquad (1.5.3\#eq.7b)$$

where the last approximation is used to guarantee a positive diagonal (i=j) when a Levenberg-Marquardt solver is used to locate the zeros of the non-linear function (1.5.3#eq.7a).

The EWMA model (1.5.2#eq.5) has one free parameter ($p_1 = \lambda$) that can be estimated anew after every trading day. Follow the links in the document on-line to study how the *derivatives of the variance*

$$\nu_n = \lambda \nu_{n-1} + (1-\lambda) u_{n-1}^2$$

$$\frac{\partial \nu_n}{\partial \lambda} = \lambda \frac{\partial \nu_{n-1}}{\partial \lambda} + \nu_{n-1} - u_{n-1}^2 \qquad (1.5.3\#eq.8a)$$

$$\frac{\partial^2 \nu_n}{\partial \lambda^2} = \lambda \frac{\partial^2 \nu_{n-1}}{\partial \lambda^2} + 2 \frac{\partial \nu_{n-1}}{\partial \lambda} \qquad (1.5.3\#eq.8b)$$

are updated before the *Levenberg-Marquardt algorithm* [18] is used to calculate the parameter as a zero of the non-linear likelihood derivative (1.5.3#eq.7a). Note that a more simple Newton algorithm for finding roots of scalar variables could have been used instead, but would be insufficient when more than one parameter has to be estimated.

The GARCH(1,1) model (1.5.2#eq.6) has three free parameters ($p_1 = \beta,\ p_2 = \alpha,\ p_3 = \omega$) and requires the evaluation of *the gradient*

$$\nu_n = \omega + \beta \nu_{n-1} + \alpha u_{n-1}^2$$

$$\frac{\partial \nu_n}{\partial \beta} = \beta \frac{\partial \nu_{n-1}}{\partial \beta} + \nu_{n-1} \qquad (1.5.3\#eq.9a)$$

$$\frac{\partial \nu_n}{\partial \alpha} = \beta \frac{\partial \nu_{n-1}}{\partial \alpha} + u_{n-1}^2 \qquad (1.5.3\#eq.9b)$$

$$\frac{\partial \nu_n}{\partial \omega} = \beta \frac{\partial \nu_{n-1}}{\partial \omega} + 1 \qquad (1.5.3\#eq.9c)$$

Maximum likelyhood of the fit is achieved when all three components of the gradient are equal to zero, which defines the parameters using the same Levenberg-Marquardt algorithm to locate the zeros of (1.5.3#eq.7a).

The `MKTSolution` applet on-line illustrates how the estimation works for the price history of the Asea Brown Boveri share during 2001-2003, rapidly switching between different regimes that are rather stable to produce the final estimate of $\lambda = 0.8891$ for the EWMA, model and $\omega = 0.000162,\ \alpha = 0.03134,\ \beta = 0.8000$ for the GARCH(1,1) model.

Market data: EWMA and GARCH(1,1) parameter estimation

1. Compare the volatility of the ABB share obtained for the EWMA model with a constant $\lambda = 0.94$ or a maximum likelihood estimate of the parameter. Identify regimes where the estimate may be above or below 0.94.
2. Study the volatility obtained for the GARCH(1,1) model, checking whether it is possible to define a long-term average, evaluate the impact of recent events and tell how quickly they are forgotten.
3. Which value of the volatility would you use to predict the financial risk in the year following this sequence?

It turns out parameter estimation is an important but rather delicate task[4] in the sense that the result depends strongly on the choice of the time window and abrupt changes

[4] The result is printed in the Java-console (Netscape open `Communicator ->Tools ->Java console`).

in the price history can lead to significant changes in the model. This is of course what the estimation is meant to do, but it is important to make sure that the values that are predicted are not only mathematically correct, but also financially meaningful.

Knowing the present value of the variance model parameters, it is possible to forecast the financial risk into the future. Substituting the long term average $\omega = V(1-\alpha-\beta)$ into the recursive definition (1.5.2#eq.6), exercise 1.11 shows that the expected value k days into the future becomes

$$E[\sigma_{n+k}^2] = V + (\alpha+\beta)^k(\sigma_n^2 - V) \qquad (1.5.3\#eq.10)$$

For the EWMA model $\alpha + \beta = 1$ so that the expected future variance rate is equal to the present value. For the GARCH model, $\alpha + \beta < 1$ the second term decreases in importance for an increasing number of days k, showing that the variance exhibits a *mean reversion* towards the level V at a rate $1 - \alpha - \beta$. An average of the variance term structure

$$\bar{\sigma}^2 = \frac{1}{N}\sum_{k=0}^{N} E[\sigma_{n+k}^2] \qquad (1.5.3\#eq.11)$$

is then generally used to parametrize option pricing models.

1.6 Computer quiz

1. Do you have to register to study this course on-line?
   (a) All the material can be accessed free of charge.
   (b) Register to gain access to restricted material and pedagogical support.
   (c) All the services on this website are for registered users only.

2. A small trading volume in your favorite stock tells you that:
   (a) you may have to wait some time before you can find a buyer.
   (b) the stock is over-valued and the investors wait for the price to drop.
   (c) the market is liquid with many buyers and sellers at the same time.
   (d) the investors may be on holidays.

3. Bonds are often less volatile and less risky than stocks
   (a) because bonds holders are reimbursed before stock owners in a bankruptcy.
   (b) because bonds generally pay a fixed coupon.
   (c) because they can always be sold at least for their net asset value.

4. How does the volatility change in a stock market crash?
   (a) It rises.
   (b) It doesn't change.
   (c) It first falls and then rises.

5. Does a high yield portfolio always imply a large volatility?
   (a) Not for investors who keep only one high performance asset.
   (b) Yes, high yields necessarily imply large price correlations and volatility.
   (c) Not necessarily, if the asset prices are anti-correlated.

6. Cost averaging strategies
   (a) reduce the transaction costs.
   (b) average out the transaction costs.
   (c) force investors to buy in falling markets.

7. A market is efficient if ◇
   (a) the investors pay minimal fees.
   (b) the prices have equal chances to rise and fall.
   (c) a large number of arbitragers exploit opportunities to make easy money.

8. Any point (σ_b, E_b) on the efficient frontier in (1.4#fig.1) satisfies the conditions ◇
   (a) The volatility sigma-b is always between sigma-1 and sigma-2.
   (b) The expected return E-b is always between E-1 and E-2.
   (c) The expected return E-c is always larger than the risk-free return r.

1.7 Exercises

1.00 E-learning. Familiarize yourself with the electronic submission of assignments, the discussion forum and the rest of the problem based learning environment. Learn to carefully read and answer all the questions.

1. Follow the TEX link for a list of mathematical symbols and type a small text with formulas. From small mistakes, learn to interpret the compiler error messages.
2. Follow the Tag link for a description of the VMARKET applet and preset your default parameters to prepare for the simulation of the price of a European vanilla put option struck for EUR 100, with one year to the expiry date, an underlying now valued at EUR 10 in a market with 30% volatility and short term interest rate of 4%. Can you figure out a simple approximation if you neglect the volatility?
3. Read the rules governing the discussion forums and introduce yourself to the classroom, by telling a few words about your background and interests. If possible, submit a digital picture of yourself from `USER:Profile`.
4. Follow the `USER:Download` link and select `solutions` to send a copy of your solution to the printer. Under `WORK:Assignments`, click the `CheckMe` button and press `Submit` to notify the teacher to check your solution.

1.01 Model portfolio. Use the portfolio manager from UBS, historical values from the markets and justified assumptions to create a model investment for EUR 10000.

1. Combine assets from the stock and bond markets so as to maximize the expected return from a mixed strategy, when an equal amount of high, medium and low risk investments are made over a time span of 3 months, 2 and 16 years.
2. Use the MKTSolution applet to plot the historical value of your assets. From the typical fluctuations and growth rate, try here only to estimate and justify the probability that you may recover less than your initial investment after 3 months, 2 and 16 years–to be compared with a quantitative solution from exercise 1.04.
3. Study one share in details and prepare yourself to implement the hedging strategies that are be discussed in this course.

1.02 Cost averaging. Use the historical values from your model portfolio (exercise 1.01) to evaluate the cost of buying the same assets with 2-3 different cost averaging strategies. Do you expect any difference? Why?

1.03 Performance of an investment. Update the data for bonds (1.3#tab.1) and stock (1.3#tab.2) using recent quotes that you can find on the Internet. Compare the performance from an investment performed on Feb 23, 2002 by distributing EUR 6000 homogeneously across all the securities in the tables with an investment in Swedish government bonds only.

1.04 Investment risk.$^{\diamond}$ Use both historical data and justified assumptions to quantify the investment risk in your model portfolio in exercise 1.01.

1. Estimate the volatility, the drift and the Sharpe ratio for each security independently and for the portfolio as a whole.
2. Execute a Monte-Carlo simulation with 100 walkers to calculate the probability of losing money on your investment after 3 months, 2 years and 16 years.

1.05 *Volatility measurements.* $^\diamondsuit$ Use the MKTSolution applet to calculate the volatility of the General Motors (GM) share during the year 2001. Compare the result obtained at the year end using the UWMA, EWMA and GARCH models and check what happens if you reduce the measurement period down to 6 and 3 months. From your measurements, try to forecast the volatility one year ahead and compare your prediction with the implied volatility from the option market, which turned out to be 31.3% in December 2002.

1.06 *Stock valuation.* $^\diamondsuit$ Financial statements disclosed to the shareholders' assembly often include earnings in the form of a dividend D per share and the company growth G. Using a yield Y to discount all the future earnings from the company, derive a formula to calculate a fair price P per share based on the earning estimates. Consider the case of a young company growing at a constant pace during the first n years and a mature company where the growth stopped. Compare with P/E ratios from the markets. Hint: remember the geometrical series $\sum_{i=0}^{n-1} x^k = (1-x^n)/(1-x)$.

1.07 *Zero rate of coupon bearing bonds.* $^\diamondsuit$ Calculate the yield to maturity for a bond with a principal of EUR 100 presently valued at EUR 90, which matures in 5 years and pays a 3% semi-annual coupon starting on the third year. Hint: assume a simple compounding to obtain an analytical solution using a geometrical series or solve the non-linear equation $f(x) = 0$ numerically using Newton iterations $x_{i+1} = x_i - f(x)/f'(x)$.

1.08 *UWMA for the volatility of interest rates.* $^\spadesuit$ Assume a normal distribution of incremental changes in the interest rates and derive a formula for an unbiased estimate of the volatility. Implement this in the MKTSolution applet and calculate the volatility at the end of 2001 from historical prices of the 10 years US treasury bond. Compare with the value obtained assuming log-normal increments.

1.09 *EWMA for the drift of shares.* $^\spadesuit$ Implement an EWMA to measure the drift from the historical price of shares in Cisco. Justify your choice for the parameter λ.

1.10 *GARCH variance targeting.* $^\spadesuit$ Consider a so-called *variance targeting* model, where the variance is first independently calculated from the historical data and then used in GARCH to estimate only two instead of three parameters. Start with an analytical derivation of the gradients and implement your model in the applet. Compare your with those that are obtained using a three parameters estimation.

All these problems can be edited and submitted for correction directly from your web browser, selecting *WORK:assignments* from the course main page.

1.8 Further reading and links

- **Theory.**
  Modern Portfolio Theory: Money-Chimp[5], Fama$^\diamondsuit$[8], Sharpe$^\diamondsuit$[21].
  Finance: Brealy and Myers$^\diamondsuit$ [4].
  Microeconomics: Luenberger$^\spadesuit$ [14].
  Macroeconomics: Heilbroner$^\diamondsuit$ [10].

- **Money market, funds.**
  Market: Financial Times[6], UBS[7], CSFB[8], Deutsche Bank[9], Handelsbanken[10], SEB[11].

- **Bonds.**
  Introduction: Money-Chimp[12], Investing-In-Bonds[13].
  Quotes: Financial Times[14], Yahoo[15], UBS[16], Bloomberg[17]
  Ratings: Standard & Poor[18], Moody's[19], KVM[20].

- **Stock.**
  Introduction: Money-Chimp[21].
  Quotes: Financial Times[22], Yahoo[23], UBS[24], Bloomberg[25].
  Ratings: Morgan Stanley[26], Merill Lynch[27], Goldman Sachs[28], Deutsche Bank[29].

- **Historical values.**
  Data: this course website, Financial Times[30], Bloomberg[31], CBOE volatility[32].

[5] http://www.moneychimp.com/articles/risk/
[6] http://www.ft.com/markets
[7] http://www.ubs.com
[8] http://www.csfb.com
[9] http://www.deutsche-bank.de
[10] http://www.handelsbanken.se
[11] http://www.seb.se
[12] http://www.moneychimp.com/articles/finworks
[13] http://www.investinginbonds.com
[14] http://www.ft.com/markets
[15] http://bond.yahoo.com
[16] http://quotes.ubs.com
[17] http://www.bloomberg.com
[18] http://www.standardandpoors.com
[19] http://www.moodys.com
[20] http://www.kvm.com
[21] http://www.moneychimp.com/articles/valuation/
[22] http://www.ft.com/markets
[23] http://finance.yahoo.com
[24] http://quotes.ubs.com
[25] http://www.bloomberg.com
[26] http://www.ms.com
[27] http://www.ml.com
[28] http://www.gs.com
[29] http://www.deutsche-bank.de
[30] http://www.ft.com/markets
[31] http://www.bloomberg.com
[32] http://www.cboe.com/

1.9 Quick intermediate evaluation form

Your opinion is precious. Please fill in the anonymous evaluation form in the web edition. Thank you very much in advance for your collaboration.

2 A VARIETY OF SECURITIES

2.1 The stock market and its derivatives

2.1.1 Shares and market indices

Small companies generally start developing a product using private resources and sometimes a limited amount of *venture capital*. If all goes well, they may under circumstance decide to go to the stock market for the quest of new capital, which will allow them to grow more rapidly than what they could achieve by simply re-investing their own earnings.

Investment banks assist them in the *initial public offering (IPO)*, when the company value (often estimated for the future potential more than the present earnings) is divided in a number of shares that are proposed to the investors on the stock market. By selling a fraction of their company, the original owners realize a *capital gain*, but also give up part of the control and future earnings to other shareholders. After a rapid and rather systematic evolution (depending on how well the investment bank succeeds in aligning the initial offering with the market expectations), the share price starts a dominantly random evolution in agreement with Fama's efficient market hypothesis introduced in section 1.4.

Previous experiments with the VMARKET applet suggested that possible realizations for the price of a share can be simulated by adding small increments to the initial price that is known. To be precise, the *market (or spot)* value can never be predicted with certainty, but an *expected value* can nevertheless be calculated, provided that the distribution of increments reproduces the market characteristics.

In addition to the deterministic growth (`Drift` parameter μ) and the random component associated with risk (`Volatility` parameter σ), statistical analysis unveils a significant difference between the stock and the bond prices: the share price increments have a *log-normal distribution*, while the spot rate increments tend to have a more *normal distribution.* In other words, a share presently at EUR 10 is as likely to double in value to EUR 20 as it is to divide by two down to EUR 5. This in contrast with interest rates at 10%, which are as likely to rise (to 15%) or fall (to 5%) by the same amount. The VMARKET applet on-line illustrates the difference between the two distributions, assimilating the random horizontal motion of a red dot with the price of a share in a volatile market.

Virtual market experiments: log-/normal price increments

1. Switch between a log-normal (`LogNkappa=1`) and a normal (`LogNkappa=0`) distribution of the increments and try to qualify the difference between the two evolutions.
2. Increase to `Volatility=3` and `Step 1` log-normal increment at a time. Take a few measurements showing that the jumps are often larger for high prices (to the right) than for low prices (to the left); they are on the contrary symmetric with a normal distribution.
3. Increase the `Volatility` further and check for both distributions if the price can ever become negative. Note that the numerical model produces wrong answers for large increments and large time steps; under realistic conditions, the volatility rarely approaches unity.

Masters: distribution of increments. $\diamondsuit$

In the next chapter, we will study more carefully how the increment dS describing the random component of an asset price S is proportional to a the stochastic increment dX

$$dS \propto dX \qquad \text{normal} \qquad (2.1.1\#\text{eq}.1)$$

$$\frac{dS}{S} \propto dX \qquad \log-\text{normal} \qquad (2.1.1\#\text{eq}.2)$$

where the increment dX can be evaluated as a random number drawn from a normal distribution with zero mean $\mathcal{N}[0,\sigma](x)$ (1.5.3#eq.4) to simulate relatively large time steps longer than one month. Very little programming experience is needed to understand how the log-normal walk (the position or the price) of a single particle has been implemented in VMARKET applet using the scheme

```
double mu      = runData.getParamValue("Drift");
double sigma   = runData.getParamValue("Volatility");

for(int j=0; j<numberOfRealisations; j++){
  currentState[j][0] += currentState[j][0] *
        ( mu*timeStep +
          random.nextGaussian()*sigma*Math.sqrt(timeStep) );
}
```

For those who are not familiar with Java or C++ coding conventions, note that `x+=1.0` increments the variable `x` by the real value one and `j++` increments the variable j by the integer value one after using it for evaluation. Choosing the input parameter `Drift=0.0` in the applet also sets `mu=0.0`, so that the normally distributed random number $\mathcal{N}[0,1]$ obtained from the function `random.nextGaussian()` is here simply scaled by the current position to simulate a log-normal distribution of the increments without drift.
For time steps shorter than one month, the random component of asset price increments is sometimes modeled with a Lévy stable symmetrical distribution

$$L_\alpha(x,\Delta t) \equiv \int_0^\infty \exp(-\gamma\Delta t q^\alpha)\cos(qx)dq \qquad (2.1.1\#\text{eq}.3)$$

The probability $P(0) \equiv L_\alpha(0,\Delta t) = \Gamma(1/\alpha)/(\pi\alpha\sqrt[\alpha]{\gamma\Delta t})$ with an index α=1.40 have been used in Ref.[15] to first calculate the scaling factor γ=0.00375 and then reproduce nearly three orders of magnitude of the "fat tails" of a *leptokurtotic* distribution in (2.1.1#fig.1), here measured from spot price increments over time intervals as short as one minute.

At least some investors have to believe that the price of a share will rise more rapidly than the return they can earn overnight from a deposit at a spot rate, which carries little or no risk at all. A *risk premium* in the range 3-8% is usually added to the 0-2% spot rate to account for a positive drift proportional to the share value (e.g. `Drift=SpotRate+0.04`).

A fixed dividend payment D per share is often made each year after the shareholder's assembly; for simplicity, this can be modeled as a continuous dividend yield $D_0 = D/S$ in the range 1-4% and contributes with a negative drift to the spot price, since the payments reduce the total value of the company (e.g. `Drift=SpotRate-Dividend`, see exercise 2.02).

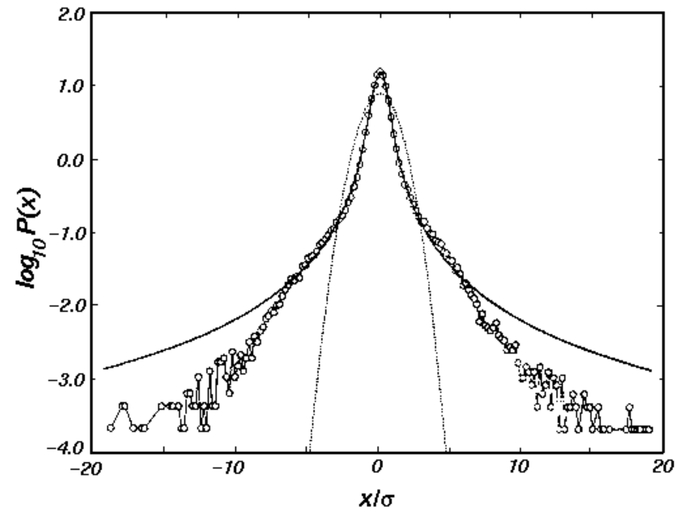

Figure 2.1.1#fig.1: Distribution of the random variable dX observed to drive market price increments (circles) for a time interval as short as 1 min in a comparison with Lévy stable (solid line) and normal distributions $\mathcal{N}[0,\sigma]$ (dots). From ref.[15].

Virtual market experiments: expectations

1. Assume reasonable values `Drift=0.1`, `Volatility=0.5` and develop an intuition for what is a typical evolution of shares quoted on NASDAQ during `RunTime=1.0` year. Measure the final price of a realization by clicking in the plot area and monitor the status field at the bottom of your browser.

2. Decrease the number of `Walkers=100` and perform 2-3 simulations counting the number of prices that drop below the initial value. Quants compare with the area below the black line, which accumulates the realizations in `MeshPoints`-1=20 bins of unit length and should be interpreted as the probability density of reaching those prices. Explain your observations...

3. Switch to (`DistribFct`, without a star) and study the difference between the stock and bond markets using log-/normal distributions of the increments (`LogNkappa=0` or `1`); press `Toggle Display` to re-normalize.

Before we conclude this short introduction on the modeling of the prices on the stock market, simply note that any combination of shares can be used to form a weighted average called *market index*: the most famous such as the Dow Jones and NASDAQ 100 in the US, the FTSE 100 (Financial Times Stock Exchange, pronounce "footsee") in the UK and the Nikkei 225 in Japan combine the largest and most prestigious companies (so called "blue chips" – the name stems from the game of poker where the blue chip have the highest value) in a country and provide a measure of the economic growth of a nation.

2.1.2 Forward contract and futures markets

A *Forward contract* is the simplest form of a *contingent claim* that can be derived from an asset, since it does not contain any element of choice. Two parties agree, on a future *delivery date* T, to exchange an *underlying asset* for a predetermined amount of cash called the *delivery price* K. The underlying can be any kind of asset (e.g. commodities, shares, currencies) that has a fluctuating spot price $S(t)$; on the delivery date T, the terminal payoff $\Lambda(S)$ is simply calculated from the difference between the spot and the delivery price

$$\Lambda_{\text{long}} = S - K, \qquad \Lambda_{\text{short}} = K - S. \qquad (2.1.2\#\text{eq.1})$$

The value (2.1.2#eq.1, left) plotted in (2.1.2#fig.1, left) shows that a *long forward position* (where the holder **has the right and the obligation** to buy the underlying for a price K) increases in value and becomes profitable when the underlying exceeds the delivery price; the maximum losses in a long position occur if the underlying loses all of its market value $S = 0$ and the contract obliges the holder to buy for the delivery price $\Lambda = -K$.

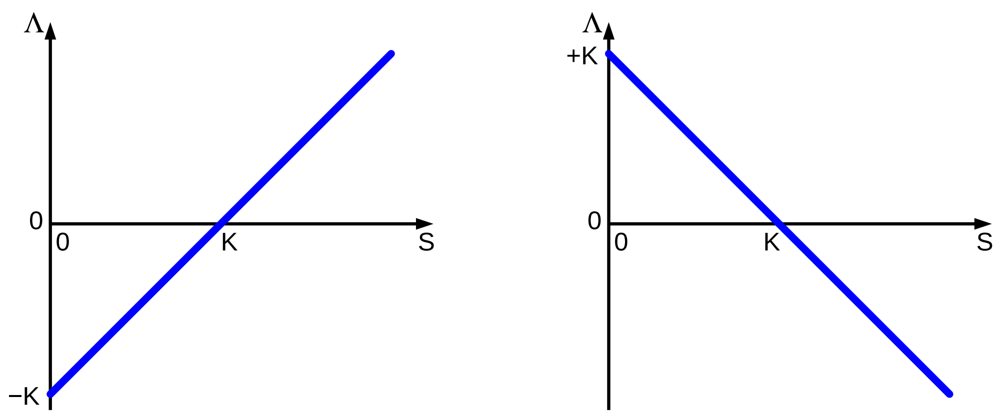

Figure 2.1.2#fig.1: Terminal payoff diagrams $\Lambda(S) = V(S,T)$ of forward contracts struck for a delivery at a price K on a date T; the value of a long (left) and a short position (right) is plotted as a function of possible realizations of the underlying spot price S.

The opposite is true for the party who enters a *short forward position* (right): the holder has both the right and the obligation to sell the underlying with a maximum profit of $\Lambda = +K$ and potential losses that are unlimited if the underlying becomes arbitrarily expensive $S \gg K$. To avoid the unnecessary exchange of cash on the day $t_0 < T$ when the contract is written, the delivery price is sometimes chosen equal to the *forward price* $F(t_0, T)$, which, by definition, makes the initial value of the contract worthless $K = F(t_0, T) = S(t_0)$.

A *futures contract* is a special type of forward contract with standardized delivery dates and sizes that allow trading on an exchange: (2.1.2#tab.1) shows an example of a commodity future that enables the owner of a contract to buy one tone of wheat some time in the future. A system of *margin requirements* is designed to protect both parties against default: instead of realizing the profit or the loss at the expiry date, futures are evaluated every day and *margin payments* are made across gradually over the lifetime of the contract. Despite these differences, futures prices can be shown to be equal to the forward prices if both parties can be trusted and the interest rate is fixed.

Delivery date	Settlement price	High	Low	Volume	Open interest
Nov	59.90	60.75	59.90	160	950
Jan	62.25	62.25	62.25	50	1550
Mar	64.10	64.75	64.10	20	900
May	65.75	66.55	65.75	140	3410

Table 2.1.2#tab.1: Futures of wheat (GBP/tone) quoted on Oct 22, 2002 in the press

2.1.3 Plain vanilla options

To avoid margin payments every day and allow investors who are not members of a clearing house to use derivatives, financial institutions created a new type of security they called *options*. As the name suggests, an option confers **the right and no obligation** for the *holder* (the buyer) to exchange an *underlying* asset (e.g. a share) for a fixed price some time in the future. Of course, the *writer* (the seller) enters an obligation towards the holder, but the writer is generally a large financial institution who is also a member of a clearing house.

In their most basic form (or "flavor"), financial derivatives are commonly called *vanilla*:[33] a plain vanilla *call* (alternatively *put*) option confers its holder the right to buy (alt. sell) the underlying for a fixed amount of cash K called *exercise* or *strike* price. Depending on whether the market value of the underlying S is higher or lower than the strike price K when the option reaches the *expiry date* T, the option holder can choose to either *exercise* the option and buy (alt. sell) the underlying for a price K, or let the option expire worthless.

The terminal payoff $\Lambda(S) = V(S,T)$ plotted in (2.1.3#fig.1) for all the possible realizations of the underlying spot price S is similar to the forward contract (2.1.2#fig.1), except that with no obligation, the option expires worthless and can never become negative.

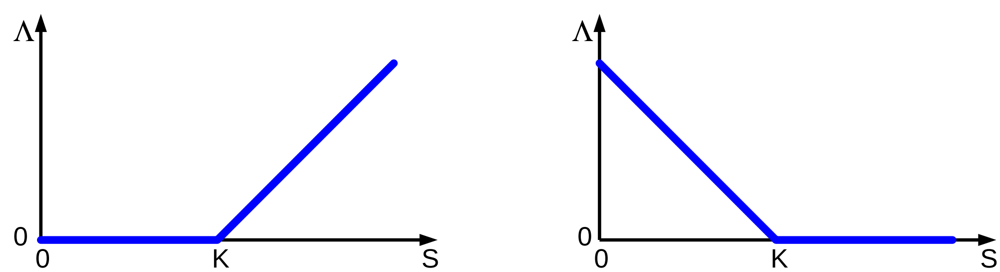

Figure 2.1.3#fig.1: Terminal payoff diagrams $\Lambda(S) = V(S,T)$ showing the value of plain vanilla call (left) and put options (right) as a function of possible realizations of the underlying share price S at the expiry time T.

A vanilla call, which carries the right to buy the underlying for a price K, has a finite value only if the underlying is more expensive on the market; the risk-free profit that can be made by exercising the call option (spending $-K$ to buy the underlying and immediately sell it for a higher price S) is given by the difference $S - K$ if this is positive and zero

[33] as for the ice-cream used as a basis for more elaborate desserts – a reminder that physicists in the early 1980's took their humor from colorful and charming discussions concerning elementary particles to better payed dinners in finance.

otherwise. Similarly, a put option has a finite value provided that its holder can sell the underlying to the writer for a price K that is higher than the spot price on the market $-S$. Mathematically,

$$\Lambda_{\text{call}} = \max(S-K,0), \qquad \Lambda_{\text{put}} = \max(K-S,0). \qquad (2.1.3\#\text{eq.1})$$

Because of the fluctuations in the underlying spot price $S(t)$, the value of an option $V(S(t),t)$ before it expires is generally different from the terminal payoff. By definition, the *intrinsic value* of an option at a time $t<T$ is defined from the terminal payoff as if the option would expire now with the current price of the underlying $V(S(t),T)$. Moreover, call and put options are said to be *out-of-the-money* if they have no intrinsic value and *in-the-money* if they have a large intrinsic value. If $S \approx K$, they are *at-the-money* and that is where their spot price is generally quoted in the press. For example, take one of the two Marconi call options quoted on Feb 23, 2002 by the Financial Times and reproduced in (2.1.3#tab.1).

Option	Strike	Calls			Puts		
(*stock price)	–	May	Aug	Nov	May	Aug	Nov
Hilton	200	17.5	23	26			13.5
(*215 1/2)	220	6.5	13	16	15.5	20	24
AstraZeneca	3500	167.5	267.5	336	129	197	249
(*3519)	3600	116.5	216	284	179.5	245.5	295.5
Marconi	15	4.5	6	7	3.5	4.5	5
(*16 3/4)	20	3	4.5	5.5	7	8	8.5

Table 2.1.3#tab.1: Options traded in London and quoted on Feb 23, 2002 in the press

An investor who speculates on a solid rebound could buy 100 Marconi shares for GBP 1675; alternatively, he could buy 100 call (options are usually traded in units of 100) for GBP 3 each, giving him the right to buy the shares later in May for a total of GBP 2000. If the stock prices double until May (the precise expiration date is on the Saturday immediately following the third Friday of the expiration month), the net benefit from exercising the options to buy 100 shares for 20 and immediately sell them for 33 1/2 will be GBP 3350-2000=1350, a larger return on investment (1350/300=4.5) than the doubling that would have been achieved by using shares alone. If the price of the share remains below 20, however, the holder of calls with a strike at 20 will however never exercise his rights and will eventually loose all the investment made when buying the options, i.e. GBP 300.

This shows how speculators can use options to achieve larger gains for a higher risk, using an effect called *gearing*. Just the opposite can be achieved with *hedging*, where the negative correlation between an asset and its derivatives is exploited in the form of an insurance reducing the investment risk at the expense of for a lower expected return. To show an extreme case of hedging, imagine a portfolio that is long one asset, long one put and short one call with the same strike price K and expiry time T. This combination corresponds to what is called the **put-call parity relation**

$$\Pi(T) = S(T) + \Lambda_{\text{put}} - \Lambda_{\text{call}} = S(T) + \max(K-S(T),0) - \max(S(T)-K,0) = K, \quad \forall S \qquad (2.1.3\#\text{eq.2})$$

and shows that the risk from the uncertain evolution of a spot price $S(t)$ can be eliminated completely in favor of a guaranteed payoff K. Hedging is particularly important for companies that work with expensive raw materials such as gold: the right combination of options allows them to secure their activity without having to take the financial risk from volatile markets.

In general, the right combination of assets (e.g. shares) and derivatives (e.g. call or put options) can be used to expose a portfolio to any level and type of risk chosen by the investor and reap the benefit from the payoff that reflects the investor's opinion. The plots in (2.1.3#fig.2) show only at the option expiry how each term (or *option series*, i.e. options having the same strike price and expiry date) contributes to the put-call parity relation (2.1.3#eq.2) and cancels the investment risk.

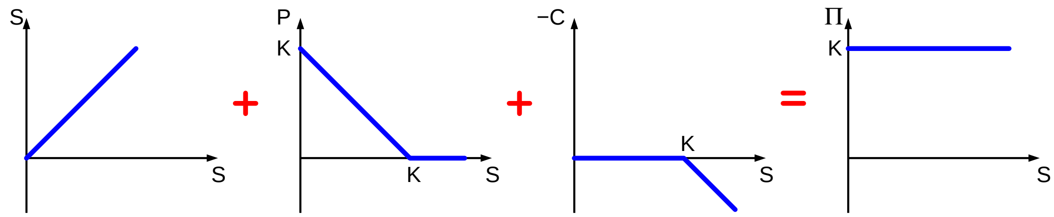

Figure 2.1.3#fig.2: Terminal payoff diagrams illustrating the put-call parity relation.

More complicated payoffs can be obtained by combining vanilla options from the same *class* (i.e. same type, but different strike price and expiry dates, exercise 2.05-2.07) or even with hybrid underlyings that have only partly correlated prices. For example, combining the right amount of put options on the NASDAQ top 100 index (a symbol called QQQ) with shares from IBM, it is in principle possible to make a profit if IBM shares fall, but less than the rest of the technology market. However, remember that individual investors who are not member of a clearing house are only permitted to write *covered* options, where every short position such as the call ($-\Lambda_{\text{call}}$) in the put-call parity relation has to appear in a combination with a long position in the underlying ($+S$).

Finally, note that different exercise styles do affect the price of an option $V(S,t)$ before it expires $t < T$: in chapter 4, we will first study the *European style* where the options can be exercised only on the expiry date and later in chapter 6, we will extend the models to deal with the *American style* where the options can be exercised anytime up to the expiry date.

2.1.4 Exotic options ♢

People generally refer to an *exotic option* when the contract is not a plain vanilla put or call that are traded on an open exchange and is instead traded over-the-counter (OTC).

Binary or digital options may be the simplest form of exotic contracts: they only differ from the vanilla options by the terminal payoff $\Lambda(S)$ that can be any positive function of the asset price S. Some binaries can be obtained from the superposition of vanilla options: *straddles*, *bullish / bearish vertical spreads* and *butterfly spreads* are the subject of exercises 2.05-2.07. Others have payoffs that remind well known functions, such as the *cash-or-nothing call* reproducing the Heavyside H(x)

$$\Lambda_{\text{cash-or-nothing}} = b\mathcal{H}(S-K) \qquad \text{(2.1.4\#eq.1)}$$

and the *supershare* reminding the Dirac delta function

$$\Lambda_{\text{supershare}} = \frac{1}{d}[\mathcal{H}(S-K) - \mathcal{H}(S-K-d)] \qquad \text{(2.1.4\#eq.2)}$$

with terminal payoff diagrams illustrated in (2.1.4#fig.1). This generalization does

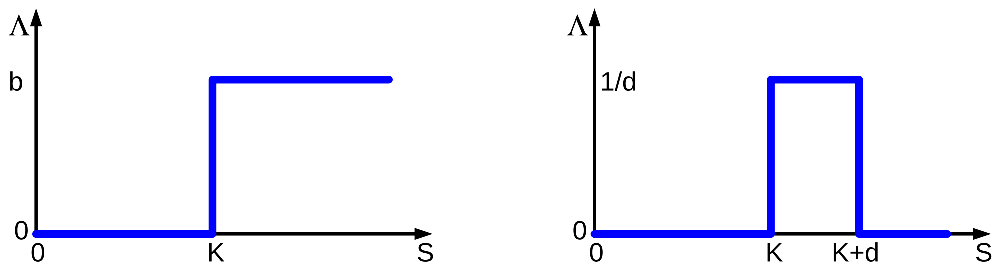

Figure 2.1.4#fig.1: Example of binary / digital options with a general terminal payoff $\Lambda(S)$: a cash-or-nothing call (left) and a super-share (right).

not present any formal difficulty, but the discontinuities in terminal payoff $\Lambda(S)$ do however seriously stretch the non-arbitrage arguments that will be used to derive the Black-Scholes equation in chapter 3. Indeed, a large amount of cash would be required to hedge small changes in the price of the underlying as the option jumps from zero to a finite value, only to fall back to zero shortly afterwards.

Compound options can be understood as options on options. In the simplest case, this involves only put and call options and leads to four types compound options. For example, the *call-on-put* carries the right at time T_1 to purchase for a price K_1 a put option $P_2(K_2, T_2)$. The coming chapters will show how to calculate the value of a put option before it expires; denoting this as $V_{\mathrm{P}_2}(S, T_1)$, the payoff at expiry T_1 gets

$$\Lambda_{\mathrm{call-on-put}} = V(S, T_1) = \max(V_{\mathrm{P}_2}(S, T_1) - K_1, 0). \qquad (2.1.4\#eq.3)$$

Since only one random variable governs the underlying asset price S and its derivatives, the value of a compound option can be calculated by solving first for the value of the option that may be bought or not, e.g. $P_2(K_2, T_2)$; inserting this solution into (2.1.4#eq.3), the value of the compound option is then obtained as usual.

Chooser options are an extension of compound options, giving its holder the right at time T_1 to purchase for an amount K_1 either a call $C_2(K_2, T_2)$ or a put $P_2(K_2, T_2)$. Going through all the possibilities, the same reasoning shows that at expiry T_1

$$\Lambda_{\mathrm{chooser}} = V(S, T_1) = \max(V_{\mathrm{P}_2}(S, T_1) - K_1, V_{\mathrm{C}_2}(S, T_1) - K_1) \qquad (2.1.4\#eq.4)$$

Barrier options are characterized by a a condition set on the existence of the option. When triggered, the right to exercise the option either appears (*in*) or disappears (*out*) if the asset price is above (*up*) or below (*down*) a prescribed *barrier* B:

- up-and-in options come into existence if $S \geq B$ before expiry,
- up-and-out options cease to exist if $S \geq B$ before expiry,
- down-and-in options come into existence if $S \leq B$ before expiry,
- down-and-out options cease to exist if $S \leq B$ before expiry.

Barrier options can be further complicated by making the knockout boundary a function of time B(t) or by having a rebate if the barrier is activated. In the latter case, the holder of the option receives a specified amount if the barrier is reached.

Asian options have a payoff that depends on the price history of the underlying via some kind of average. Different definitions use a continuous $S(t)$ or a discrete sampling of the price history $\{S(t_1), S(t_2)...S(t_N)\}$ and involve an arithmetic

$$\bar{S} = \frac{1}{\Delta t}\int_{t-\Delta t}^{t} S(\tau)d\tau \qquad \bar{S} = \frac{1}{N}\sum_{j=1}^{N} S(t_j) \tag{2.1.4\#eq.5}$$

or geometric average

$$\bar{S} = \exp\left[\frac{1}{\Delta t}\int_{t-\Delta t}^{t} \log S(\tau)d\tau\right] \qquad \bar{S} = \left[\prod_{j=1}^{N} S(t_j)\right]^{1/N} \tag{2.1.4\#eq.6}$$

to define the strike price on the expiry date T. An *average strike call*, for example, is structurally similar to a vanilla call, with a payoff equal to the difference between the asset price at expiry and its average if the difference is positive and zero otherwise

$$\Lambda_{\text{average-strike-call}} = \max(S - \bar{S}, 0). \tag{2.1.4\#eq.7}$$

Such a product can be used to average out the price of an underlying without the need for continuous re-hedging.

Lookback options are similar in spirit as Asian options, except that the strike price is a suitable definition of the maximum or minimum of the underlying price history

$$\Lambda_{\text{lookback-call}} = \max(S - \min_{0\leq\tau\leq t} S(\tau), 0). \tag{2.1.4\#eq.8}$$

Such options can result in extremely advantageous payoffs and can therefore be very expensive: think of an option that allows the holder to buy the underlying at a low and sell it at a high.

Russian options are an example of perpetual options with an American exercise style: at any time, Russian options pay out the maximum realized asset price up to that date.

A variety of option can moreover be constructed by combining several exotic features and the list presented here far from exhaustive.

2.1.5 LEAPS and warrants

For the holders, *long-term equity anticipation securities (LEAPS)* and *warrants* have a strong similarity with European call options, with the slight difference that they have usually a much longer time to expiry of up to 10 years and can sometimes be exercised intermittently on several occasions before they expire.

Rather than referring to an underlying that already exists, companies can however issue a new share each time a warrant is exercised, leading to a dilution of the underlying asset value on exercise. Warrants are a convenient way for companies to raise new capital and are sometimes distributed as an incentive for company executives to link their benefit with the appreciation of the share value sought by the shareholders.

2.2 The credit market and its derivatives

2.2.1 Interest rates: treasury note, LIBOR, credit spread

To compensate for the risk of not getting the back the money, investors in the credit market ask for higher interest rate when the credit worthiness of a borrower deteriorates.

Always at the bottom, the central bank has virtually no risk of defaulting because it can always print money if it needs to: it pays the so-called *treasury rate (TR)* to commercial banks, in return for the margin deposit the latter have to make in order to obtain a banking liscence. The chairman of the Federal Reserve Bank (Fed), the European Central Bank (ECB) and indeed every central bank are responsible for setting the interest rate to steer the economy. For example, by raising the treasury rate, the central bank makes the money more valuable for commercial banks, who in turn, pay a higher interest rate to attract more money from their customers. This is how the central bank tightened its monetary policy and reduced the flow of capital to fight inflation in the early 1990's. On the contrary, (2.2.1#fig.1) suggest that historically low rate have been used during 2002–2004 to stimulate economic growth.

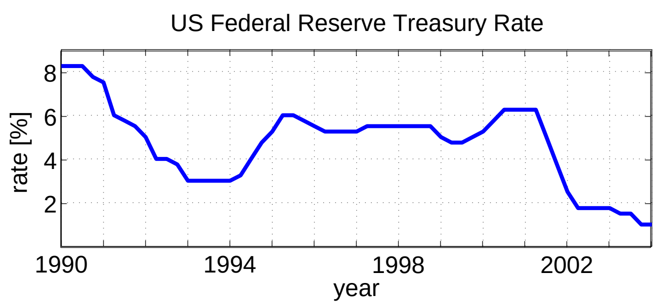

Figure 2.2.1#fig.1: Treasury rate set by the US Federal Reserve bank

For relatively short times (overnight up to 12 months), high-credit financial institutions can borrow money in the inter-bank interest rate market (such as the *London Inter-Bank Offered Rate or LIBOR*), at a rate that is only marginally higher than the treasury rate.

Not to be mixed up with the central bank, the government often borrows money for a longer time to finance big construction projects. Rather than the credit worthiness, it is the expected long-term average rate that generally decides on the spot rate investors are willing to pay. Have a look at the MKTSolution applet on-line to verify how the yield from the *10 years US Treasury bill* clealy follows the trend set by the central bank, with a minium yield of 3.1% in June 2003 when the Treasury rate reached the minimum of 1%. The credit spread of 2100 bps (*basis points* or hundredth of one percent, here 2.1%) does however change on a daily basis and discounts the investors expectation of future movements from the Fed.

Virtual market experiments: interest rates

1. Follow the MKTSolution link and discuss the correlation between the US treasury rate and the 10 and 30 years treasury bills.
2. Which of the three rates is the most / least volatile?

2.2.2 Underlying discount bonds and forward rates ♢

The introductory section 1.3 suggested how fixed income securities, which pay a stream of coupons some time in the future $t_i > t$, are a form of contingent claim that can always be replicated with a combination of zero-coupon bonds $AP(t, t_i)$. Rather than the interest rate, it is the present value of such zero-coupon bonds that is traded on the bond market, with a spot price for each maturity date that is determined by the offer and demand from the investors. Given the similarity with the stock market, it is not surprising that most of the derivatives that have been discussed for shares can be generalized for bonds.

For simplicity, the principal is often normalized to unity $A = 1$, and the *discount bond* $P(t, T)$ is used as a building block for more elaborate products. The *discount function* $P(0, T)$ in particular measures the present value of one unit due at a later time T; (2.2.2#fig.1) shows an example at a time when the treasury rate was relatively low and the market expects rising interest rates.

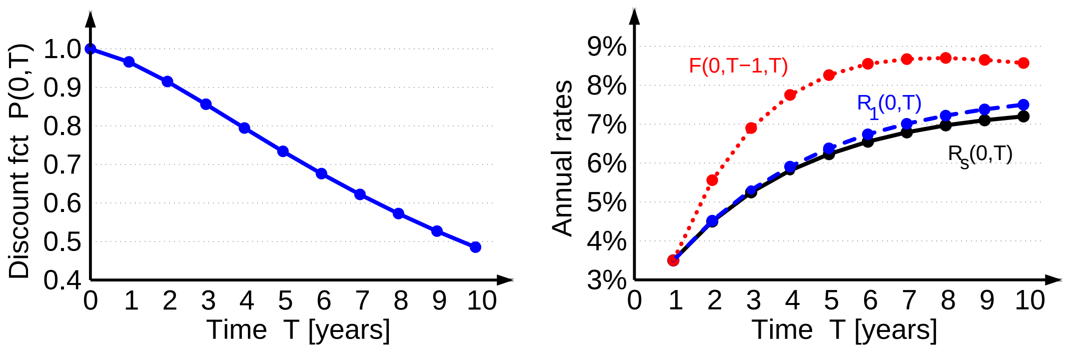

Figure 2.2.2#fig.1: On the left, a discount function $P(0, T)$ from the market with up to $T = 10$ years maturity; on the right the corresponding zero-yield rates using a simple $R_s(0, T)$ (line) and a discrete one year compounding $R_1(0, T)$ (dashes) together with the one year forward rates $F(0, T-1, T)$ (dots).

For a short time, the *spot rate* $r(t)$ taken e.g. from the inter-bank market is nearly constant and the yield can be calculated without compounding $R_s(t, T)$ in (2.2.2#eq.1, left). For longer periods, a compounded calculation has to be used $R_m(t, T)$ in (2.2.2#eq.1, right) and is often replaced by a continuous compounding with a rate $R(t, T) = \exp[Y(t, T)] - 1$ calculated from the discount factor (2.2.2#eq.1, bottom)

$$P(t, T) = \frac{1}{1 + R_s(t, T)(T - t)} \qquad P(t, T) = \frac{1}{[1 + R_m(t, T)/m]^{T-t}}$$

$$P(t, T) = \exp[-Y(t, T)(T - t)] \tag{2.2.2\#eq.1}$$

Plotted as a function of the time to maturity $R(0, T)$, these yield curves are often called the *term structure of interest rates* and can directly be constructed from the price of discount bonds quoted on the market (2.2.2#fig.1, 2.2.2#tab.1, exercise 2.09). Depending on whether the treasury rate is below or above the market expectations for the longer term interest rates, the term structure can have either a positive slope (as in fig.2.2.2#fig.1, right) or a negative slope.

From the ratio between values of the discount function in the future, it is convenient to define the implied *forward rates*, which correspond to the interest payed today (or any time $t < T_1 < T_2$) for a discount bond with a maturity T_2 and starting in the future T_1

$$F(t, T_1, T_2) = \frac{P(t, T_1)/P(t, T_2) - 1}{T_2 - T_1}$$

$$F(t, T, T + \Delta t) = -\frac{\ln[P(t, T + \Delta t)/P(t, T)]}{\Delta t} \qquad (2.2.2\#\text{eq.2})$$

As expected, this definition recovers the present value for $F(t, t, T) \equiv R(t, T)$. Examples of forward rates starting after a delay d are displayed in (2.2.2#fig.2) and have been derived from the same discount function that was used previously in (2.2.2#fig.1, 2.2.2#tab.1).

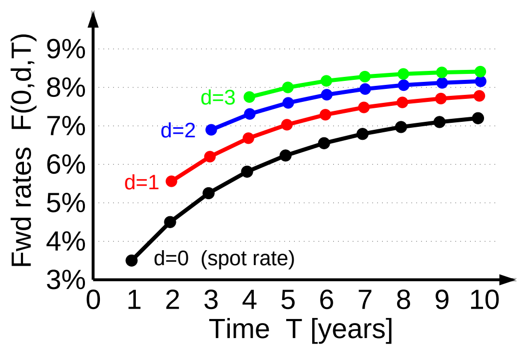

Figure 2.2.2#fig.2: Forward rates $F(0, d, T)$ starting after a delay d plotted as a function of the time to maturity T.

T [years]	P(0,T)	R_s(0,T)	R_1(0,T)	F(0,T-1,T)	F(0,1,T)	F(0,2,T)	F(0,3,T)
1	0.9662	0.0350	0.0350	0.0350	–	–	–
2	0.9153	0.0450	0.0452	0.0556	0.0556	–	–
3	0.8563	0.0525	0.0531	0.0690	0.0620	0.0690	–
4	0.7947	0.0581	0.0591	0.0775	0.0668	0.0731	0.0775
5	0.7340	0.0623	0.0638	0.0826	0.0703	0.0760	0.0800
6	0.6762	0.0655	0.0674	0.0855	0.0729	0.0781	0.0817
7	0.6222	0.0679	0.0701	0.0867	0.0748	0.0796	0.0828
8	0.5725	0.0697	0.0722	0.0870	0.0761	0.0806	0.0835
9	0.5269	0.0710	0.0738	0.0865	0.0771	0.0812	0.0839
10	0.4853	0.0720	0.0750	0.0857	0.0778	0.0816	0.0841

Table 2.2.2#tab.1: Example of a discount function $P(0, T)$ and the corresponding present $R(0, T)$ and forward rates $F(0, d, T)$ starting after a delay d for a maturity date T.

Because of the uncertainty associated with the credit worthiness of long term borrowers and the seemingly random changes of the central bank policies, the price of a discount bond $P(t, T)$, the yield $Y(t, T)$ and the forward rates $F(t, T_1, T_2)$ are all random functions of time via the spot rate $r(t)$; which will be discussed further in chapter 3. Nevertheless, is it possible for loan takers to protect themselves against unpredictable changes in the interest rate? Yes, using the so-called swaps and forward rate agreements.

2.2.3 Interest rate swaps and forward rate agreements ◇

A plain vanilla *interest rate swap* is a contract whereby two parties agree to exchange, at known dates in the future, a fixed for a floating set of interest rate payments without ever exchanging the *notional principal* A. The *fixed leg* of the swap replicates the *coupons* (1.3#eq.5) payed at the end of every *accrual period* spanning from the *reset time* to the *payment time* $[t_i; t_i + \tau_i]$

$$\text{fixed} = AK\tau_i \qquad (2.2.3\#\text{eq.}1)$$

using a fixed interest rate K that is initially agreed upon when the swap is purchased. The *floating leg* consists of payments that also occur at a time $t_{i+1} = t_i + \tau_i$

$$\text{float} = Ar_i\tau_i \qquad (2.2.3\#\text{eq.}2)$$

using however the unknown spot rate $r_i = r(t_i, t_{i+1})$ that prevails at some future times t_i.

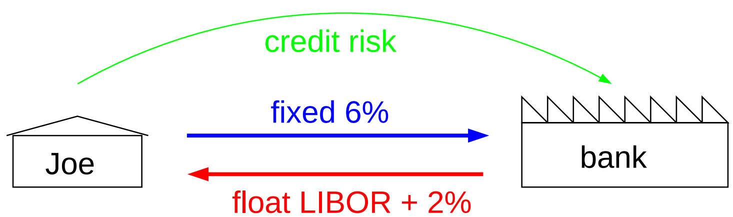

Figure 2.2.3#fig.1: Sketch of an example showing the cash flows when a bank takes the credit risk from a loan taker and agrees to pay 2% in excess of the floating spot rate in exchange of a fixed interet payments of 6%.

The present value of both legs can be discounted back in time using discount bonds to get

$$\begin{aligned} PV(\text{fixed}) &= AK\tau_i P(t, t_{i+1}) \\ PV(\text{float}) &= Ar_i\tau_i P(t, t_{i+1}) &= Ar_i\tau_i \frac{1}{1 + r_i\tau_i} \end{aligned} \qquad (2.2.3\#\text{eq.}3)$$

where a simple compounding has been assumed to substitute the spot rate for the discount bond using (2.2.2#eq.1). Now compare the latter with a portfolio long one bond $P(0, t_i)$ and short another with longer maturity $P(0, t_{i+1})$. At time t_i, the portfolio value is

$$\Pi(t_i) = P(t_i, t_i) - P(t_i, t_{i+1}) = 1 - \frac{1}{1 + r_i\tau_i} = \frac{r_i\tau_i}{1 + r_i\tau_i} \qquad (2.2.3\#\text{eq.}4)$$

or indeed the same, to a normalizing constant A, as the floating leg in (2.2.3#eq.3)

$$r_i\tau_i P(0, t_{i+1}) = P(0, t_i) - P(0, t_{i+1}) \quad \Rightarrow \quad r_i = \frac{P(0, t_i)/P(0, t_{i+1}) - 1}{\tau_i} \qquad (2.2.3\#\text{eq.}5)$$

After identification with the definition of simply compounded forward rates (2.2.2#eq.2), this shows that the a priori unknown values of **future spot rates have the same value today as the projected forward rates** $r_i = F_i = F(0, t_i, t_{i+1})$.

An *equilibrium swap rate* can therefore be calculated in the form of a weighted average of forward rates, making the values of the floating and the fixed legs equal when the contract is initially written at t=0

$$\sum_i PV(\text{float}) = \sum_i AF_i\tau_i P(0,t_{i+1}) = \sum_i AK\tau_i P(0,t_{i+1}) = \sum_i PV(\text{fixed}) \qquad (2.2.3\#\text{eq}.6)$$

$$K = \sum_i w_i F(0,t_i,t_{i+1}) \ , \qquad w_i = \frac{A\tau_i P(0,t_{i+1})}{\sum A\tau_i P(0,t_{i+1})} \qquad (2.2.3\#\text{eq}.7)$$

Notice that no assumption about the random evolution of spot rates has been made, the combination of long and short bonds being amenable to a purely deterministic evaluation in a manner similar to what was has been found for the put-call parity relation (2.1.3#eq.2). By definition, a one period swap is sometimes called *forward rate agreement*: an X's/Y's FRA refers to an interest rate swap starting in X and finishing in Y months and has a present value given by the difference between the floating and the fixed legs

$$PV(\text{FRA}) = A[P(0,t_i) - P(0,t_{i+1})] - AKP(0,t_{i+1})\tau_i \qquad (2.2.3\#\text{eq}.8)$$

The total amount of cash payed after each accrual period $[t_i;t_{i+1}]$ depends on the difference between the settlement rate R_i and the forward rate K; after a simply compounded discounting, the cash flow at a time t_i from the seller to the buyer amounts to

$$A\frac{(R_i - K)\tau_i}{1 + R_i\tau_i} \qquad (2.2.3\#\text{eq}.9)$$

Here is an example showing the entire sequence of events:

Wed 02-Feb-00	2's/5's FRA contract written at 6% for EUR 1 Mio
Fri 31-Mar-00	settlement rate determined at 5% (3 months forward LIBOR for the period Tue 04-Apr-00 to Wed 05-Jul-00) Settlement amount given by (2.2.3#eq.9) 1000000 × (-0.01×92/360)/(1+0.05×92/360) = - 2523.31
Wed 05-Jul-00	buyer pays seller EUR 2523.31

Beware of the dealers jargon, which is opposite for bonds and swaps: *bid* means to buy fixed in bonds and sell fixed in swaps, whereas *offer* means to sell fixed in bonds and buy fixed in swaps.

To conclude this section with a little review, it should now be clear that for the holder of a swap, the earnings increase (alt. drops) when the spot rate evolves above (alt. drops below) the projected forward rates. At the same time, the market data in (1.3#tab.1) illustrates how an increasing spot rate produces a rise in the *par coupon* (particular coupon that prices the bond today exactly at *par* – i.e. for a present value equal to the nominal principal) when the bond trades at a *discount* (alt. *premium*).

2.2.4 Bond options: caps, floors and swaptions ◇

In the same manner as stock market derivatives have been introduced in section 2.1.3 for an underlying share, different types of credit market derivatives confer the holder a right to buy or sell the earnings from interest rates. The simplest is the *bond option*, which confers its holder the right to buy or sell an underlying discount bond with a maturity T_B that is necessarily longer than the option expiry $T < T_B$. Using the notation $V_B(r,t,T_B)$ for the value of the discount bond before the maturity (something that we will calculate later in chapter 5) the terminal conditions for plain vanilla call or put options are simply defined by

$$\begin{aligned} \Lambda_{\text{call-bond}}(r,T) &= \max(V_B(r,T,T_B) - K, 0) \\ \Lambda_{\text{put-bond}}(r,T) &= \max(K - V_B(r,T,T_B), 0) \end{aligned} \qquad (2.2.4\#\text{eq.}1)$$

Directly related is the *interest rate cap* (alt. *floor*), which can be understood as a form of insurance against underlying floating rates moving above (alt. below) a certain level. Imagine a loan for an amount A, where the floating rate $r_i = r(t_i, t_i + \tau_i)$ resets to LIBOR at the end of every period and leads to payments $\text{float}(t_{i+1}) = Ar_i\tau$. To guarantee that these payments do not exceed (alt. drop below) a certain level, the loan can be supplemented with a cap (alt. floor), which is itself composed of a series of *caplets* (alt. *floorlets*) having all the same cap-/floor rate K, and paying the difference when the floating rate moves beyond:

$$\begin{aligned} \Lambda_{\text{floorlet}}(t_{i+1}) &= \max(K - r_i, 0)\tau_i \\ \Lambda_{\text{caplet}}(t_{i+1}) &= \max(r_i - K, 0)\tau_i \end{aligned} \qquad (2.2.4\#\text{eq.}2)$$

No-arbitrage arguments again show that the a priori unknown future values of the rates resetting at the end of the time interval have to be equal to the projected forward rates $r_i = F(0, t_i, t_{i+1})$. Exercise 2.11 shows that a floorlet is closely related with a call option on a discount bond and a caplet is closely related with the corresponding put option. By holding a cap and shorting a floor with different rates $K_{\text{floor}} < K_{\text{cap}}$ you get a *collar*, which guarantees that the interest rate remain within a pre-determined interval. Using the same rates $K_{\text{floor}} = K_{\text{cap}}$ guarantees the payment of a fixed rate leading to the cap-floor parity

$$\text{cap} - \text{floor} = \text{swap} \qquad (2.2.4\#\text{eq.}3)$$

Options can also be defined on credit derivatives: a European swap option or *swaption* carries the right to enter a swap (i.e. switch from variable to fixed interest rates) at a predetermined rate K':

$$\begin{aligned} \Lambda_{\text{payer-swaption}}(t_{i+1}) &= \max(K - K', 0)B \\ \Lambda_{\text{reciever-swaption}}(t_{i+1}) &= \max(K' - K, 0)B \end{aligned} \qquad (2.2.4\#\text{eq.}4)$$

where $B = \sum_k P(t_i, t_{i+k})\tau_k$. Options on caps-/floors can be defined in the same manner as above and are called *captions* and *floortions*.

2.3 Convertible bonds

A *convertible bond* has many features of a regular bond with payment of coupons at regular intervals, except that the holder has the right anytime to exchange the principal for a given asset. When it reaches maturity at time T, the convertible bond returns an amount A unless the owner has converted the bond into n shares of the underlying with a total value nS. Immediately before maturity, the payoff is described by

$$\Lambda_{\text{convertible}} = \max(A, nS). \qquad (2.3\#\text{eq.}1)$$

Although the terminal payoff in (2.3#fig.1) does not present much difficulty, the valuation of a convertible bond before the expiry is substantially complicated by the long time spans under consideration: a proper model involves two imperfectly correlated random variables describing fluctuations in the stock $S(t)$ and bond prices $r(t)$.

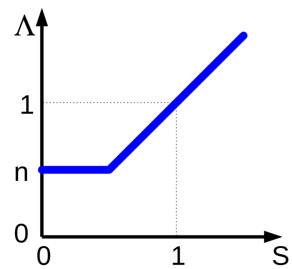

Figure 2.3#fig.1: Terminal payoff diagram for a convertible bond with a principal A as a function of possible realizations of the underlying share price S.

2.4 Hedging parameters, portfolio sensitivity ◊

The put-call parity in (2.1.3#eq.2) shows how a particular combination of options with the underlying can be used to exactly cancel the investment risk in a very simple situation. The same trick could in principle be used for all the securities in a portfolio

$$\Pi = \sum_i S_i \qquad (2.4\#eq.1)$$

Such a hedging strategy is however neither practical nor does it in general produce the desired effect: remember that, to achieve a certain return, investors need to take a limited amount of risk – albeit in a controlled manner that can be monitored to make sure that in the long term the investment survives even large market fluctuations. To quantify the sensitivity to small changes of a limited number of parameters, the portfolio value is often expanded into a Taylor series and the dominant terms labelled using Greek letters.

The largest contribution is usually *delta* and measures how the value of the portfolio changes with the value of each individual asset

$$\Delta_i = \frac{\partial \Pi}{\partial S_i}, \qquad (2.4\#eq.2)$$

Gamma quantifies smaller effects due to the curvature

$$\Gamma_i = \frac{\partial^2 \Pi}{\partial S_i^2} \qquad (2.4\#eq.3)$$

and *vega* (not a Greek letter) measures the sensitivity to changes in the volatility

$$\mathcal{V}_i = \frac{\partial \Pi}{\partial \sigma_i}. \qquad (2.4\#eq.4)$$

An extension including variation from any subset of the assets is straightforward using gradients in multiple dimensions. The so-called *theta* measures the decay of the value with time

$$\Theta = -\frac{\partial \Pi}{\partial t} \qquad (2.4\#eq.5)$$

and *rho* the sensitivity to small variations in the interest rate

$$\rho = \frac{\partial \Pi}{\partial r} \qquad (2.4\#eq.6)$$

Finally, when a bond paying coupons A_i at times t_i is discounted to the present value $B = \sum A_i \exp(-Yt_i)$ at a yield Y, the *duration* measures how long on average the investor has to wait before he recieves cash payments

$$D = \frac{1}{B} \sum_{i=1}^{n} t_i A_i \exp(-Yt_i) = -\frac{\partial \ln B}{\partial Y} \qquad (2.4\#\text{eq.}7)$$

By monitoring and limiting the dependences to some of these factors, hedgers can at least start to identify and reduce if not eliminate the short term risk in a portfolio.

2.5 Computer quiz

1. Can the random future price of a share be modeled mathematically?
   (a) Yes, with a normal random walk where up/down increments have equal chances.
   (b) Yes, with a log-normal walk where multiplication/division have equal chances.
   (c) Not exact values, only likely outcomes to reach a certain value.
   (d) No, it is only possible to model derivatives such as put and calls.

2. Selling short the underlying and buying a put deep in-the-money differ in that
   (a) if the share rises, you win with the share and loose with the option.
   (b) if the share rises, you loose with the share and win with the option.
   (c) a much larger gearing is achieved with the option.
   (d) the potential losses are limited with the option, but not with the share.

3. The holder of a European call has the possibility of
   (a) making arbitrarily large profits and limited losses.
   (b) making arbitrarily large losses and limited profits.
   (c) selling his option on the market before it expires.
   (d) exercising the option before it expires.

4. Exotic options are generally$^\diamondsuit$
   (a) created by a broker OTC for two clients independently of the rest of the market.
   (b) valued using mathematical models in the absence of an efficient market.
   (c) available to small individual investors.

5. An exponential decrease of discount function $P(0,T)$ corresponds to a$^\diamondsuit$
   (a) linear rise of the spot rate in time.
   (b) constant spot rate in time.
   (c) linear drop of the spot rate in time.

6. The projected forward rates $F(t, t_1, t_2)^\diamondsuit$
   (a) can always be calculated from the spot rate.
   (b) can always be calculated from the yield curve.
   (c) are upward sloping when the spot rate is high.
   (d) are upward sloping when the spot rate is low.

7. To hedge a long position in a discount bond you can$^\diamondsuit$
   (a) buy the same type of bond with a different maturity.
   (b) sell the same type of bond with a different maturity.
   (c) make an offer for a swap.
   (d) buy a caplet.

8. The value of a portfolio having positive Θ, Vega and $\Delta = \Gamma = \rho = 0^\diamondsuit$
   (a) does not change in a crash and rises smoothly afterwards.
   (b) drops in a crash and remains constant afterwards.
   (c) rises in a crash and decays smoothly afterwards.

2.6 Exercises

2.01 Modeling interest rates. Use the historical data available on the Internet to characterize the interest rate evolution from the 10 years US treasury bill. Adjust the parameters in the VMARKET applet to reproduce a 6 months simulation using a normal distribution of the spot rate increments. Explain and justify with words how you choose the parameters.

2.02 Forecast for NASDAQ.$^\diamondsuit$ Estimate the probability that the NASDAQ market index will be a factor two higher / lower when the markets close on December 31, 2010.

1. Use historical data over a reasonable time period to split the evolution into a superposition of a systematic drift and a random component with a volatility.
2. Set the default parameters in the VMARKET to mimic 100 possible evolutions of the spot level until the target date is reached. Count the number of realizations where the level is a factor two higher / lower and divide by the total to obtain an estimate.

Document which models and assumptions you are using to obtain your solution.

2.03 Resistance/support levels.$^\spadesuit$ The irrational behavior of traders on the stock market results in "resistance" and "support" levels (also called "barriers") for prices that are difficult to cross–for example when the Dow Jones market index reaches the value 10000. Propose a model to simulate the dynamical properties of such a market with a 6% drift and 40% volatility during nine months, assuming that the index starts 10% below the barrier. Compare the probability distribution obtained with-/out a barrier and choose the parameters to illustrate the effect with an accuracy of at least 20%. To be continued in exercise 4.11.

2.04 Dividend yield.$^\spadesuit$ Explain how you can account for the payment of a fixed dividend per share to mimic the spot price of a dividend paying asset S. Perform the modifications in the Java template proposed for VMARKET applet. Hint: start by assuming a continuous payment in time $D(S,t)dt \equiv D_0 dt$. If you want, you may use the `UserDouble` parameter to define a rate at which discrete payments occur.

2.05 Vertical spread, cash-or-nothing. Explain how you can combine assets with calls and puts that have different exercise prices and the same time to expiry to reproduce the terminal payoff diagrams illustrated below using simple ASCII graphics.

1. The *bearish vertical spread* is called *bearish* to say that the investor benefits from a fall in the price, *vertical* to say that there are two prices involved and *spread* because it is made up of options of the same type.
2. The *cash-or-nothing put* is obtained when $K_2 \to K_1 = K$ at a constant payoff b in-the-money. What view of the market is expressed? Give an example of how an investor could use such an option.

```
V(S,T) ^                                 V(S,T) ^
       |                                        |
 K2-K1 +---------                             b +----------+
       |         \                              |          |
       |          \                             |          |
       |           \                            |          |
       |            \                           |          |
       |             \                          |          |
       |              \                         |          |
     0 +--------+------+---->                 0 +----------+------->
       0        K1     K2   S                   0          K       S
```

```
Figure: Terminal payoff diagram for a bearish vertical spread (left)
        and a cash-or-nothing put option (right)
```

2.06 ***Straddle options.*** Draw the terminal payoff diagram obtained in a portfolio that is (a) short one share S and long two calls with an exercise price K, and (b) long one call and one put with an exercise price K. Use the quotes from table 2.1.3#tab.1 to calculate the cost of setting-up each portfolio; what are the largest possible losses?

2.07 ***Butterfly spead options.*** Draw the terminal payoff diagram obtained in a portfolio that is long one call with an exercise price K_1, long one call with an exercise price K_2 and short two calls with an exercise price K. What view about the market does this investment strategy reflect if $K_1 < K < K_2$?

2.08 ***Delta-hedging.***$^\diamond$ Combine a share S and an option expiring at a time T with an exercise price E to limit the potential losses to a fraction f of the initial investment. Discuss this *delta-hedging* strategy in a comparison with a hedging with three products in the put-call parity; justify your arguments with a sensitivity analysis in terms of the hedging parameters in sect.2.4.

2.09 ***Credit market fundamentals.*** Use the market data available from the Internet to measure the present discount function with up to 10 years maturity in a country you choose. Calculate the implied yield curves using both a simple and a continuous compounding and compare the one year forward rates now with those from the 1990's in (2.2.2#fig.1).

2.10 ***Interest rate swap.***$^\diamond$ Use the discount function (2.2.2#tab.1) to calculate the value of a bond that matures in 4 years and pays a 5% fixed annual coupon. Determine the equilibrium rate for a swap that pays a *par coupon* that prices the underlying bond today exactly at par with the nominal principal at the maturity date.

2.11 ***Call on discount bond is a floorlet.***$^\diamond$ Show that the payoff from a call expiring at a time T and struck at $K' = 1/(1+K\tau)$ on an underlying discount bond maturing at time $T+\tau$, is proportional to a floorlet resetting at time T and paying at $T+\tau$.

All these problems can be edited and submitted for correction directly from your web browser, selecting *WORK:assignments* from the course main page.

2.7 Further reading and links

- **Theory.**
  General: Hull$^\diamondsuit$[11], Cox and Rubenstein$^\diamondsuit$ [6].
  Stock: Wilmott$^\spadesuit$ [24].
  Bonds: Rebonato$^\spadesuit$[19].

- **Stock options.**
  Quotes: CBOE[34], UBS[35], CSFB[36].
  Markets: Eurex[37], ISE[38], Stockholmborsen[39], Pari-mutuel digital call auction PDCA[40].
  New products: Derivatives on economic statistics[41].

- **Bond options.**
  Quotes: Eurex[42], UBS[43].

2.8 Quick intermediate evaluation form

Your opinion is precious. Please fill in the anonymous evaluation form in the web edition. Thank you very much in advance for your collaboration.

[34] http://www.cboe.com
[35] http://quotes.ubs.com
[36] http://www.csfb.com
[37] http://www.eurexchange.com
[38] http://www.iseoptions.com
[39] http://www.stockholmborsen.se
[40] http://www.longitude.com/
[41] http://www.gs.com/econderivs/
[42] http://www.eurexchange.com
[43] http://quotes.ubs.com

3 FORECASTING WITH UNCERTAINTY

3.1 Option pricing for dummies

Because investors have different opinions about the value of every security that is traded in open markets, the spot price does not evolve smoothly with time and *cannot be predicted with any certainty*. Nevertheless, sect.2.1.1 showed that *possible realizations can be simulated* by adding price increments that are typical of volatilities from the past. This chapter examines increasingly sophisticated models to forecast market prices, using them first to estimate the average terminal payoff from financial derivatives, and later to calculate what is a fair price for an option before it expires.

For a qualitative understanding, you can think of an option as a form of insurance covering a financial risk without having any obligation: clearly, an insurance gets increasingly valuable when the market becomes risky and volatile... but how large should this risk premium be? To show here with a simple example how the price of options can be calculated, imagine a vanilla put expiring in 3 months ($T = 0.25$ years) with a strike price $K = 10$, for an underlying currently valued at $S_0 = 9$ EUR and risk free interest rate of 3% ($r = 0.03$).

In a first study, consider the case without drift nor volatility ($\mu = \sigma = 0$) for which the forecast price simply remains constant at $S = 9$ EUR: the terminal payoff from the put option (2.1.3#eq.1) can then easily be calculated three months into the future as suggested by (3.1#fig.1) and yields $\Lambda = \max(10 - 9, 0) = 1$ EUR:

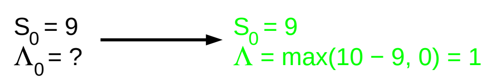

Figure 3.1#fig.1: Sketch showing an evolution of the underlying price in absence of drift and volatility: the forecast remains constant $S = S_0 = 9$ EUR, which makes it easy to calculate the terminal payoff from a put option with a strike of $K = 10$ EUR and 3 months to expiry.

The general **no-arbitrage argument** states that without taking any risk, a security has to grow exactly at the risk free interest (spot) rate. Indeed, if the security grew more quickly, investors could make a risk less profit by borrowing money at the spot rate, buy large amounts of that security and sell it later for a higher price; on the contrary, if the security grew more slowly, investors could make a risk less profit by selling short large amounts of that security and re-invest the proceeds for the higher yield of the spot rate. This **shows that a risk less investment always grow exactly at the risk free interest rate**. In absence of volatility, the terminal payoff from the option can therefore be discounted back in time using the risk free rate during the entire lifetime of the option. This finally yields the present value of the option $\Lambda = 1 \times \exp(-0.03 \times 0.25) = 0.9925$.

In a second study, repeat the calculation using what appears to be a simplistic model of the uncertainty, where the forecast price can take only two distinct values.[44] Starting from the initial price $S_0 = 9$ EUR, the sketch in (3.1#fig.2) shows how, after three months, the price of underlying can either move up to $Su = 12$ or down to $Sd = 6$ EUR. The corresponding terminal option payoff are easily calculated and yield $\Lambda_u = 0$ and $\Lambda_d = 4$.

[44] Quants: as a matter of fact, the central limit theorem shows that for a large number of infinitesimally short time steps, the price increments from a binomial distribution converge to the same terminal payoff as the more sophisticated Monte-Carlo models where the price increments are drawn from a normal distribution.

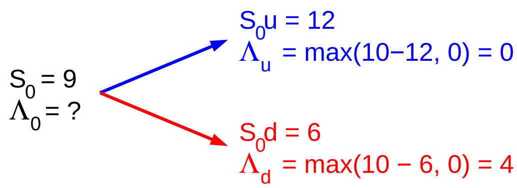

Figure 3.1#fig.2: Sketch showing two possible realizations of the market, with a share price presently at $S_0 = 9$ EUR that can either to move up $Su = 12$ or down $Sd = 6$ EUR at a time in the future when the put option is known to expires with a strike $K = 10$.

Dealing with an uncertain outcome, the general strategy is to **eliminate the risk** by setting-up a perfectly hedged portfolio that combines an (a priori unknown) amount Δ of the underlying security with a (negatively correlated) derivative. The initial value of the portfolio $\Pi_0 = S_0\Delta + \Lambda$ evolves to new values until the expiry date, depending whether the underlying moves up or down. Choosing exactly the right hedging factor Δ, it is however possible to force both outcomes to be equal, which in effect makes the investment risk less

$$\underbrace{12\Delta}_{\text{up}} = \underbrace{6\Delta + 4}_{\text{down}} \qquad \Rightarrow \qquad \Delta = 2/3. \tag{3.1\#eq.1}$$

Indeed, the forecasted value of the portfolio in both cases becomes

$$\Pi_u = 12 \times \frac{2}{3} = 8, \qquad \text{and} \qquad \Pi_d = 6 \times \frac{2}{3} + 4 = 8. \tag{3.1\#eq.2}$$

Repeating the general *no-arbitrage argument*, this a portfolio earns exactly the risk free interest rate and can be discounted back during the entire lifetime of the option $\Pi_0 = 8 \times \exp(-0.03 \times 0.25) = 7.94$. This has to be equal to the initial value of the portfolio:

$$7.94 = \underbrace{9 \times \frac{2}{3}}_{S_0\Delta+\Lambda_0+\Lambda_0} \qquad \Rightarrow \qquad \Lambda_0 = 1.94 \tag{3.1\#eq.3}$$

The example illustrates how the price of an option can be calculated before the expiry date. It also shows that the price can be dramatically different when accounting for the uncertain evolution of the underlying. The next section takes these ideas further and shows how the price levels can be chosen to match the volatility observed in real markets.

3.2 Simple valuation model using binomial trees ♢

A single step binomial forecast provides only a crude approximation for the fair price of an option before it expires. To increase the accuracy of the model, an obvious improvement would be to extend the number of possible outcomes; the evolution of the forecast price could also be modeled more accurately by allowing them to reverse trends during the calculation. Both can be achieved by dividing the lifetime of the option $[0;T]$ into a number smaller time intervals of duration Δt and performing the calculation recursively with the *binomial tree* model sketched in (3.2#fig.1). For each step starting with the present value of the underlying S_0, two new forcasts are obtained by multiplying the value on each node by the factors u or d to mimic possible movements up or down until the entire lifetime of

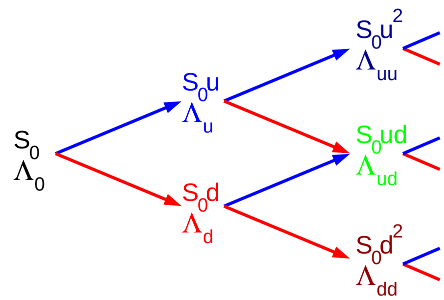

Figure 3.2#fig.1: Sketch with a sequence of binomial steps of limited duration showing that the possible realizations of the underlying take the shape of a tree spanning over the entire lifetime of the option.

the option is covered by the tree. A perfectly hedged portfolio is then constructed starting from every branching point closest to the expiry date (work backwards from the right of (3.2#fig.1), by combining an amount delta of the underlying with a (conventionally short) position of a (positively correlated) option. By demanding that the portfolio be risk free, the movement up or down produce the same return and a new value is obtained for delta

$$S_0 u\Delta - \Lambda_u = S_0 d\Delta - \Lambda_d \qquad \Rightarrow \qquad \Delta = \frac{\Lambda_u - \Lambda_d}{S_0(u-d)} \tag{3.2\#eq.1}$$

Since a perfectly hedged portfolio carries no risk at all, the standard *no-arbitrage argument* shows that it can be discounted back one step in time using the risk free interest rate r. This discounted value (3.2#eq.2, left hand side) has to be equal to the cost of setting up the portfolio before the step is taken (right hand side):

$$(S_0 u\Delta - \Lambda_u)\exp(-r\Delta t) = S_0\Delta - \Lambda_0 \tag{3.2\#eq.2}$$

Substituting the hedging factor delta (3.2#eq.1) and rearranging the terms, this yields an expression to calculate the fair value of an option one step back at a time

$$\Lambda_0 = [p\Lambda_u + (1-p)\Lambda_d]\exp(-r\Delta t) \qquad \text{with} \qquad p = \frac{\exp(r\Delta t) - d}{u-d} \tag{3.2\#eq.3}$$

The parameter p can be interpreted as the probability of the forecast price moving up and $(1-p)$ the probability that it will move down in the tree. The scaling factors (u, d) control the amplitude of the change and have to be carefully chosen

$$u = \exp(+\sigma\sqrt{\Delta t}) \qquad \text{and} \qquad d = \exp(-\sigma\sqrt{\Delta t}) \tag{3.2\#eq.4}$$

to reproduce the drift and the volatility observed in the real markets (quants read below).
Although the importance will only appear later, simply note here that the expected value of the underlying calculated using the probability (3.2#eq.3)

$$E[S] = pS_0u - (1-p)S_0d = p(u-d)S_0 + S_0s = S_0\exp(r\Delta t) \tag{3.2\#eq.5}$$

grows, on average, exactly at the risk free interest rate. Using the probability (3.2#eq.3) therefore implies that the return on the underlying stock is equal to the risk free rate $\mu = r$.

Quants: matching the parameters (u,d) with drift and volatility. ♠
For clarity, distinguish the probability of a price moving up in the tree p from the probability of the price moving up in the real world q. In the presence of drift, the real world price of the underlying grows exponentially (3.2#eq.6, left hand side), which should be reproduced by the expectation $E[S]$ from the price forecast in the tree (right hand side):

$$S_0 \exp(\mu\Delta t) = qS_0 u + (1-q)S_0 d \qquad \Rightarrow \qquad q = \frac{\exp(\mu\Delta t) - d}{u-d} \qquad (3.2\#eq.6)$$

In the same manner, the real world variance (square of volatility, left) has to be matched with the variance $\mathrm{Var}[S] = E[S^2] - (E[S])^2$ from the price forecast in the tree (right):

$$\sigma^2\Delta t = qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 \qquad (3.2\#eq.7)$$

Substitute the real world probability (3.2#eq.6) into (3.2#eq.7)

$$(u+d)\exp(\mu\Delta t) - ud - \exp(2\mu\Delta t) = \sigma^2\Delta t \qquad (3.2\#eq.8)$$

and expand to first order in the small time steps by writing $\exp(\mu\Delta t) \approx 1 + \mu\Delta t$. The symmetric solution $u = 1/d$ is generally chosen and has been given in (3.2#eq.4).

To summarize, calculations using binomial trees can be organized as follows

1. Divide the entire lifetime of the option into a finite number of steps N, ranging from only a few (by hand) up to 30 (using a computer to evaluate 31 possible outcomes that are connected with $2^{30} \approx 1$ billion possible paths).

2. Forecast the underlying forward in time (trunk→leaves), choosing (u,d) according to (3.2#eq.4) to reproduce the historical volatility observed in a real market.

3. Work backward in time (trunk←leaves) starting from the terminal option payoff; for every neighboring branching point, calculate the hedging delta (3.2#eq.1) and the option price at the previous time step (3.2#eq.3). In the case of American options, substitute the (larger) *intrinsic value* that can be obtained from an early exercise when the calculated price drops below this intrinsic value.

4. The final result is obtained on the trunk of the tree and is an approximation of the fair value of the option before the expiry date, with an accuracy proportional to $1/\sqrt{2^N}$.

Involving only simple mathematics, binomial trees are ideally suited to develop an intuition for option pricing (exercise 3.02, 3.03). Some practitioners use trees to evaluate option prices with a computer: the forthcoming sections argue that differential calculus provides a better framework to account for the features in exotic contracts. Indeed, without these features, a computer is not really needed, since the price will be calculated from an analytic solution of the Black-Scholes differential equation that we are about to derive.

3.3 Improved model using stochastic calculus

3.3.1 Wiener process and martingales ♠

Although it is not possible to predict with any certainty the spot price X of an asset in an efficient market, sect.2.1.1 demonstrated in that possible realizations can be simulated with their probability of occurrence by summing price increments dX over small steps in time dt. Separating the *deterministic* (μdt) from the remaining *random* component (σdW), the evolution of the random variable X satisfies a *stochastic differential equation*

$$\frac{dX}{X^\kappa} = \mu dt + \sigma dW(t) \qquad (3.3.1\#eq.1)$$

where κ=0 (alt. 1) chooses between the normal (alt. log-normal) distribution of increments discussed previously in (2.1.1#eq.1). Integrating over time, the first term yields a uniform (alt. exponential) growth with a rate μ that accounts for example for a continuous payment of a fixed dividend (alt. a compounded interest rate). The second term reproduces a random walk proportional to the market volatility σ, using a so-called *Wiener process* that has the following properties

1. The Wiener increment $dW(t) = W_{t+dt} - W_t$ over a time step dt is a random variable drawn from a normal distribution with zero mean and a time-step variance $\mathcal{N}[0, dt]$ (1.5.3#eq.4). Extensions to other distributions (e.g. 2.1.1#eq.3) are possible.

2. The Wiener increment is independent of the past. Using the definition (1.5.3#eq.2) for the correlation, this is satisfied when $E[dW(t_1)dW(t_2)] = 0, \forall t_1 < t_2$.

A convenient way of writing this is

$$dW(t) = \zeta\sqrt{dt} \qquad \text{with} \qquad \zeta \in \mathcal{N}(0,1) \qquad (3.3.1\#eq.2)$$

where different realizations of the random variable $dW(t)$ are generated using random numbers ζ that are normally distributed. This construction provides the mathematical foundation for the *Monte-Carlo* simulations, where the mean value of the increment

$$E[dX] = E[X^\kappa(\mu dt + \sigma dW(t))] = \mu X^\kappa dt \qquad (3.3.1\#eq.3)$$

and the variance

$$\text{Var}[dX] = E[dX^2] - E[dX]^2 = E[(X^\kappa \sigma dW(t))^2] = \sigma^2 X^{2\kappa} dt \qquad (3.3.1\#eq.4)$$

are matched with historical data to forecast possible evolutions into the future. Third or even higher order moments of the probability distribution can in principle also be matched; experience, however, shows that little is to be gained from such a procedure.

A better description of the market is obtained by performing a *principal components analysis* [19], where several imperfectly correlated random variables are identified and then superposed to drive increments of the form

$$dX = \sum_i X_i^\kappa[\mu_i dt + \sigma_i dW_i(t)] \tag{3.3.1\#eq.5}$$

$$\begin{aligned} E[dW_i(t_1)dW_i(t_2)] &= \quad 0 \qquad\qquad t_1 < t_2 \\ E[dW_i(t)dW_j(t)] &= \quad \rho_{ij}dt \end{aligned} \tag{3.3.1\#eq.6}$$

where $\rho_{ij} \in [-1;1]$ is a correlation factor. The first component typically accounts for 80-90% (and the first three for up to 95-99%) of the variance observed in the market price of forward rates [19].

Note that the forecast value for the price of a share $(S + dS)$ or an interest rate $(r + dr)$ constructed using Wiener increments depends only the present value: this independence from the past is known in mathematics as the *Markov property*. Also note that a suitable choice of a *numeraire* can always be found to normalize random variables and make them risk-neutral by scaling out the drift observed in the real world: called *martingales*, such variables play an important role in the construction of financial models.

3.3.2 Itô lemma ♠

Despite the short-hand differential notation that has been used so far, the stochastic differential equation (3.3.1#eq.1) is formally defined only in its integral form: in other words, a probability weighted average has to be carried out before the random sampling from the Wiener process acquires any significance. The stochastic or Itô calculus dealing properly with the extensions of the usual Riemann integrals to non-smooth differentials $dW(t)$ leads to the Itô lemma and draws on mathematics that goes beyond the scope of this course. The same result can however be derived from a Taylor expansion in multiple dimensions, keeping terms up to $\mathcal{O}(dt)$ and $\mathcal{O}(dW^2)$ and applying the special rules for stochastic calculus:

$$\begin{aligned} dt^2 &= 0 & dW_i(t)^2 &= dt \\ dtdW_i(t) &= 0 & dW_i(t)dW_j(t) &= \rho_{ij}dt \end{aligned} \tag{3.3.2\#eq.1}$$

After substitution of the value of the stochastic differential (3.3.1#eq.1), this leads directly to *Itô's lemma*, here given for the function of only one stochastic variable $X(t)$

$$\begin{aligned} df(X(t),t) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}dX(t)^2 + \ldots \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}X^\kappa[\mu dt + \sigma dW(t)] + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}X^{2\kappa}[\mu dt + \sigma dW(t)]^2 \\ &= \underbrace{\left[\frac{\partial f}{\partial t} + \mu X^\kappa \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 X^{2\kappa}\frac{\partial^2 f}{\partial X^2}\right]dt}_{\text{deterministic component}} + \underbrace{\frac{\partial f}{\partial X}\sigma X^\kappa dW(t)}_{\text{random component}} \end{aligned} \tag{3.3.2\#eq.2}$$

In words, the Itô lemma states that the differential of a stochastic function is the superposition of a deterministic component proportional to the time step dt, and a random component proportional to the Wiener increment $dW(t)$. Remember that the factor $\kappa = 0$ or 1 here chooses between a normal or log-normal distribution of the price increments dX and can also take other values if this is found to be appropriate.

3.3.3 Evaluate an expectancy or eliminate the uncertainty ♠

The Itô lemma shows how it is possible to superpose infinitesimal increments df to mimic the evolution of the value of a financial derivative $f(X(t),t)$, which is a known function of the stochastic spot price $X(t)$. Starting from an initial (alt. terminal) value that is known at a time T, a finite number of incremental changes dV can in be accumulated to approximate a single possible outcome at a later (alt. earlier) time: the implementation of the so-called *Monte-Carlo method* will discussed later with a practical example (sect.4.5). At the end, the fair price for the derivative is calculated as the **expectancy from a large number of possible outcomes**, i.e. by performing a statistical average where each payoff is properly weighted with the number of times this value has been reached.
The main drawback of a statistical method is the slow convergence ($\propto 1/\sqrt{N}$) with the number of samples. The problem can be traced back to the difficulty of integrating the stochastic term in the Itô differential (3.3.2#eq.2). By combining anti-correlated assets, it is however possible to reduce the amount of fluctuations in a portfolio. In fact, it is possible to completely **eliminate the uncertainty through delta-hedging**, in effect transforming the stochastic differential equation (SDE) into a partial differential equation (PDE) that is much simpler to solve. For that

1. Create a portfolio, combining one derivative (e.g. an option) of value $f(X(t),t)$ with a yet unspecified, but constant number $-\Delta$ of the underlying asset. The initial value of this portfolio and its incremental change per time-step are

   $$\Pi = f - \Delta X, \qquad d\Pi = df - \Delta dX \tag{3.3.3\#eq.1}$$

   where the Itô differential (3.3.2#eq.2) can be used to substitute df and the stochastic differential (3.3.1#eq.1) for dX.

2. Choose the right amount Δ of the underlying so as to exactly cancel the random component, which is proportional to $dW(t)$ in the Itô differential

   $$\Delta = \frac{\partial f}{\partial X} \tag{3.3.3\#eq.2}$$

   With this choice, the total value of the portfolio becomes deterministic, i.e. the remaining equation has no term left in $dW(t)$.

3. No-arbitrage arguments show that without taking any risk, the portfolio has to earn the same as the risk-free interest rate $r(t)$

   $$d\Pi = r\Pi dt \tag{3.3.3\#eq.3}$$

   Indeed, if this was not the case and the earnings were larger (alt. smaller), arbitrageurs would immediately borrow money from (alt. lend money to) the market until the derivative expires and make a risk less profit from the difference in the returns.

4. Reassemble the small deterministic incremental values into a partial differential equation, which can be solved more efficiently to obtain the present value of the derivative $f(X(t),t)$.

Of course, the amount Δ will change after a short time and the portfolio has to be continuously re-hedged to obtain a meaningful value for the derivative–which is not quite possible in the real world. Two examples illustrate the procedure in the coming sections, using delta-hedging to calculate the price of derivatives in the stock and the bond markets.

3.4 Hedging an option with the underlying (Black-Scholes) ◇

Take a log-normal distribution of the price increments dS in (3.3.1#eq.1) modeling the price evolution of a share S with a volatility σ in an efficient market where all the risk-free portfolios earn the same deterministic spot rate r. For simplicity, neglect the payment of dividends $D_0 = 0$ and assume that the underlying trades continuously without transaction costs $\nu = 0$ (extensions are considered in exercises 3.03 & 3.04).

Now construct a portfolio combining an option (or indeed any kind of stock market derivative) with a value $V(S(t), t)$ and an unspecified, but constant number $(-\Delta)$ of the underlying share. The initial value of this portfolio and its incremental change per time-step are

$$\Pi = V - \Delta S, \qquad d\Pi = dV - \Delta dS \tag{3.4\#eq.1}$$

Using Itô's lemma (3.3.2#eq.2) to substitute for the stochastic value of the option price increment dV and choosing $\kappa = 1$ for a log-normal walk with (3.3.1#eq.1), this yields

$$d\Pi = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \mu S \Delta\right] dt + \sigma S \left[\frac{\partial V}{\partial S} - \Delta\right] dW(t) \tag{3.4\#eq.2}$$

The random term in $dW(t)$ can now be eliminated by continuously hedging the portfolio so as to maintain the number of shares equal to $\Delta = \partial V / \partial S$. With this choice, the portfolio becomes deterministic (risk-free) and, using the standard *no-arbitrage arguments*, must earn the same amount as the risk-free return $r\Pi dt$ payed when an amount of cash Π is invested at a spot rate r during a small time dt. Equating both earnings

$$d\Pi = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt = r\Pi dt \tag{3.4\#eq.3}$$

leads directly to the celebrated Black-Scholes equation [3]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \tag{3.4\#eq.4}$$

which shows that the return of a delta-hedged portfolio (first two term) is exactly balanced by the return on a bank deposit (last two terms). The equation describes the price of stock options (call, put and indeed any derivative security where the price $V(S, t)$ depends only on the spot price and time) in a market that is parametrized by the volatility σ and the spot rate r. An important finding is that the price of an option does not explicitly depend on how rapidly the underlying asset grows: **the drift μ does not explicitly appear in the Black-Scholes equation**. The reason is that the option price has been calculated relative to the underlying, so that the drift is here already accounted for in an implicit manner.

In mathematics (3.4#eq.4) is known as a partial differential equation (since it has partial derivatives up to first order in time ∂_t and second order in the asset price ∂_S^2) of backward parabolic type (backward because both these derivatives have the same sign, parabolic since this combination of coefficients yields exactly one characteristic). In physics, it bears a strong resemblance with the heat equation, except that here it is the price of an option that is diffusing with respect to the underlying rather than heat diffusing in space.

To solve the Black-Scholes equation backward in time, a *final condition* needs to be imposed at expiry $t = T$ reproducing the terminal payoff $V(S, T) = \Lambda(S), \forall S$ discussed in chapter 2. *Boundary conditions* have to be imposed on the price boundaries and are obtained from additional no-arbitrage arguments. A vanilla put option, for example, is increasingly unlikely to be exercised when the price of the underlying gets very large, so that the value of a put $V(S,t) \to 0, \forall t$ when $S \to \infty$. On the lower boundary, (3.3.1#eq.1) shows that the log-normal price increments of the underlying vanish $dS \to 0$ when $S \to 0$, so that the problem becomes deterministic: a put option that is certain to be exercised for a price K at the expiry can be discounted back to the present value $V(0,t) = K\exp[-r(T-t)]$. Similar arguments will be used to derive the conditions for a call option in section 4.4.

3.5 Hedging a bond with another bond (Vasicek) ♠

In contrast to the price of a share (which can never drop below zero because of the regulations of bankruptcies), it is in principle possible to live with negative interest rates: these have even been observed in Switzerland in the 1960s, albeit only for a short period. This motivates the development of stochastic models for long term interest rates where the random walk of the spot rate increments dr is based on (3.3.1#eq.1) and the distribution chosen somewhere between $\kappa = 0$ (normal walk assumed by Vasicek [23]) and $\kappa = 1/2$ (assumed by Cox, Ingersoll and Ross [5]).

Here we follow the classical derivation from Vasicek using $\kappa = 0$ and apply Itô's lemma to calculate the stochastic price increment for a bond $P(t,T)$ of maturity T

$$dP(t,T) = \underbrace{\left[\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 P}{\partial r^2} + \mu_s \frac{\partial P}{\partial r}\right]}_{\mu(t,T)} dt + \underbrace{\left[\sigma_s \frac{\partial P}{\partial r}\right]}_{\sigma(t,T)} dW(t) \qquad (3.5\#eq.1)$$

The drift and volatility of a bond (μ, σ) depend on the spot rate $r(t)$ and can be parametrized using market data (μ_s, σ_s). Having no anti-correlated underlying as for the case of stock options, the trick here is to create here a portfolio that is long one bond $P(t,T_1)$ and short a number $(-\Delta)$ of bonds $P(t,T_2)$ with a different maturity $T_1 < T_2$. The portfolio value and its incremental change per time step become

$$\Pi(t) = P(t,T_1) - \Delta P(t,T_2) \qquad (3.5\#eq.2)$$

$$d\Pi(t) = [\mu(t,T_1) - \Delta\mu(t,T_2)]\,dt + [\sigma(t,T_1) - \Delta\sigma(t,T_2)]\,dW(t) \qquad (3.5\#eq.3)$$

Choosing $\Delta = \sigma(t,T_1)/\sigma(t,T_2)$ to eliminate the random component, the portfolio becomes deterministic and, using no-arbitrage arguments, earns exactly the risk-free spot rate

$$d\Pi(t) = dP(t,T_1) - \Delta dP(t,T_2) = r(t)\Pi(t)dt = r(t)dt\,[P(t,T_1) - \Delta P(t,T_2)] \qquad (3.5\#eq.4)$$

Substitute the value for Δ, insert (3.5#eq.1) for the increments and move all the terms with the same maturity on the same side of the equation to obtain

$$\frac{\mu(t,T_1) - r(t)P(t,T_1)}{\sigma(t,T_1)} = \frac{\mu(t,T_2) - r(t)P(t,T_2)}{\sigma(t,T_2)} \equiv \lambda(t,r), \qquad \forall T_1, T_2 \qquad (3.5\#eq.5)$$

This shows that the so-called *market price of risk* $\lambda(t,r)$ is independent of the maturity and can therefore be used to parameterize the market. Rewriting the bond drift $\mu(t,T)$ in (3.5#eq.1) in terms of the market price of risk (3.5#eq.5), the properly hedged portfolio (3.5#eq.2) finally yields the bond pricing equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 P}{\partial r^2} + (\mu_s - \lambda\sigma_s)\frac{\partial P}{\partial r} - rP = 0 \qquad (3.5\#eq.6)$$

which can be solved using the normalized terminal payoff at maturity $P(T,T) \equiv 1, \forall r$ as the terminal condition. Boundary conditions depend on the model for the spot rate, e.g.

$$\begin{aligned} \mu_s(r,t) &= a(t)r - b(t) \\ \sigma_s(r,t) &= \sqrt{c(t)r - e(t)} \end{aligned} \qquad (3.5\#\text{eq.}7)$$

which can be used with $P(r,t) \to 0, \forall t$ when $r \to \infty$ and keeping $P(r,t)$ finite for small r.

Using (3.5#eq.6) to re-write the deterministic component of a single bond (3.5#eq.1), Itô's lemma can be put into another form

$$dP = \left[\lambda\sigma_s \frac{\partial P}{\partial r} + rP\right] dt + \sigma_s \frac{\partial P}{\partial r} dW(t) \qquad (3.5\#\text{eq.}8)$$

or

$$dP - rPdt = \sigma_s \frac{\partial P}{\partial r} \left[\lambda dt + dW(t)\right] \qquad (3.5\#\text{eq.}9)$$

showing on the left hand side that a higher return can be earned exceeding the risk-free interest rates, provided that the investor accepts a certain level of risk $dW(t)$. Indeed, the portfolio grows by an extra λdt per unit of risk dW. This justifies the interpretation of λ as the market price of risk, with investors that are either risk seeking or risk averse depending whether λ is positive or negative.

3.6 Computer quiz

1. By combining anti-correlated securities, a portfolio becomes
   (a) more risky
   (b) more predictable
   (c) more profitable

2. No arbitrage arguments state that
   (a) arbitrage can never exceed the risk free interest rate
   (b) arbitrageurs immediately seize opportunities for making risk-free profits
   (c) without using arbitrage opportunities, a portfolio grows at the risk-free rate

3. Which of the following random variables are martingales? ♠
   (a) betting on "heads" when you flip a coin
   (b) the value of a share
   (c) the daily price increment to the value of a share in a mature company
   (d) the daily increment to the short term interest rate
   (e) playing Russian roulette

4. A discrete rather than continuous delta-hedging of the portfolio ♠
   (a) reduces the expected return of the portfolio
   (b) increases the expected return of the portfolio
   (c) increases the amount of risk in the portfolio

5. A negative market price of risk λ signifies that ♠
   (a) the underlying is cheap, signalling a good buying opportunity
   (b) the stock market is more volatile than the bond market
   (c) the bond market will outperform the stock market
   (d) the investors expect the underlying to under perform the spot rate

6. The coefficients (σ, r, etc) in the Black-Scholes and Vasicek equations ♠
   (a) have been assumed constant
   (b) can be arbitrary deterministic functions of time and the stochastic variable
   (c) can be arbitrary stochastic functions

3.7 Exercises

3.01 Price estimate for a European call. Repeat the option pricing calculation from sect.3.1, using the same parameters for the case of a call option. Verify that the put-call parity relation (2.1.3#eq.2) remains satisfied, provided that the guaranteed payoff is properly discounted at the risk free interest rate.

3.02 Convergence study using trees.$^\diamond$ Use 2 trees to improve the accuracy of the price estimate obtained for the put option in sect.3.1 by dividing the option lifetime first into 2 and later into 3 steps. Which is the best estimate and what is its accuracy?

3.03 American put calculated with a tree.$^\diamond$ Use a two-step tree to calculate the value of an American put option expiring in 2 years with a strike price of 104 EUR, if the underlying is currently trading for 100 EUR with a volatility of 300% and the risk free interest rate is 5%.

3.04 Stock option with a dividend yield.$^\spadesuit$ Show that the Black-Scholes equation can be extended to account for dividend paying assets

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0$$

where D_0 is the dividend yield modeling a deterministic and continuous payment proportional to the underlying value. Use no-arbitrage arguments to determine if and how this affects the initial and the boundary conditions for a European vanilla call option on a share. *Hint: examine first how the payment of a dividend affects the stochastic evolution of the underlying price and repeat the analysis in sect.3.3 remembering that the value of the asset drops according to the dividend payment.*

3.05 Stock option with transaction costs.$^\spadesuit$ Reproduce Leland's model accounting for transaction costs in the form of commissions, using a delta-hedging procedure to arrive at the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi}} \frac{k\sigma S^2}{\sqrt{\delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| + rS \frac{\partial V}{\partial S} - rV = 0$$

Allow for a fractional number of assets to be bought ($\nu > 0$) or sold ($\nu < 0$) with commissions that are proportional to the monetary value $k|\nu|S$. The portfolio is periodically re-hedged after a finite time δt small enough to justify a leading order expansion of the transaction volume $\nu = \Delta(S + \delta S, t + \delta t) - \Delta(S, t)$. Conclude by setting the expected return of the portfolio equal to that of a bank deposit. *Hint: for a normally distributed number $\phi \in \mathcal{N}(0,1)$, the expectancy $\mathcal{E}[|\phi|] = \sqrt{2/\pi}$.*

3.06 Bond option with forward contracts.$^{\spadesuit\spadesuit}$ Having shown the close relation between a call on a discount bond and a floorlet (exercise 2.11), this is an extension of the syllabus leading to Black's formula for the price of such a product.

Consider a portfolio with an option O expiring at t with a strike K on a discount bond $P(0,T)$ with a maturity $T > t$; for hedging purposes, add an amount a of discount bonds $P(0,t)$ and another b of forward contracts $[P(0,T) - P_0]$ maturing at T with a strike at P_0. Choose $P(0,t)$ as numeraire and write the portfolio value

$$\Pi(t) = fP(0,t) + aP(0,t) + b[Q(t,T) - Q_0]P(0,t)$$

Derive the stochastic variation induced by log-normally distributed increments $dP(0,t)$, $dQ(t,T)$ with a correlation factor ρ_{PQ} modeling the dependence between bonds of different maturities. Show that the choice $x=-f$, $y=-\partial f/\partial Q$ makes the differential deterministic; starting from $Q_0 = Q(0,T)$ to make the initial holding worthless, the price of the portfolio cannot increase with time and yields the option value

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_Q^2 Q(t,T)^2 \frac{\partial^2 f}{\partial Q^2} = 0$$

This can be solved analytically by choosing appropriate initial / boundary conditions and yields

$$\begin{aligned} \text{Call}[P(t,T)-P(0,T),K] &= [P(t,T)N(h_1) - KN(h_2)] \\ h_{1,2} &= \left[\ln\frac{Q(t,T)}{X} \pm \frac{1}{2}\sigma_Q^2 t\right] \Big/ \sigma_Q\sqrt{t} \end{aligned}$$

where $N(x) = [1+\text{erf}(x)]/2$ is the cumulative normal distribution.

3.07 Convertible bond option.♠♠ Calculate the value of a convertible bond $V(S,r,t)$ governed by two imperfectly correlated random variables. Assume that the asset value follows a log-normal distribution of the price increments dS and that the interest rate is described with a normal distribution of spot rate increments dr. Discuss the boundary conditions to be applied on the two dimensional domain (S,r).

All these problems can be edited and submitted for correction directly from your web browser, selecting *WORK:assignments* from the course main page.

3.8 Further reading and links

- **Trees.**
  Hull◇[11].
- **Option pricing.**
  Hull◇[11], Wilmott♠[24], Neftci♠[17], Björk♠♠[2], Duffie♠♠[7], Rebonato♠♠[19].
- **Stochastic differential equations (SDEs).**
  Kloeden♠♠[13], Van Kampen♠♠[22], Sczepessy♠♠[20].
- **Partial differential equations (PDEs).**
  Numerical methods for PDEs[45]♠[12] and references therein.

3.9 Quick intermediate evaluation form

Your opinion is precious. Please fill in the anonymous evaluation form in the web edition. Thank you very much in advance for your collaboration.

[45] http://www.lifelong-learners.com/pde

4 EUROPEAN OPTION PAYOFF DYNAMICS

4.1 Plain vanilla stock options

Following a rather descriptive chapter 2 where the terminal payoff of an option was only defined on the expiry date, increasingly sophisticated methods have been introduced in the previous chapter 3 to calculate the fair value of an option *before it expires*. Using these tools, we about to explore how the price of a financial derivative evolves with time. Rather than limiting the analysis to simplistic models or restricting the audience to so-called "rocket scientists", we will take advantage here of numerical experiments that can be performed using the VMARKET applet on-line: your task will be to run the simulations, edit the parameters and analyze the output to develop your intuition, using the same methods that are used by the professionals. The second part of this chapter is more advanced and deals with the implementation of financial models using both analytic and numerical methods.

4.1.1 The European Black-Scholes model for dummies

Remember how the price of an option has been calculated in sect.3.1: with a small twist, this will show you how sophisticated Monte-Carlo simulations are carried out in practice. Consider a European call giving its holder the right to buy a share for a strike of $K = 10$ EUR when the contract expires in 3 months or $T = 0.25$ year. For simplicity, assume that the terminal price of a share presently valued at $S_0 = 9$ EUR can take only one of two uncertain values with equal probabilities: $S_0 d = 6$ and $S_0 u = 12$ EUR. Using this probabilistic knowledge, it is easy to calculate the expected value at the expiry by weighting the terminal payoff (2.1.3#eq.1) from each realization with the probability factor 1/2:

$$\frac{1}{2} \times \max(6 - 10, 0) \quad + \quad \frac{1}{2} \times \max(12 - 10, 0) \quad = \quad 0 + 1 = 1.$$

Discounting back a quarter of year with a 3% continuous interest rate ($r = Y = 0.03$ in 1.3#eq.6), you immediately get the fair present value $1 \times \exp[-0.25 \times 0.03] = 0.9925$ EUR.

Even if the principle is correct, we showed in sect.3.2 that the forecasting values (3.2#eq.4) have to be carefully chosen to reproduce the market volatility, for example 40% ($\sigma = 0.4$): this yields $(u, d) = \exp(\pm 0.4 \times \sqrt{0.25}) = (1.221, 0.819)$ and risk-neutral world probabilities (3.2#eq.3) where a movement up $p = (\exp[0.03 \times 0.25] - 0.819)/(1.221 - 0.819) = 0.469$ is slightly less likely than a movement down $(1 - p) = 0.531$. These parameters can again be used to calculate the expected payoff for different market values of the underlying share S_0

$$p \times \max(S_0 u - K, 0) \quad + \quad (1 - p) \times \max(S_0 d - K, 0).$$

After discounting by the same factor 0.9925, we get the present values $V(S_0, t = 0)$ shown in (4.1.1#tab.1) with the corresponding terminal payoff $V(S_0, t = T) = \max(S_0 - K, 0)$.

S_0	7	8	9	10	11	12	13
$V(S_0, t = 0)$	0	0	0.46	1.03	1.60	2.17	3.07
$V(S_0, t = T)$	0	0	0	0	1	2	3

Table 4.1.1#tab.1: Present and terminal value of a call obtained using a binomial step.

Almost the same procedure is used in a Monte-Carlo calculation, except that for a higher accuracy, the lifetime of the option is divided into smaller steps: increments are accumulated to **simulate possible realizations** starting from the initial (spot) price of the underlying (*log-normal walk* with shares in sect.2.1.1) and **set the drift equal to the spot rate** (*risk-neutral evolution* observed for trees in sect.3.2). After each time step, the arithmetic average from all the possible terminal payoffs is used to estimate the mean price of the option on the expiring date and is discounted back in time to plot the fair value of an option having a lifetime equal to the run time. The VMARKET applet in the document on-line shows an example with (`Volatility=0.4`, `Drift=0.03`), where new increments are generated every trading day (the duration of one step is Δt=1/252=0.00397 year) and are accumulated to forecast two possible realizations of the underlying spot price ($S(t)$, red dots) starting from an initial 10 EUR (coincides with the `StrikePrice`). After one step backward in time ($T-\Delta t$ or `Time=0.003` displayed on the top of the applet measures the lifetime ranging from $t=0$ to T), the fair value of the option $V(S,T-\Delta t)$ is plotted (in black) together with the terminal payoff $V(S,T)$ (in grey). Running the simulation for 3 months (`Time=0.25`), the prices obtained using the Monte-Carlo method can be cross-checked with the value obtained from the binomial step (4.1.1#tab.1): they are quite different!

Virtual market experiments: price of call option using Monte-Carlo

1. Execute a few runs to verify that the same initial condition yield different solutions three months before the option expires–a financial non-sense!
2. Increase the number of random samples of the underlying and convince yourself that the solution converges to a reasonable value above approximatively `Walkers=1000`. N.B.: select `Monte-Carlo` to avoid plotting the prices in red.
3. Back to `Walkers=1`, explain the option payoff dynamics resulting from the movements of a single realization by reviewing all the arguments above.

The experiments show that the numerical accuracy of the Monte-Carlo calculation can be improved by increasing the number of realizations: the values obtained approach those given in (4.1.1#tab.1) without reproducing them exactly. The difference is particularly striking for low values of the underlying $S \leq 8$, where the binomial step gives options that are worthless, while the Monte-Carlo method converges to small but finite values. As you may have guessed, also binomial trees are only an approximation of the true solution, with an accuracy that improves when the number of steps is increased–resulting in a larger number of forecasting prices. As a matter of fact, both methods converge to the same value in the limit of small time steps and a large number of realisations: this value is the same as the one that has first been obtained by Black & Scholes [3] by solving (3.4#eq.4).

Congratulations: you probably solved your first option pricing equation and hopefully even understood what you were doing! Of course, analytical minded persons may say that a formula is more general and provides a better understanding. In this syllabus, we argue the opposite: formulas, just like computers, are only tools to obtain solutions from a certain model of the reality. Analytical and computational tools are both perfectly adequate if they are used in a knowledgable manner: they are often favourably compared to ensure that the solution is not affected by different assumptions made during the derivation of the models.

Before tackling these issues, let us first develop an intuition for the financial parameters and study with experiments how they affect the option payoff before the expiry date.

4.1.2 Parameters illustrated with VMARKET experiments

Apart from the *terminal conditions* that specify the value of an option when it expires (K for a vanilla option or `StrikePrice` in the applet) and the *numerical parameters* that specify the precision of the calculation (such as the `TimeStep`, `Walkers` and `MeshPoints`), the Black-Scholes model depends on only four *financial parameters*:

- The *time to the expiry* ($T-t$ or `RunTime`) is usually expressed in a fraction of a year, e.g. 0.25 for three months or a quarter of year to the expiry date.
- The *short term interest or spot rate* (r or `SpotRate`) is specified as a annual fraction of the capital investment, e.g. 0.05 for a risk free return of five percent per year.
- the *dividend yield* (D_0 or `Dividend`), here modeled with a continuous payment that is proportional to the underlying price, e.g. 0.04 for a dividend paying four percent of the share value during one year,
- the *volatility of the underlying* (σ or `Volatility`) estimated as the standard deviation of closing prices in sect.1.5, e.g. 0.5 for 50 percent for a volatile share.

To clearly visualize the payoff dynamics from the Black-Scholes model, it is useful to increase the interest rate and the dividend yield to artificially large values and study with the VMARKET applet on-line how the parameters affect the price of a European call struck for 10 EUR six months before expiry. The evolution of the payoff $V(S,t)$ should be far from obvious, but can be disentangled by investigating the effect of each parameter separately.

Virtual market experiments: volatility

1. Set `SpotRate=0`, `Dividend=0` and observe how the price changes as the time runs backwards, i.e. for an increasing amount of time to the expiry date. Click inside the plot area and use the coordinates $(S;V(S,t))$ displayed in the browser status field and the Java console to measure the price of the call at-the-money as a function of time. You can use `Step 10` to simulate an evolution backwards with multiples of ten days and collect the quantitative data on a sheet of paper. How do the prices move in reality when the time runs forward and approaches the expiry date?
2. Run the applet with a fixed time to expiry (e.g. `RunTime=0.5`) and measure the price for a `Volatility` ranging from zero to about one. How does the price of the option at-the-money depend on the volatility?
3. Repeat both experiments after switching to a `Put` option.

These experiments show that the main effect of the volatility is to “smear out sharp edges”, i.e. where the vanilla call and put options are at-the-money $S \approx K$. This phenomenon, known as *diffusion* in engineering sciences, is strongest at the begining of the simulation when the option is close to expiry date. It is the result of unpredictable market fluctuations: even if the value of the underlying share is below the strike price of a call $S < K$ before the expiry date, there is a finite chance that the market price will suddenly rise above that value, which would allow the holder of a call option to make a final profit $\max[S(T)-K,0]$. Such a right to make a potential profit without any obligation has of course a finite value, which decreases as the time approaches the expiry date.

Virtual market experiments: interest rate

1. Observe how the prices change with an increasing lifetime $T-t$ for the option. How do the prices move in reality when the calendar time t runs forward?

2. Run the applet with a fixed option lifetime (e.g. `RunTime=0.5`) and measure the price of a put option deeply in-the-money as a function of the `SpotRate` parameter. What happens to the payoff when the underlying share $S = 0$, i.e. the company goes bankrupt? Why?

3. Repeat the experiments after switching to a `Call` option and explain the qualitative difference in financial terms.

The effect of the risk-free interest rate can be understood from the drift that affects any type of investment: to finally coincide with the exercise price K on the expiry date, the strike price has to be discounted back in time (1.3#eq.6) to $K \times \exp[-r(T-t)]$. This is clearly visible in the applet, where the value at-the-money shifts to lower prices as the simulation runs backward in time. With a drift that is proportional to the strike price, the interest rate appears to have its largest relative effect when the option is at-the-money while the underlying is kept fixed; this is somewhat misleading, since the underlying should also grow by the same amount but is here used as a parameter. In fact, the graph could be continuously renormalized with the same amplification factor for the share, strike and option value–e.g. introducing a new currency after every time step, so that the graph would not evolve anymore at all.

Virtual market experiments: dividend yield

1. Set the `SpotRate=0` and study the payoff from a call option when the underlying pays a `Dividend=0.4`; compare with the effect from an interest rate.

2. Repeat the experiment switching to a `Put` option; explain why the payoff V(0,t) remains constant for an underlying company that goes bankrupt (S=0).

3. Set `SpotRate=Dividend=0.4` and try to explain what happens when switching to both `Call` and `Put` options.

Hopefully, these last experiments contribute more to your understanding than your confusion: with payments that are proportional to the underlying, the dividend yield continuously reduces the value of the share by the same amount; this results in a drift along the horizontal axis (in the opposite direction from the effect of interest rates) and appears as if the share prices were *amplified* when the time runs backwards. If the interest rate is equal to the dividend yield, the drifts in the horizontal direction cancel out and all that remains is the effect from the discounting at a risk-free interest rate.

With a good intuition for each parameter taken separately, it is a good exercise to now return to the first applet and discuss the main features that characterize an option payoff when all the parameters are combined into one calculation. Also, remember that unrealistically large parameters have been used in this section to exaggerate the effect from each parameter; realistic values will be used for an real option pricing calculation in the next section.

4.1.3 Application, time value and implied volatility

We are ready now to use the VMARKET applet and compare the fair value obtained from a model with the market price of products that are sold by financial institutions. For example, take the European call on the Swiss Market Index (SMI) expiring on Dec 19, 2002 with a strike at 7000. Nine months before the expiry, the underlying index was trading at 6610 with a market volatility around 18% . Under reasonable assumptions in Switzerland of a 2% risk-free rate and a 2% average dividend yield for the shares that constitute the SMI index, the VMARKET applet on-line calculates the fair price for this option according to the Black-Scholes model.[46] After interpolation, the fair price (CHF 251.6) is encouragingly close to the offer from Crédit Suisse First Boston CSFB (500 x 0.51 = CHF 255.0). Considering the crude approximations we made for the input parameters and the limitations of the Black-Scholes model, this agreement may however well be fortuitous: in fact, there is no guarantee that market prices coincide with *any model at all*–the offer and demand from traders on the option markets need not to be rational!

You may well ask now why people use financial modeling... It turns out that the predicted values are nevertheless often in the right range. Modeling is particularly useful to *estimate what should be a fair value* when there are not a sufficient number of buyers / sellers to make a market, for example when an option is offered for the first time or when an exotic option is tailored by a financial institution (the *market maker*) to meet the specific needs of only two clients. Simple products such as the vanilla call above have more than 100 million options listed on the Swiss exchange: this is enough to set a price only by offer and demand. Instead of calculating the option price as a function of the volatility, the Black-Scholes model is then often used as a market standard to calculate an *implied volatility*, i.e. the volatility that has to be used in the model to reproduce the price from the market.

Virtual market experiments: application in a real market

1. Keeping the same `Volatility=0.18`, calculate the fair value of a call with a `StrikePrice=7500`; compare with the market price (500 x 0.22 = CHF 110).
2. Reduce the `Volatility` until the calculated value matches the market price; compare with the 17% implied volatility calculated by CSFB.
3. From the two values above, estimate the *Vega* measuring the sensitivity of this option to changes in the volatility. You can use a finite difference approximation to calculate the derivative from *Vega=(V(0.18)-V(0.17))/(0.18-0.17)*.

Keeping the same expiry date, the implied volatility can be measured for different strike prices $\sigma(K)$; this curve is traditionally called the *smile*, but has a shape that really depends on the market conditions and can equally well be a frown (exercise 4.04). Although there is no guarantee to make a profit from the so-called *volatility trading*, some investors buy options with a low- and short options with a high implied volatility: their bet is that market forces will eventually move the prices of options so as to make implied volatilities comparable in the future.

[46] For an approximative output, click inside the plot area to measure the payoff $V(S,t)$ around the co-ordinate 6610. For a complete printout, switch from `Double-click below` to `Print data to console`, set `TimeStep=0` and press `Step 1`; prices can be read from the Java-console (with Netscape open `Communicator ->Tools ->Java console`) where `x[]` is the price of the underlying, `f0[]` is the intrinsic value and `f[]` the solution. Don't forget to switch back to `Double-click below` to avoid overflowing your console...

To complete this analysis dealing with the European option payoff dynamics, simply note that the difference between the present and the intrinsic (or final) value of an option is traditionally called *time value*. The simulation using the VMARKET applet on-line shows that the time value is usually negative for a put option in-the-money and is sometimes strictly positive for a call option.

Virtual market experiments: time value

1. Try to find a combination of financial parameters that leads to a negative time value for the call option and a positive time value for the put option.

2. Can you produce a positive time value for both the call and the put option?

4.2 Exotic stock options

The large number of exotic contracts outlined in chapter 2 is often divided in two categories depending on the terminal payoff, which can be a function of the underlying asset history or not. *Binary*, *compound* and *chooser options* are *path independent*, with a payoff that is entirely determined only by the market condition on the expiry date irrespective of how the market gets there. On the contrary, *barrier*, *loockback*, *Asian*, *Russian* and *American options* are all *path dependent*, because their terminal payoff depends in a non-trivial manner on the price history of the underlying. This section shows one example from each category, leaving a more detailed analysis of the American exercise style for chapter 6.

4.2.1 Binary options

Binary or digital options are a straight forward extension of a plain vanilla contract with a more general terminal payoff $\Lambda(S)$: as a consequence, the solution methods and the payoff enjoy many of the same features that have already been discussed for the put / call option. The VMARKET applet on-line shows the evolution of a super-share option for an increasing time to the expiry date.

Virtual market experiments: exotic binary options

1. Set `Volatility=0` and keep a finite `SpotRate`, `Dividend` to show how large oscillations appear in the proximity of the sharp edges. These oscillations lead to negative option prices and have no financial justification: they are an artifact of the numerical solution and should be avoided.

2. Modify the `StrikePrice`, `Shape0` and `Shape1` parameters defining the center, the height and the width of the box function to calculate the present value of a cash-or-nothing put option that pays EUR 1 if the underlying presently trading for 10 EUR rises to EUR 12 in six months time. Assume a 3% spot rate, 40% volatility and no dividend payment.

3. Switch from `SuperShr` to `VSpread` and use the shaping parameters to reproduce the same payoff starting from a vertical spread option. Note that the payoff from a vertical spread can be replicated with the payoff from two plain vanilla options (exercise 2.05).

Apart from stretching the validity of the numerical method, the sharp edges in the terminal payoff do seriously question whether it is practically possible to perform the delta-hedging in order to eliminate the uncertainty close to the expiry date. Indeed, the option value jumps every time the underlying moves across an edge, so that a large number of underlying shares have to be bought / sold to keep the portfolio value deterministic. Transaction costs play an increasingly important role and have to be taken into account (exercise 3.05).

4.2.2 Barrier options

More than an option on its own, a barrier is a feature that can be added to most of the contracts, including binary options that have just been discussed. Remember that an "in-barrier" option typically expires worthless unless the underlying crosses the barrier once at least during the option lifetime. With these specifications, it becomes important to keep track of the underlying asset price history and is most conveniently implemented by "tagging" each of the possible realizations in a Monte-Carlo simulation. A fair value for the barrier option is then obtained from the average payoff where only tagged realizations finally contribute to the sum. The VMARKET applet on-line shows the result in the case of an down-and-in barrier put option.

Virtual market experiments: exotic barrier options

1. Move the in-barrier from below to `Barrier=0.1` or 10% above the initial asset value. Can you see any difference in the option prices? Try to find a reason why an investor may want to buy such an option.

2. Move the `Barrier=-0.3` or 30% below the initial asset value and check how the price becomes ill defined since few realizations ever make it to the barrier. Knowing that the relative precision of a Monte-Carlo calculation is proportional to $1/\sqrt{N}$, estimate the number of random walkers N that are needed to achieve a 10% precision.

3. Reload the initial applet parameters and switch from `inBarrier*` to `outBarrier*` to experiment with the payoff of an down-and-out barrier option. How are the in- and out-barriers related?

4. Change the terminal option payoff and study the evolution of prices in the case of a vertical spread call featuring an in-barrier.

You probably found and verified in your the experiments that "in-" and "out-" barriers are complementary: the sum of both gives the same price as the option without a barrier.

4.3 Methods for European options: analytic formulation

A judicious combination of options and shares has been used in chapter 3.3.3 to eliminate the uncertainty in a portfolio, suggesting that the price of an option can be calculated by solving a partial differential equation. So far, this has been carried out using the VMARKET applet without paying much attention to the different methods that have to be employed. These methods are the subject of the rest of this chapter, showing with some mathematical details how to implement and use them within their validity limits. More advanced methods will be discussed later when dealing with bonds and American options, but could very well be used also for European options.

The first method uses analytical tools to produce an explicit solution in the form of the Black-Scholes formula. A considerable amount of algebra is required for the derivation and is only accessible to graduates from quantitative fields. The Black-Scholes formula, however, has a much broader appeal and is often used to calculate the implied volatility of prices that are traded on the markets. Unfortunately the analytical method is difficult to generalize beyond the simplest plain-vanilla or binary options.

4.3.1 Transformation to log-normal variables ♠

The log-normal distribution of the price increments ($\kappa = 1$ in 3.3.1#eq.1) chosen to derive the Black-Scholes equation (3.4#eq.4) shows that the asset price S and the time t are in fact not a natural choice of variables for the price of an option that expires at a time $t = T$. This motivates a transformation from financial variables $V(S,t)$ to log-normal variables $v(x,\tau)$ defined by

$$S = K\exp(x), \qquad t = T - 2\tau/\sigma^2, \qquad V = Kv(x,\tau) \qquad \text{(4.3.1\#eq.1)}$$

Substitute these in the Black-Scholes equation (be careful with the second derivative)

$$\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} - (k_2 - 1)\frac{\partial v}{\partial x} + k_1 v = 0 \qquad \text{(4.3.1\#eq.2)}$$

showing that only two dimensionless parameters in fact characterize the problem

$$k_1 = \frac{2r}{\sigma^2}, \qquad k_2 = \frac{2(r-D)}{\sigma^2} \qquad \text{(4.3.1\#eq.3)}$$

With a little more algebra, you can verify that further scaling by

$$V = Kv = K\exp\left[-\frac{1}{2}(k_2-1)x - \left(\frac{1}{4}(k_2-1)^2 + k_1\right)\tau\right]u(x,\tau) \qquad \text{(4.3.1\#eq.4)}$$

transforms Black-Scholes into a normalized diffusion equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(4.3.1\#eq.5)}$$

which bears a strong resemblance with the *heat-equation* from engineering sciences. This equation has to be solved for $x \in [-\infty;\infty]$, $\tau \geq 0$ using boundary $u(-\infty,\tau)$, $u(+\infty,\tau)$ and initial conditions $u(x,0)$ that have to be derived from no-arbitrage arguments with financial variable $V(S,t)$ via the transformations (4.3.1#eq.1 and 4.3.1#eq.4). .

4.3.2 Solution of the normalized diffusion equation ♠

Readers interested in solving the diffusion equation (4.3.1#eq.5) analytically are likely to be familiar with the Fourier-Laplace transform that is commonly used to solve initial and boundary value problems. Others may skip the whole derivation and verify that the final result (4.3.2#eq.11) indeed satisfies the Black-Scholes equation (3.4#eq.4).

Start by transforming (4.3.1#eq.5) with a Laplace transform in time

$$\int_0^\infty d\tau \exp(i\omega\tau) \left[\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right] = 0, \qquad \Im m(\omega) > 0 \tag{4.3.2\#eq.1}$$

Note that the condition $\Im m(\omega) > 0$ is important to ensure causality. Integrate the first term by parts and substitute a Dirac function $\delta(x-\xi)$ for the initial condition $u(x,0)$

$$\begin{aligned} u\exp(i\omega\tau)|_0^\infty + \int_0^\infty d\tau \exp(i\omega\tau)\left[-i\omega u - \frac{\partial^2 u}{\partial x^2}\right] &= 0 \\ -\delta(x-\xi) + \left[-i\omega u(x,\omega) - \frac{\partial^2 u(x,\omega)}{\partial x^2}\right] &= 0 \end{aligned} \tag{4.3.2\#eq.2}$$

The notation $u(x,\omega)$ refers to the Laplace transform of $u(x,\tau)$. Spatial derivatives are dealt with a Fourier transform

$$\begin{aligned} \int_{-\infty}^{\infty} dx \exp(-ikx)\left[-\delta(x-\xi) - i\omega u(x,\omega) + k^2 u(x,\omega)\right] &= 0 \\ \exp(-ik\xi) + i\omega u(k,\omega) - k^2 u(k,\omega) &= 0 \end{aligned} \tag{4.3.2\#eq.3}$$

and yields an explicit solution in Fourier-Laplace space

$$u(k,\omega) = \frac{-i\exp(-ik\xi)}{\omega + ik^2} \tag{4.3.2\#eq.4}$$

The pole in the complex plane for $\omega = -ik^2$ needs to be taken into account when inverting the Laplace transform in a causal manner

$$\begin{aligned} u(k,\tau) &= \int_{-\infty+iC}^{+\infty+iC} \frac{d\omega}{2\pi} \exp(-i\omega\tau)\left(\frac{-i\exp(-ik\xi)}{\omega+ik^2}\right) \qquad C > 0 \\ &= \frac{-i\exp(-ik\xi)}{2\pi} 2\pi i \exp(-k^2\tau) = \exp(-ik\xi)\exp(-k^2\tau) \end{aligned} \tag{4.3.2\#eq.5}$$

where the residue theorem has been used to evaluate the complex integral, closing the contour in the lower half plane where the phase factor $\exp(-i\omega\tau)$ decays exponentially. Invert the Fourier transformation

$$\begin{aligned} u(x,\tau) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(ikx)\left[\exp(-ik\xi)\exp(-k^2\tau)\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp ik(x-\xi)\exp(-k^2\tau) \end{aligned} \tag{4.3.2\#eq.6}$$

and use the formula (3.323.2) from Gradshteyn & Ryzhik [9]

$$\int_{-\infty}^{\infty} dx \exp(-p^2x^2)\exp(\pm qx) = \frac{\sqrt{\pi}}{p}\exp\left[\frac{q^2}{4p^2}\right] \qquad p > 0 \tag{4.3.2\#eq.7}$$

here with $p = \sqrt{\tau}$ and $q = i(x-\xi)$ to write down a solution of the diffusion equation

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] \qquad (4.3.2\#eq.8)$$

This shows that a Dirac function $\delta(x-\xi)$ assumed as initial condition in (4.3.2#eq.2) spreads out into a Gaussian as time evolves forward. A superposition of a whole series of Dirac functions can now be used to decompose any arbitrary initial condition $u_0(\xi)$

$$u(x,0) = \int_{-\infty}^{\infty} u_0(\xi)\delta(\xi-x)d\xi = \int_{-\infty}^{\infty} u_0(\xi)\delta(x-\xi)d\xi \qquad (4.3.2\#eq.9)$$

and after evolving each Dirac functions separately using (4.3.2#eq.8), can again be superposed at a later time when $\tau \geq 0$:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} d\xi u_0(\xi) \exp\left[-\frac{(\xi-x)^2}{4\tau}\right] \qquad (4.3.2\#eq.10)$$

Transforming back into financial variables (4.3.1#eq.1), some algebra finally yields a general formula for the price of a binary option with a terminal payoff $\Lambda(S)$

$$V(S,t) = \frac{\exp[-r(T-t)]}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} \Lambda(X) \exp\left[-\frac{\left(\ln(\frac{X}{S}) - (r-D-\frac{\sigma^2}{2})(T-t)\right)^2}{2\sigma^2(T-t)}\right] \frac{dX}{X} \qquad (4.3.2\#eq.11)$$

4.3.3 Black-Scholes formula ♢

In the case of plain vanilla call and put options, the price can be evaluated in terms of the cumulative normal distribution $N(x)$ and yields the well known Black-Scholes formula

$$V_{\text{call}}(S,t) = SN(d_1) - K\exp[-r(T-t)]N(d_2) \qquad (4.3.3\#eq.1)$$

$$d_{1,2} = \frac{\ln(S/K) + (r-D\pm\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad (4.3.3\#eq.2)$$

$$V_{\text{put}}(S,t) - V_{\text{call}}(S,t) + S = K\exp[-r(T-t)] \qquad (4.3.3\#eq.3)$$

Remember that S denotes the (spot) price of an underlying share that pays a dividend D and has a historical volatility σ, K is the strike price of the option evolving in time $t \in [0;T]$ from the present to the expiry date and r the risk-free interest (spot) rate. Note that the last relation (4.3.3#eq.3) is nothing more than the put-call parity previously obtained in (2.1.3#eq.2), where the guaranteed payoff has been discounted back in time to achieve the risk free return of the spot rate. The cumulative normal distribution is related with the so-called error function $N(x) = \frac{1}{2}(1+\text{erf}(\frac{x}{\sqrt{2}}))$, which is available in Matlab and can be approximated with 6 digits accuracy using the polynomial expansion [1]

$$N(x) \approx \begin{cases} 1 - N'(x)\left(a_1k + a_2k^2 + a_3k^3 + a_4k^4 + a_5k^5\right) & \text{when } x \geq 0 \\ 1 - N(-x) & \text{when } x < 0 \end{cases}$$

$$\text{where } N'(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \quad \text{and} \quad k = \frac{1}{1+\gamma x} \qquad (4.3.3\#eq.4)$$

with the coefficients $\gamma = 0.2316419$, $a_1 = 0.319381530$, $a_2 = -0.356563782$, $a_3 = 1.781477937$, $a_4 = -1.821255978$, $a_5 = 1.330274429$.

4.4 Methods for European options: finite differences (FD)

The difficulty of extending analytical derivations much further motivates developments of computational tools, which calculate the fair value of an option directly with a computer. Finite differences are commonly used, because it is relatively simple to formulate and implement an algorithm that converges to the solution. In this section, we will see that the simplest form results in explicit 2-levels schemes that remain of limited practical interest. Rather than extending the theory with the so-called implicit methods, we will wait to develop a more general framework in chapter 5, and show how the implicit Crank-Nicholson method with finite differences is in fact a special case of a more general and more robust formulation using finite elements.

4.4.1 Naive implementation using financial variables ♢

Start with a finite number prices $\{V_j^T\}$ that are all known from the terminal condition $V_j^T = V(S_j, T) = \Lambda(S_j)$ with a regular sampling of the underlying asset $S_j = j\Delta S$. The idea behind the finite difference method is to approximate the infinitesimal changes in the Black-Scholes equation by small (but finite) differences and to generate new prices $\{V_j^{t-\Delta t}\}$ in a sequence of small steps taken backward in time until the solution is found. It is not difficult to propose approximations of the partial derivatives directly from the definition

$$\begin{aligned}\frac{\partial V}{\partial S}(S_j,t) = \lim_{\delta S\to 0}\frac{V(S_j+\delta S)-V(S_j)}{\delta S} &= \frac{V_{j+1}-V_j}{\Delta S}+\mathcal{O}(\Delta S^2) \qquad \text{forward}\\ &= \frac{V_{j+1}-V_{j-1}}{2\Delta S}+\mathcal{O}(\Delta S^3) \qquad \text{centered}\\ &= \frac{V_j-V_{j-1}}{\Delta S}+\mathcal{O}(\Delta S^2) \qquad \text{backward}\end{aligned} \tag{4.4.1\#eq.1}$$

Convince yourself that if you subtract two Taylor expansions around the asset price S_j

$$V_{j+1} = V(S_j+\Delta S) = V(S_j) + \Delta S\frac{\partial V}{\partial S} + \frac{1}{2}\Delta S^2\frac{\partial^2 V}{\partial S^2} + \mathcal{O}(\Delta S^3) \tag{4.4.1\#eq.2}$$

$$V_{j-1} = V(S_j-\Delta S) = V(S_j) - \Delta S\frac{\partial V}{\partial S} + \frac{1}{2}\Delta S^2\frac{\partial^2 V}{\partial S^2} - \mathcal{O}(\Delta S^3) \tag{4.4.1\#eq.3}$$

the quadratic terms cancel out, showing that a higher precision in $\mathcal{O}(\Delta S^2)$ is achieved when using a centered scheme. Precision is however not the only requirement and it turns out that only backward (alt. forward) differences are stable for the approximation of the time derivative when the scheme runs backward (alt. forward). An approximation for the second derivative is obtained by subtracting $2V_j$ from the sum of two Taylor expansions above

$$\frac{\partial^2 V}{\partial S^2} = \frac{V_{j+1}-2V_j+V_{j-1}}{\Delta S^2} + \mathcal{O}(\Delta S^3) \tag{4.4.1\#eq.4}$$

Substituting finite differences for all the partial derivatives in the Black-Scholes equation without dividend (extension considered in exercise 4.07)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

this immediately leads to

$$\frac{V_j^t - V_j^{t-\Delta t}}{\Delta t} + \frac{1}{2}\sigma^2 (j\Delta S)^2 \frac{V_{j+1}^t - 2V_j^t + V_{j-1}^t}{\Delta S^2} + r(j\Delta S)\frac{V_{j+1}^t - V_{j-1}^t}{2\Delta S} - rV_j^t = 0 \quad (4.4.1\#eq.5)$$

where the time derivative (first term) has been approximated with a difference backward from the time level t for stability reasons and the price derivative (third term) has been centered on $S_j = j\Delta S$ for a higher precision. After a few rearrangements, the unknown "new" option value at a time level $t - \Delta t$ are obtained explicitly in terms of the known "old" values at time level t

$$\begin{aligned} V_j^{t-\Delta t} &= V_{j+1}^t \frac{\Delta t}{2}(\sigma^2 j^2 + rj) \\ &+ V_j^t \left[1 - \Delta t(\sigma^2 j^2 + r)\right] \\ &+ V_{j-1}^t \frac{\Delta t}{2}(\sigma^2 j^2 - rj) \end{aligned} \quad (4.4.1\#eq.6)$$

The value of an option can therefore be approximated before the expiry date, using a sequence of small steps to evolve the solution backward until the chosen time is reached.

To complete the formulation, the solution has to be specified on the domain boundaries $S_0 = 0$ and $S_n = n\Delta S$; in the same way as for the terminal conditions, these *boundary conditions* depend on the type of contract. For example, the underlying price at $S = 0$ doesn't change in the case of log-normal price increments, showing that the value of a put option is certain to be exercised with a payoff equal to the strike price K. This can be discounted back from the expiry time T at a rate r. On the contrary, the put option is increasingly unlikely to be exercised if the underlying asset value moves above the strike price and looses all of its value when $S \to \infty$. The boundary conditions for a put option without dividends (extension in exercise 4.07) are

$$V_{\text{put}}(S,t) = \begin{cases} K\exp[-r(T-t)] & \text{for} \quad S = 0 \\ 0 & \text{for} \quad S \to \infty \end{cases} \quad (4.4.1\#eq.7)$$

A similar reasoning leads to the boundary conditions for a call option

$$V_{\text{call}}(S,t) = \begin{cases} 0 & \text{for} \quad S = 0 \\ S - K\exp[-r(T-t)] & \text{for} \quad S \to \infty \end{cases} \quad (4.4.1\#eq.8)$$

This scheme that has been implemented in the VMARKET class `FDSolution.java` as

```
double timeStep = runData.getParamValue("TimeStep");
double strike   = runData.getParamValue("StrikePrice");
double rate     = runData.getParamValue("SpotRate");
double divid    = runData.getParamValue("Dividend");
double sigma    = runData.getParamValue("Volatility");
double sigmaSq  = sigma*sigma;
for (int j=1; j<n; j++) {                          //Explicit 2 levels
  fp[j]=f[j+1]* 0.5*timeStep*(sigmaSq*j*j + rate*j)
       +f[j  ]*(1.0-timeStep*(sigmaSq*j*j + rate  ))
       +f[j-1]* 0.5*timeStep*(sigmaSq*j*j - rate*j);
}                                                  //Boundary condition
if (isCall)      { fp[0]=0;  fp[n]=n*dx[0] -strike*Math.exp(-rate*time);
} else if(isPut) { fp[0]=strike*Math.exp(-rate*time);   fp[n]=fp[n-1];
} else           { fp[0]=fp[1];  fp[n]=fp[n-1]; }
```

showing clearly how the option values at the new time level `fp[j]` are explicitly calculated in terms of the old values `f[j]`. To limit the required size of the simulation domain around the strike price, it is marginally better to replace the Dirichlet condition `fp[n]=0` with a Neumann condition `fp[n]=fp[n-1]`. This is what has been used above for the put option and in the default, so as to accommodate in a simple manner for more general payoffs from binary options. The VMARKET applet on-line shows an application using this very simple model.

Virtual market experiments: naive FD scheme

1. Switch from `Put` to `Call`, `SuperShr` and `VSpread` to verify that the same scheme works also for binary options.

2. Imagine how the solutions would look like if Dirichlet conditions were used instead of Neumann conditions; note that neither converges to the correct value at the boundaries where the mesh is truncated.

3. Switch back to `Put` option and try to accelerate the calculation with a slightly larger `TimeStep=5E-4`. Check how an instability grows for the largest values of the underlying and becomes visible after `Time=0.22`.

4. Increase the time step above `TimeStep=2.74E-3` and switch between different initial conditions to verify how, for large steps, the instability always appears around the largest value where the initial condition is not zero.

The mere simplicity of this naive formulation makes this explicit finite difference scheme attractive; two major problems, however, should lead you to reconsider an early judgment. First, the regular spacing of the grid is ill suited to accommodate a log-normal distribution of price increments: the relative numerical accuracy in $\mathcal{O}(\Delta S^2)/S$ rapidly decreases and becomes insufficient when the value of the underlying becomes small. The second problem is the numerical instability, which appears when the solution evolves too rapidly. It turns out that the upper limit on the time step depends on the relative changes that are possible for the largest values of the underlying–even if the option price is negligible there. These drawbacks motivate a transformation to log-normal variables in the same manner as for the analytical solution in section 4.3.1.

4.4.2 Improved scheme using log-normal variables ♠

Instead of solving Black-Scholes (3.4#eq.4) with finite differences in the financial variables (S,t), apply the transformations (4.3.1#eq.1, 4.3.1#eq.4) and solve the normalized equation (4.3.1#eq.5) with finite differences in log-normal variables (x,τ).
Apart from a numerical accuracy in $\mathcal{O}(\Delta x^2)$, which gets independent of the underlying value S, the transformation has the additional advantage of evolving the numerical solution everywhere at the same rate, so that the time step is everywhere limited by the condition for the normalized diffusion equation:

$$\frac{\Delta\tau}{\Delta x^2} < \frac{1}{2} \qquad (4.4.2\#eq.1)$$

The finite difference solution is extremely simple to calculate using a difference forward from the time level τ (4.4.1#eq.1 since a scheme backward in t runs forward in τ) and a second derivative centered on x_j (4.4.1#eq.4):

$$\frac{u_j^{\tau+\Delta\tau} - u_j^\tau}{\Delta\tau} - \frac{u_{j+1}^\tau - 2u_j^\tau + u_{j-1}^\tau}{\Delta x^2} = 0 \qquad (4.4.2\#eq.2)$$

$$u_j^{\tau+\Delta\tau} = u_j^\tau + \Delta\tau \frac{u_{j+1}^\tau - 2u_j^\tau + u_{j-1}^\tau}{\Delta x^2} \qquad (4.4.2\#eq.3)$$

The only difficulty (which results in quite a bit of coding) is that the regular spacing $x_j = j\Delta x$ assumed for the finite differencing becomes exponential in financial variable $S = K\exp(x)$. This is not a problem for the put and call options (exercise 4.08), since the terminal and boundary conditions are known explicitly in log-normal variables

$$u_{\text{put}}(x,0) = \max\left(\exp\left[\frac{1}{2}(k_2-1)x\right] - \exp\left[\frac{1}{2}(k_2+1)x\right], 0\right) \qquad (4.4.2\#eq.4)$$

$$u_{\text{put}}(x,\tau) = \begin{cases} \exp\left[\frac{1}{2}(k_2-1)x + \frac{1}{4}(k_2-1)^2\tau\right] & x \to -\infty \\ 0 & x \to +\infty \end{cases} \qquad (4.4.2\#eq.5)$$

The transformation is however an additional source of imprecision when a general payoff from a binary option has first to be transformed from regularly spaced financial variables $V(S_j,0)$ to log-normal variables on an inhomogeneous grid $u(x_j,0)$, before it is interpolated onto a regular grid $u(x_j,0)$ using a linear approximation

$$u(x) = u_{k-1} + (x - x_{k-1})\frac{u_k - u_{k-1}}{x_k - x_{k-1}} \qquad (4.4.2\#eq.6)$$

The evolution can be calculated keeping only financial variables, but an interpolation is again needed when the solution is finally plotted in the financial world. This finally yields a rather clumsy scheme that has been implemented in the VMARKET class `FDSolution.java`

```
double x0, x1, f0, f1, xk;                          //Change variables
double tau = 0.5*sigmaSq*time;                      // f(x,t) ->
double dtau= 0.5*sigmaSq*timeStep;                  //    fm(xk,tau)
double xk0 = Math.log(x[1]/strike);                 // lognormal mesh
double xkn = Math.log(x[n]/strike);
double dxk =(xkn-xk0)/(n-1);
double k1  = 2*rate/sigmaSq;
double k2  = 2*(rate-divid)/sigmaSq;
double k2m1= k2-1.;
double k2p1= k2+1.;
int    j,k;
//=== Interpolate only once from (x,t) to lognormal variables (xk,tau)
if (time<=timeStep) {
  if(isPut) { //Initialize fm[] directly as put-option
    for (k=1; k<=n; k++) {
      xk=xk0+(k-1)*dxk;
      fm[k]=Math.max(0., Math.exp(0.5*k2m1*xk) -
                         Math.exp(0.5*k2p1*xk)  );
    }
  } else if (isCall) { //Left as an exercise
  } else { //Interpolate fm[] from IC in f[]
    j=1; ; x0=xk0;
```

```
        f0=f[1]/strike*Math.exp(0.5*k2m1*xk0);
        x1=x0; f1=f0;
        for (k=1; k<n; k++) { //Loop over lognormal mesh index
          xk=xk0+(k-1)*dxk;   // given xk, find x0,x1 | x[j] < xk < x[j+1]
          while (xk>=x1) { j++; x0=x1; f0=f1; x1=Math.log(x[j]/strike); }
          f1=f[j]/strike*Math.exp(0.5*k2m1*x1);
          fm[k]= f0 + (xk-x0)*(f1-f0)/(x1-x0);
        }
        fm[n]= fm[n-1] + dxk*(fm[n-1]-fm[n-2]);
      }
    } else { //Retrieve fm[] from previous time step
    }
    //===== Solve diffusion equation with an explicit 2 levels scheme
    double D = dtau/(dxk*dxk);
    for (j=2; j<n; j++)
      f[j]= fm[j] + D*(fm[j+1]-2.*fm[j]+fm[j-1]);
    if (isPut) {
      f[1]= Math.exp(0.5*k2m1*xk0+0.25*k2m1*k2m1*tau);
      f[n]= f[n-1];
      fp[0]=strike*Math.exp(-rate*time);
//  } else if (isCall) { //Left as an exercise
    } else {
      f[1]= f[2];
      f[n]= f[n-1];
      fp[0]=fp[1];
    }
    //===== Interpolate rest from lognormal to financial mesh variables
    k=1; x0=x[0]; x1=x0; f0=fp[0];
    xk=xk0; f1=f[1]*strike*Math.exp(-0.5*k2m1*xk-(0.25*k2m1*k2m1+k1)*tau);
    for (j=1; j<n; j++) { //Loop over financial mesh index
      while (x[j]>=x1){
        k++;x0=x1;f0=f1;xk=xk0+(k-1)*dxk;x1=strike*Math.exp(xk);}
      f1=f[k]*strike*Math.exp(-0.5*k2m1*xk-(0.25*k2m1*k2m1+k1)*tau);
      fp[j]= f0 +(x[j]-x0)/(x1-x0)*(f1-f0);              //Lin interpol in x
    }
    if (isPut) {
      fp[n]=f[n]*strike*Math.exp(-0.5*k2m1*xkn-(0.25*k2m1*k2m1+k1)*tau);
//  } else if (isCall) { //Left as an exercise
    } else {
      fp[0]=fp[1];
      fp[n]=fp[n-1];
    }
```

Virtual market experiments: log-normal FD scheme

1. Press `Step 1` after initializing different payoffs to visualize the initial error induced by the interpolation to log-normal variables and back.
2. Reduce the interpolation error by increasing `MeshPoints`; note that you then have to reduce `TimeStep` in agreement with (4.4.2#eq.1).
3. Switch from `European logn` to `European` and compare the results from the naive scheme with the present one using log-normal variables.

4.5 Methods for European options: Monte-Carlo sampling (MCS)

Monte-Carlo sampling is perhaps the easiest method to understand and implement; it offers considerable flexibility when dealing with path dependent exotic options and is generally adopted for problems involving more than three independent random driving factors. The main drawback of the method is the slow convergence, scaling with the inverse square root of the number of samples and starting from what is often a large initial error. It is therefore not uncommon to use simulations with a million samples to guarantee a precision better than one percent. Clearly, this is prohibitively computer intensive for the valuations of the simple options that can be calculated in a different manner.

4.5.1 Forecast possible realizations of the underlying asset ♠

From the definition of stochastic increments in sect.3.3.1, it is easy to propose an approximation for the uncertain evolution of the underlying asset price S, using a finite number of steps in time Δt, which result in correspondingly small increments in the price ΔS

$$\frac{\Delta S}{S^\kappa} = \mu \Delta t + \sigma \zeta \sqrt{\Delta t} \qquad (4.5.1\#\text{eq.}1)$$

As previously, (μ, σ) are the drift and the volatility measured from the market, $\zeta \in \mathcal{N}(0,1)$ is a normally distributed random number generated anew for each step and the parameter κ is chosen to reproduce a normal walk with $\kappa = 0$ or a log-normal walk with $\kappa = 1$.

Starting from (`k=0,...,numberOfRealisations-1`) samples modeling each one possible evolution of the asset (`currentState[k][0]`) and an equal number of markers (`mark[k][0]`) to account for barriers in exotic options, the initialization in `SamplingSolution.java` has been implemented as

```
numberOfRealisations = runData.getParamValueInt("Walkers");
strike   = runData.getParamValue("StrikePrice");
kappa    = runData.getParamValue("LogNkappa");
                                                          //Separable
if( (Math.abs(kappa)<0.001) ||(Math.abs(kappa-1.)<0.001)){
  currentState = new double[numberOfRealisations][1];
  for (k=0; k<numberOfRealisations; k++) currentState[k][0]=strike;
                                                          //Barriers
  if(scheme.equals(vmarket.MCIN)||scheme.equals(vmarket.MCINPP)){
    mark = new double[numberOfRealisations][1];
    for (k=0; k<numberOfRealisations; k++) mark[k][0]=0.;
  } else if (scheme.equals(vmarket.MCOUT)||scheme.equals(vmarket.MCOUTPP)){
    mark = new double[numberOfRealisations][1];
    for (k=0; k<numberOfRealisations; k++) mark[k][0]=1.; }
```

If the parameter `kappa` is sufficiently close to log-/normal with $\kappa \simeq 0$ or 1, the first four lines initialize the array `currentState[k][0]` with `numberOfRealisations` samples of one single price stored in a one dimensional array with an idle index `[0]`; the entire array is (arbitrarily or, rather, for plotting) initialized with the strike price `currentState[k][0]=strike`. The value of the selector `scheme` decides if the modeling of an in-/out-barrier option with-/out particle plotting requires the creation of an additional marker array `mark[k][0]`, which has to be initialized with the corresponding behavior. Not shown in the code above but visible in the VMARKET listing, is that a parameter `kappa` sufficiently different from zero or one can be used to initialize a two dimensional array `currentState[k][j]` with `j=0,..., mesh.size()-1` different prices; these are evolved in parallel if the problem is

not separable–e.g. when the increments depend in a non-trivial manner on the asset price.
Starting from an initial value, possible future realizations of the asset prices are calculated with a sequence of small steps in time using (4.5.1#eq.1); in the case of a log-normal random walk ($\kappa = 1$), new values for `currentState[k][0]` are calculated in `MCSSolution.java` repeating for each step

```
double timeStep = runData.getParamValue("TimeStep");
double strike   = runData.getParamValue("StrikePrice");
double mu       = runData.getParamValue("Drift");
double divid    = runData.getParamValue("Dividend");
double sigma    = runData.getParamValue("Volatility");
double barrier  = runData.getParamValue("Barrier");

if(Math.abs(kappa-1.)<0.001){              //Separable log-normal
  for(int k=0; k<numberOfRealisations; k++)
    currentState[k][0]+= currentState[k][0]*( (mu-divid)*timeStep +
            random.nextGaussian()*sigma*Math.sqrt(timeStep) );

                                                  //Barriers
  if (scheme.equals(vmarket.MCOUT) || scheme.equals(vmarket.MCOUTPP)){
    for(int k=0; k<numberOfRealisations; k++)
      if ((barrier >= 0. && currentState[k][0]-strike > strike*barrier)||
          (barrier <  0. && currentState[k][0]-strike < strike*barrier)  )
        mark[k][0]=0.;
  } else if (scheme.equals(vmarket.MCIN) || scheme.equals(vmarket.MCINPP)){
    for(int k=0; k<numberOfRealisations; k++)
      if ((barrier >= 0. && currentState[k][0]-strike > strike*barrier)||
          (barrier <  0. && currentState[k][0]-strike < strike*barrier)  )
        mark[k][0]=1.;
  }
}
```

The first four lines compute the deterministic `(mu-divid)*timeStep` and the random component `random.nextGaussian()*sigma*Math.sqrt(timeStep)` of the evolution, which are easily identified as the right-hand side of (4.5.1#eq.1). Further scaling by the underlying asset price `currentState[k][0]` reproduces the log-normal distribution of the increments, which are finally accumulated with the Java operator `currentState[k][0]+=increment`. The variable `mark[k][0]` is reset to zero (alt. one) whenever the condition for an "out-" (alt. "in-") barrier is met for a given sample. Note that the position of the barrier is here defined relative to the initial condition, using a positive (alt. negative) value of the variable `barrier` to distinguish a barrier above (alt. below) the initial price of the underlying. This relative definition is here required to keep the problem separable, so that the evolution of any price S_j can be obtained from the same sequence of increments ΔS_0 normalized to $S_0 = K$ using the scaling

$$\Delta S_j = \frac{S_j}{S_0} \Delta S_0 \qquad (4.5.1\#eq.2)$$

These values need in fact not to be evaluated until the expected values of the derivatives are calculated from the underlying in a manner described in the comming section.

Using a finite time step in (4.5.1#eq.1) is of course an approximation of the stochastic differential (3.3.1#eq.1); great care has to be taken to keep the steps small enough not to create, for example, negative asset prices when $|\mu|\Delta t$ or $|\sigma|\sqrt{\Delta t}$ exceed unity in a log-normal walk. For the record, note that the stochastic differential (assuming $\sqrt{dt} \to 0$) is not an exact model neither, since a finite number of trades sets a lower limit on the duration between trades on the markets. The VMARKET applet on-line illustrates some effects of the time discretization for the case of an down-and-in barrier put option.

Virtual market experiments: Monte-Carlo random walk

1. Increase the value of the `TimeStep` parameters to create spurious negative asset values with a bad numerical approximation of a log-normal random walk.

2. Choose the largest `TimeStep` that you believe gives a financially significant result; justify your choice.

3. Switch to a normal evolution by setting `LogNkappa=0` and discuss what is now the upper limit for the time step.

Having dealt with the simulation of the underlying asset price, we now turn to the valuation of derivatives.

4.5.2 Expected value of an option from sampled data ♠

To develop your intuition, let us first define the transition probability $p[S,t;S',T]$ measuring the likelyhood that an asset evolves from the present value to the terminal value $(S,t) \rightarrow (S',T)$: weighted by the terminal payoff of an option $\Lambda(S')$, this can be used to evaluate the expected return from a particular realisation of the market. Summing the weighted returns from all possible realisations with the proper Jacobian, the present value of an option could be calculated from

$$V(S,t) = \exp[-r(T-t)] \int_0^\infty p[S,t;S',T]\Lambda(S')\frac{dS'}{S'^\kappa} \qquad (4.5.2\#eq.1)$$

where the expected terminal payoff has been discounted back to the present time t by multiplication of the factor $\exp[-r(T-t)]$. This expression can be identified with the analytical solution (4.3.2#eq.11) and shows that the price of an option can also be calculated as the present value of the expected return, using a random walk **in a risk-neutral world where the drift is replaced by the spot rate minus the dividend yield** $\mu = r - D_0$. (Note the analogy with the delta hedging, where the risk has been eliminated to obtain a Black-Scholes equation that is also independent of the drift μ.)
Instead of calculating a complicated n-dimensional path-dependent integral with transition probabilities $p[S_i,t_i;S_j,t_j|\mathcal{C}_j]$ that are subject to multiple conditions $\mathcal{C}_j$

$$V(S,t) = \exp[-r(T-t)] \int_0^\infty \frac{dS_1}{S_1^\kappa} p[S,t;S_1,t_1|\mathcal{C}_1] \int_0^\infty \frac{dS_2}{S_2^\kappa} p[S_1,t_1;S_2,t_2|\mathcal{C}_2] \ldots$$
$$\ldots \int_0^\infty \frac{dS_n}{S_n^\kappa} p[S_{n-1},t_{n-1};S_n,T|\mathcal{C}_n]\Lambda(S_n) \qquad (4.5.2\#eq.2)$$

the Monte-Carlo sampling method simply uses a large number of possible realizations as an unbiased estimator for the mean price payed when the option is exercised

$$V(S,t) = \exp[-r(T-t)]\frac{1}{N}\sum_{k=1}^{N}\Lambda(S_k) \qquad (4.5.2\#eq.3)$$

The realizations of the underlying asset prices $\{S_1, S_2, \ldots S_N\}$ are evolved using a risk-neutral random walk by setting the drift $\mu = r - D_0$. Path dependent features such as barriers can be easily be incorporated at the end, by retaining only those prices that satisfy the conditions: the terminal payoff can for example be multiplied with a marker variable that is either equal to zero or one depending whether the condition has been fulfilled or not. The scheme that has been implemented in the VMARKET class MCSSolution.java reads

```
} else if(Math.abs(kappa-1.)<0.001){        //Separable log-normal
   if (markers){
     for (k=0; k<numberOfRealisations; k++){
       f[j]+= option.getValue(currentState[k][0] *x[j]/strike) *mark[k][0];
       g[j]+= option.getValue(currentState[k][0] *x[j]/strike);
     }
   } else
     for (k=0; k<numberOfRealisations; k++)
       f[j]+= option.getValue(currentState[k][0] *x[j]/strike);
}
f[j]=Math.exp(-time*rate)*f[j]/numberOfRealisations;
g[j]=Math.exp(-time*rate)*g[j]/numberOfRealisations;
```

If the problem is separable, the random walk is first scaled according to (4.5.1#eq.2) to obtain the terminal value of the underlying S_j with `currentState[k][0]*x[j]/strike`; this is then used as an argument to accumulate the terminal payoff $\Lambda(S_k)$ from every realization using the statement `f[j]+=option.getValue()` and finally calculate the discounted average of (4.5.2#eq.3) using the last two lines. Note that two functions (`f[j],g[j]`) have been used to compare the price obtained with-/out barriers. The VMARKET applet on-line illustrates the result in the case of a simple vanilla put option.

Virtual market experiments: Monte-Carlo expectation

1. Adjust the number of `Walkers` to achieve a precision of only about 10%.
2. Change the parameter `LogNkappa=1.002` to keep a nearly log-normal distribution of the increments and yet force the applet to use a new random walk for every value of the underlying asset. The "noise" between adjacent prices can then be used as a measure or the precision of the calculation. How far were you from the previous 10% target?
3. Vary the parameter `LogNkappa` to study how different distributions of the market increments affect the price of an option.

To conclude this section with a comparison between the finite difference and Monte-Carlo methods, remember that Monte-Carlo simulations offer considerable flexibility to model path-dependent options and change the statistics of the market increments. This flexibility, however, comes at a high computing cost for reaching an acceptable precision at the percent level, this even if it is generally sufficient to calculate a single price for the option, which finite differences cannot do.

4.6 Computer quiz

1. The present value of an plain vanilla option can be calculated using
   (a) the average terminal payoff from possible realizations of the underlying
   (b) the payoff from the average possible realization of the underlying
   (c) the time averaged payoff from possible realization of the underlying

2. Approaching the expiry, the price of a vanilla call on a share without dividends
   (a) rises everywhere and particularly at the money
   (b) falls everywhere and particularly at the money
   (c) rises out-of-the-money and falls in-the money
   (d) falls out-of-the-money and rises in-the-money

3. For the same underlying and time to expiry, a larger strike price yields
   (a) a higher price for the vanilla put
   (b) a higher price for the vanilla call
   (c) a higher price for the cash-or-nothing call

4. In a risk-neutral Monte-Carlo simulation with shares is
   (a) the drift is equal to the long-term average growth of the stock market
   (b) the drift is larger than the risk-free interest rate
   (c) the drift is equal to the risk-free interest minus the dividend yield
   (d) the volatility is equal to zero

5. With a 'frown' or an inverted 'smile' in the implied volatility, the market expects
   (a) a systematic fall in the underlying (bear market)
   (b) a systematic rise in the underlying (bull market)
   (c) expects rather stable prices for the underlying

6. A negative time value is obtained from
   (a) a finite interest rates in the case of a vanilla call option
   (b) a finite dividend yield in the case of a vanilla call option
   (c) the volatility in the case of a super-share

4.7 Exercises

4.01 ***Price of a European call option.*** Calculate the price of a European vanilla call option five months before it expires with a strike at EUR 12, if the underlying share is now trading for EUR 11 in a market with 20% volatility and a spot rate of 5%. Preset the default parameters in the VMARKET applet ready to execute a Monte-Carlo simulation with 1000 walkers and perform at least three independent simulations to estimate the precision of your result. Compare the price with the value you obtain from at least one other method.

4.02 ***Limit the potential losses from a share.*** What is the fair price of an insurance that limits the possible losses from a share to 10% of the investment, if the underlying does not pay dividends and trades in a market with 30% volatility and 3% spot rate. To which extent does this protection reduce the expected return on the investment?

4.03 ***Time value of an exotic option.*** Write a table showing the time value of cash-or-nothing call options that pay EUR 1 in nine months time if the underlying share exceeds a strike 0, 10 and 33% below / above the current price of EUR 10. Use Monte-Carlo simulations to account for an in-barrier that can be independently set 0, 10 and 33% below / above the current asset price. Make sure that you achieve a precision better than 10%.

4.04 Implied volatility. Use the market data from the SMI index from June 21st, 2002 (alt. today) to calculate at least three points showing the structure of the implied volatility. Interpret your result in the light of the market conditions that prevailed (alt. today).

4.05 Hedging the shares of your portfolio. Propose a practical hedging strategy using European options limiting the downside risk from stock in your model portfolio (exercise 1.01) to 30% of the investment. Explain your risk management strategy, illustrating how the total value of the portfolio changes with different plausible scenarios; estimate the cost of this strategy in terms of the reduced return of the capital investment.

4.06 Dividend yield with Monte-Carlo. ♠ Having described the drift associated with a continuous payment of a dividend (exercise 2.04), implement a Monte-Carlo scheme to calculate the price of a vanilla call paying a dividend. Keep the default parameters and switch from (`Monte-Carlo`, `Exercise`) to (`FinElements`, `StckOption`) to compare your solution with the one obtained using finite elements.

4.07 Dividend yield with finite differences. ♠ Modify the naive finite difference discretization of the Black-Scholes equation to account for the payment of a dividend yield. Keep the default parameters and switch from (`FinDifferen`, `Exercise`) to (`FinElements`, `StckOption`) to compare your solution with the one obtained using finite elements.

4.08 Log-normal finite differences for a call. ♠ Study the finite difference scheme for a European put option in log-normal variables and extend its validity to the case of a European call.

4.09 Options devaluation model. ♠ To provide economic stability, the governments of emerging market economies sometimes "peg" their currency with a fixed exchange rate to the Euro or the Dollar. In a financial crisis, this peg gets under pressure and sometimes leads to an abrupt devaluation as experienced by Argentina in December 2001. Long periods of stability, followed by a devaluation and a period of high volatility can be modeled by adding a Poisson process to the random walk that simulates the currency spot rate S

$$\text{Before:} \quad \frac{dS}{S^\beta} = \mu_1 dt + \sigma_1 dW(t) + \begin{cases} \eta & \text{with a probability } \ \xi dt \\ 0 & \text{with a probability } \ 1 - \xi dt \end{cases}$$

$$\text{After:} \quad \frac{dS}{S^\beta} = \mu_2 dt + \sigma_2 dW(t)$$

where η is the devaluation size estimated for an annualized probability ξ. Implement this Monte-Carlo model in the VMARKET applet and calculate the price of a European call giving its holder the right to exchange Argentinian Pesos at par with US dollars in one year time. Assume plausible values for January 2001 such as $\beta = 1$, $\mu_1 = 0$, $\sigma_1 = 0.03$, $\mu_2 = -0.05$, $\sigma_2 = 0.1$ with an annualized probability $\xi = 0.3$ of devaluation by a factor two. Discuss the effect the volatilities and devaluation probability have on a currency spot rate that is out-, at- and in-the-money of the strike price.

4.10 ***Stochastic volatility model.***♠ Use a Monte-Carlo simulation in the VMARKET applet to implement the stochastic volatility model

$$\begin{cases} \frac{dS}{S} &= \mu dt + \Sigma dW_1(t) \\ d\Sigma &= a(b-\Sigma)dt + \sigma dW_2(t) \end{cases}$$

featuring two uncorrelated random numbers dW_1 and dW_2. The model assumes a log-normal evolution of asset prices $S(t)$ and a normal evolution for the volatility $\Sigma(t)$ with a mean reverting process to keep a positive volatility $\Sigma(t) \approx \sigma > 0$.

4.11 ***Trading resistance levels.***♠ Using the model for "resistance" levels previously derived and implemented in exercise 2.03, calculate the price of a vanilla put option that is *at-the-money* and 10% below a significant resistance level nine months before the expiry. Assume a 3% spot rate for a market drifting at an annual 6% and a 40% volatility. Propose a trading strategy taking advantage of a resistance level; can an individual investor benefit from this?

All these problems can be edited and submitted for correction directly from your web browser, selecting *WORK:assignments* from the course main page.

4.8 Further reading and links

- **Option pricing.**
  General: Hull♢[11], Cox and Rubenstein♢ [6].
  Stock: Wilmott♠[24].
  Bonds: Rebonato♠[19].
- **Numerial methods.**
  Finite-differences, Monte-Carlo: Jaun[47] [12]♠ and references therein, Wilmott♠[24].

4.9 Quick intermediate evaluation form

Your opinion is precious. Please fill in the anonymous evaluation form in the web edition. Thank you very much in advance for your collaboration.

[47] http://www.lifelong-learners.com/pde

5 BONDS, SWAPS AND DERIVATIVES

5.1 Discound bonds

Dealing with stock options, the small uncertain changes in the short term interest (or the spot rate) have been neglected in comparison with the much larger random changes in the price of the underlying share. As a consequence, the Black-Scholes model accounts only for the volatility of the underlying and is only applicable over a relatively short period of time with typically less than one year to the expiry. In contrast to stock options, bonds do not depend on an underlying and mature over time intervals as long as 30 years. The uncertain evolution of the spot rate is then the dominant factor that drives the bond price if the credit rating of the bond issuer doesn't change: a bond that pays a fixed annual coupon of 5% during the next 10 years is more valuable if the spot rate is forecasted to drop to 2% rather than if it is to rise to 6%... This chapter examines the effect of volatile interest rates on a variety of credit market securities, using the VMARKET to develop your intuition with numerical experiments for a number of standard models used for the credit market.

5.1.1 Term structure models for dummies

Imagine a portfolio with two identical discount bonds, except that the first $P(t, T_1)$ expires some time before the second $P(t, T_2)$. What is the effect of a market fluctuation, which suddenly rises the spot rate at a time $t < T_1 < T_2$ before the first bond reaches maturity? The bonds are correlated and both will loose some of their original value; since there is more time left for another fluctuation to step back in the opposite direction, it is reasonable to assume that the second bond with a longer time to maturity will be less affected.

Taking advantage of this correlation, Vasicek creates a portfolio with a positive holding in the first bond and a negative holding in the second. By choosing exactly the right balance, this delta-hedging cancels out the uncertain effect from fluctuations and leaves only a deterministic change in the portfolio value. This is then used to calculate the fair price of a bond. The normalized value of the discount function is of course known at the maturity $P(T, T) = 1$ and the calculation is carried out with a forecast of the interest rates backward in time to predict the fair value $P(T - t, T)$ for an increasing lifetime $T - t$.

The VMARKET applet on-line illustrates the procedure for a bond lifetime with up to `RunTime=10` years. For a given value of the spot rate r (horizontal axis, chosen to reflect the current market conditions), the discount function $P(t, T)$ is decreasing backward in time t. Indeed, investors expect a return from their investment, which shows up as a growth of the discount function when the time runs forward so as to reach exactly one at maturity. The reward can be measured using (2.2.2#eq.1) as a yield $Y(t, T) = -\log(P)/(T - t)$ and differs from the spot rate r because of the uncertain evolution of the future rates.

Virtual market experiments: evolving the yield curve

1. Press `Toggle Display` to study the evolution of the yield curve $Y(r)$ for a fixed lifetime of the bond (specified under `Time`) and the term structure of the interest rates $Y(t)$ that is plotted for the specified `SpotRate`.

2. Set `Volatility=0` and compare the output obtained for a constant interest rate with the simple discounting previously used in (1.3#eq.6).

Due to the cyclic nature of the economy and the changes in the central bank interest rates, economists generally forecast what may be the future evolution of spot rates $r(t')$ with $t' \in [t,T]$. This opinion consists of a drift ("the spot rate will fall") and a volatility ("the spot rate will fluctuate") that can be estimated from historical values (exercise 1.05).

Masters: one factor models to forecast the term structure of interest rates. ♢ A broad class of models can already be obtained using only one driving term for the uncertainty and assuming a normal distribution of the interest rate increments of the form

$$dr = \mu(r,t)dt + \sigma(t)dW(t). \qquad (5.1.1\#eq.1)$$

Contrary to stock options where the drift scales out of the Black-Scholes equation (3.4#eq.4), the **interest rate drifts play a crucial role** for the evolution of bond prices. Using the excess return $\frac{dP}{dt} - rP = (-\mu_s + \lambda\sigma_s)\frac{\partial P}{\partial r}$ when the stochastic term is neglected in (3.5#eq.6), different models have been proposed to forecast the evolution of the interest rates.

The Vasicek model accounts for a long-term average rate and investors appetite for risk

$$\frac{dP}{dt} - rP = [a(b-r) + \lambda\sigma]\frac{\partial P}{\partial r} \qquad (5.1.1\#eq.2)$$

The first term is a *mean reversion* process, where the interest rate is pulled back to the level b at a velocity a. The second term is proportional to the market price of risk λ and measures the extra return per unit risk expected by the investors (3.5#eq.9).

The Ho and Lee model uses the instantaneous forward rate $F(0,0,t)$ from the market

$$\frac{dP}{dt} - rP = \left[\frac{\partial F(0,0,t)}{\partial t} + \sigma^2 t\right]\frac{\partial P}{\partial r} \qquad (5.1.1\#eq.3)$$

to forecast a drift based on today's expectations without ever saturating.

The Hull an White model circumvents this problem with an evolution

$$\frac{dP}{dt} - rP = \left[\frac{\partial F(0,0,t)}{\partial t} + a\big(F(0,0,t) - r\big) + \frac{\sigma^2}{2a}\big(1 - \exp(-2at)\big)\right]\frac{\partial P}{\partial r} \quad (5.1.1\#eq.4)$$

which reproduces the slope of the initial instantaneous forward rates from Ho and Lee, and later revert back to the long-term average $F(0,0,t)$ with a velocity a.

The VMARKET model (c.f. `Vasicek`) uses a modulation of the market price of risk

$$\frac{dP}{dt} - rP = \big[a(b-r) + \lambda\sigma\cos(2n\pi t/T)\big]\frac{\partial P}{\partial r} \qquad (5.1.1\#eq.5)$$

to reproduce economic cycles and help you develop and intuition.

Analytical solutions can be found provided that the parameters remain constant [11, 19]. A numerical solution is however needed to account for the *volatility hump* observed in the markets (5.1.1#fig.1): the volatility starts at zero (no uncertainty with bond prices today), reaches a maximum and drops again to zero at maturity (the price equals the face value):

$$\sigma(t) \simeq \sigma_{\max}\left[1.7\left((1-t/T) - (1-t/T)^6\right)\right]. \qquad (5.1.1\#eq.6)$$

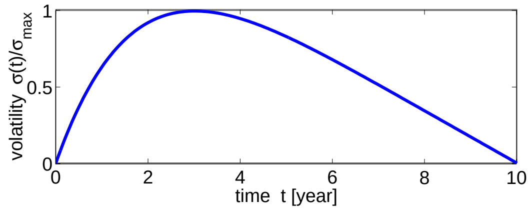

Figure 5.1.1#fig.1: Volatility hump during the 10 years lifetime of a bond.

5.1.2 Parameters illustrated with VMARKET experiments

Since the terminal value of the discount function at the maturity is simply $P(T,T) = 1$, the parameters characterize either the forecast of the spot rate or the numerical method that will be examined later in sect.5.3.1. The financial parameters that are relevant in the applet are:

- the **lifetime or maturity date** (T or `RunTime`) of a bond in years, e.g. 10 for a bond reaching its maturity in a decade,
- the **maximum volatility** (σ_{max} or `Volatility`) of a bond estimated from historical values, e.g. 0.02 for a two percent volatility peak that will be reached after one third of the bond lifetime (5.1.1#fig.1),
- the **market price of risk** (λ or `MktPriceRsk`) measuring the reward $\lambda\sigma$ expected by the investors for taking an investment risk, e.g. -0.25 in a risk averse market with little appetite for risk. In the applet, the effect is further modulated by a cosine function reproducing (n or `UserDouble`) **economic cycles** during the lifetime of the bond,
- the **mean reversion target rate** (b or `MeanRevTarg`) is the value towards which the spot rate returns to after a long time, e.g. 0.05 for a market with a 5% average rate,
- the **mean reversion velocity** (a or `MeanRevVelo`) measures the speed of the process, e.g. 0.5 [1/year] for a mean reversion taking about $1/0.5 = 2$ years.
- the **spot rate** (r or `SpotRate`) used to plot the term structure of the interest rates.

To visualize the evolution of a bond and the corresponding yield in a very simple case, the VMARKET applet on-line shows what happens in the absence of drifts (right hand side of 5.1.1#eq.5 equals zero) and without volatility. The discount function decreases exponentially backward in time $P(t,T) = \exp(-r[T-t])$ as expected for a risk free investment (1.3#eq.6). The bond yield is equal to the spot rate $Y(r) = r$ and the term structure of the interest rates is constant $Y(t) = r$.

Virtual market experiments: trivial bond

1. Vary the length of the simulation domain `MeshLength` and, by clicking in the plot area, verify that bond yield is indeed equal to the spot rate.
2. Modify the time to the maturity `RunTime` and verify that you have properly understand all three graphs that are plotted.

The second applet illustrates the effect of a large volatility σ in the spot rate and accounts for the extra return investors expect from the market through the so-called market price of risk λ. Although this is not immediately apparent in the simulation, the main effect of the volatility is to reduce the curvature of the discount function $P(r)$ by smearing out irregularities in the yield curves $Y(r)$, $Y(t)$: if the forecast rate changes rapidly, the yield curves do not follow immediately everywhere. The reward payed to the investor who accepts the risk associated with fluctuations in the spot rate is clearly visible, with an effective yield that increases with time for a positive value of the market price of risk λ.

Virtual market experiments: volatility and the market price of risk

1. Vary the amount of volatility in the spot rate and observe how the effect evolves with time. Remember that 4% volatility is huge for credit markets!
2. Change the value and the sign of the market price of risk λ associated in the applet with the `MktPriceRsk` parameter.
3. Set `UserDouble=2` to model two cycles in an economy. Try to identify when the forecasted rates are high; how is the corresponding yield?

The applet illustrates the effect of evolving drifts in the forecast rates, here modeled with two economic cycles during the lifetime of the bond: recession $\to$ cut rate $\to$ over-heated economy $\to$ rise rate... or rather the opposite when the time runs backward in the applet.

The third applet finally illustrates the effect of a mean reversion, which accounts for the tendency of the forecasted rates to fall back to a long term average value. Observe how the yield rapidly drops for large values of the spot rate, reflecting the reversion back to the long term average target.

Virtual market experiments: mean reversion

1. Change the `MeanRevVelo` parameter a to modify the typical time scale for interest rates to revert back to the target level `MeanRevTarg`.
2. Use all the forecasting parameters to approximate the price of a bond during 10 years reproducing the evolution of interest rates in your country.

Experimenting with the applet enables you to develop an intuitive understanding for the fundamental processes that characterize the credit market. The experiments also prepare you also for the inverse problem, where the term structure of the interest rates is known from the market (e.g. 2.2.2#fig.1) and the drift / volatility parameters are matched in order to extrapolate into the future (exercise 5.01).

5.2 Credit derivatives

Even if credit derivatives are not commonly traded in open markets, they are often embedded with bonds to create the flexibility needed by the lenders and borrowers. A 10 years loan, for example, offered by a bank to an individual who buys an apartment for the payment of a fixed 5% interest, can generally be cancelled without penalty at any time. To make this possible, the bank sells a bond with a 5% coupon embedded with an American call option–a product known as a *callable bond*. The money may however originally come from a deposit made at floating LIBOR rates and can be tailored to fixed rates using a *swap*.

5.2.1 Vanilla swaps

Remember from sect.2.2.3 that a swap is a contract derived from a loan, where the payments from a fixed interest rate K are exchanged for the payments from a floating interest rate. In a plain vanilla swap, the floating rate is evaluated at the end of every accrual period $[t_i; t_i + \Delta t]$ when a payment is made in compensation for the difference in rates.

To avoid difficulties with a floating rate that is only known at the end of the accrual period, the calculation proceeds backwards in time and evaluates the price (of what is sometimes called FRA) for each period separately using the same procedure as for bonds. The equilibrium swap rate K is chosen so as to make the contract worthless at the onset $V(r,T) = 0$ and the mismatch between the spot rate r and the swap rate K is accumulated over the accrual period to calculate the price of a swap having a finite lifetime $V(r, T - \Delta t)$. For a unit Notional principal, the incremental change in swap value is the difference of interest rates multiplied by the time interval $(r-K)\Delta t$. As for any asset with an investment value, the accumulated earnings or losses from the swap can themselves be viewed as bonds with a positive or negative value and can therefore be described using the Vasicek model from the previous section. In fact, a swap can be understood as a bond that starts with zero as initial value and pays a continuously compounded annual coupon $(r - K)dt$. Only one parameter is required in addition to those that have been defined in sect.5.1.2:

- the fixed **swap rate** (K or `StrikePrice`) is expressed as the relative annual return in the fixed leg, e.g. 0.04 for a predetermined swap rate of 4%.

The VMARKET applet on-line shows how the value of a swap with a fixed rate of 8% evolves as a function of the spot rate for an increasing time to the maturity. Immediately after the start of the contract, the swap acquires a negative value if the spot rate is below the swap rate because the holder of the swap has the obligation to pay the fixed rate: this is indeed more than the market is asking for and the swap holder is therefore loosing money to the writer of the swap. On the contrary, the swap acquires a positive value if the spot rate is above the swap rate: the swap holder has the right to pay only a fixed rate and is therefore earning money at the expense of the writer.

Virtual market experiments: a simple swap

1. Change the swap rate (K or `StrikePrice`) and observe what happens.
2. Explain the existence of a fixed point, for which the swap remains worthless.
3. Why does the swap value $V(r)$ bend downwards after some time?

Think of a swap as a coupon paying bond: the downward curvature of the price ($V''(r) < 0$) is the results of the exponential growth at the spot rate, which is expected for any risk free investment when the time runs forward. The opposite happens when the time is reversed and the exponential decrease of the swap price with the spot rate results in a downward curvature in the same manner as previously seen for the discount function.

The same models that have been used for bonds forecast the drift in the interest rate, but the volatility should here be modified to reflect the uncertainty of payments in the floating leg (exercise 5.07). The volatility reduces the overall curvature and therefore also reduces the value of the swap: this can be understood financially from the spot rate fluctuations above and below the swap rate, which tend to cancel out in time and reduce the value of the swap.

Virtual market experiments: forecasting rates and volatility for a swap

1. Study the effect of a typical `Volatility=0.1` observed in swap markets.
2. Account for a finite market price of risk `MktPriceRsk` or $\lambda \in [-1; 1]$ and explain the movement of the neutral point where the swap is worthless.
3. Examine the price of a swap during economic cycles setting `UserDouble=2`.
4. Reload the default parameters and adjust mean reversion parameters to reproduce a drift (5.1.1#eq.5) proportional to the spot rate. Explain why the value of the swap becomes a linear function $V \propto r$.

5.2.2 Cap-/floorlets

The cap-floor parity relation (2.2.4#eq.3) shows that both share many features with swaps. The contract is again decomposed into elementary intervals and the valuation of every caplet or floorlets is carried out backwards starting from zero, since a finite time has to pass to accumulate interest rate earnings. The incremental change reflects the difference between the spot and the cap-/floor rates; contrary to swaps, this difference can never become negative, since caps and floors do not carry any obligation. Using the Vasicek model, the contract can therefore be viewed as a bond paying a continuously compounded annual coupon of $\max(r-K,0)dt$ for caplets and $\max(K-r,0)dt$ for floorlets. By analogy with the swap rate, define the

- the interest **cap-/floor rate** (K or `StrikePrice`), expressed as the relative annual return above /below which the contract pays the rate difference, e.g. 0.04 for a cap rate of 4%.

The VMARKET applet on-line calculates the value of a caplet with a cap rate of 8%, as a function of the spot rate and an increasing time to the maturity.

Virtual market experiments: parameters of a caplet

1. Verify that holding a `Caplet` and shorting a `Floorlet` in a portfolio results in the same payoff as the swap that can be reproduced by selecting a `IRateSwap`.
2. Press `Toggle Display` to discuss in detail the similarities and the differences you notice in a comparison with the payoff from call / put options on shares. Why is the payoff deep "in-the-money" here curved?
3. Set `Volatility=0.05` and more to study how uncertainties affect the prices.
4. Examine the effect of drifts in the forecasted rates by changing parameters such as `MktPriceRsk`, `MeanRevTarg`, `MeanRevVelo`, `UserDouble`.

The same models used to forecast drifts in the interest rate can also be used here, but the volatility should be modified to reflect the uncertainty in the interest earnings (exercise 5.07).

5.3 Methods for bonds and derivatives: finite elements (FEM)

5.3.1 The Vasicek model for a bond ♠

The finite element method offers considerable advantages in terms of flexibility and robustness at the expense of a slightly more complicated formulation. For example, the FEM method does not suffer from the numerical instability that limits the time stepping in finite difference schemes, it converges very quickly in comparison with the Monte-Carlo method using only few driving factors and the next chapter will show how the formulation can easily be extended to deal with the American exercise style. Here we use the Vasicek equation (3.5#eq.6) only to illustrate one implementation and justify the VMARKET solution for the price of a bond $P(r,t)$ as a function of the spot rate and time. More details about the finite element method and its formulation can be found on-line[48].

Since the random increments of interest rates are normally distributed in the Vasicek model, there is no real advantage to transform the problem into normalized variables. We therefore start directly from the partial differential equation (3.5#eq.6)

$$\frac{\partial P}{\partial t}+\frac{1}{2}\sigma_s^2\frac{\partial^2 P}{\partial r^2}+(\mu_s-\lambda\sigma_s)\frac{\partial P}{\partial r}=rP$$

Multiply by an arbitrary *test function* $Q(r)$, integrate over the domain $\Omega=[r_-;r_+]$ where the solution is sought and formulate a *variational principle*

$$\int_{r_-}^{r_+} drQ\left[\frac{\partial P}{\partial t}+\frac{1}{2}\sigma_s^2\frac{\partial^2 P}{\partial r^2}+(\mu_s-\lambda\sigma_s)\frac{\partial P}{\partial r}-rP\right]=0 \qquad \forall Q\in\mathcal{C}^1(\Omega) \qquad (5.3.1\#eq.1)$$

It turns out that this variational problem is equivalent to the original equation provided that the equation is satisfied for all the test functions that are "sufficiently general". Galerkin's method of choosing the same functional space as the solution is an excellent starting point: here it is sufficient to keep piecewise linear function $\mathcal{C}^1(\Omega)$. Indeed, after integration by parts of the second order term

$$\int_{r_-}^{r_+} dr\left[Q\frac{\partial P}{\partial t}-\frac{\sigma_s^2}{2}\frac{\partial Q}{\partial r}\frac{\partial P}{\partial r}+(\mu_s-\lambda\sigma_s)Q\frac{\partial P}{\partial r}-rQP\right]=\frac{\sigma_s^2}{2}Q\frac{\partial P}{\partial r}\bigg|_{r_-}^{r_+}=0 \qquad \forall Q\in\mathcal{C}^1(\Omega) \qquad (5.3.1\#eq.2)$$

only first order derivatives remain, which can be evaluated using linear functions. The surface term that has been produced is used to impose so-called *natural boundary conditions*: the contribution from the upper boundary vanishes if r_+ is large enough, since $Q(r)\sim P(r,t)\to 0$ for $r\to+\infty$ when P,Q are chosen from the same functional space. *Essential boundary conditions* will be imposed on the lower boundary to normalized the discount function where r_- is small, so that the surface term can here simply be neglected. Now discretize the time into small steps using a backward difference for the first term and a partly implicit evaluation for the rest $P=\theta P^{t-\Delta t}+\bar{\theta}P^t$ where $\theta=1-\bar{\theta}\in[1/2;1]$

$$\int_{r_-}^{r_+} dr\left[Q\frac{P^t-P^{t-\Delta t}}{\Delta t}+\theta\left(-\frac{\sigma_s^2}{2}\frac{\partial Q}{\partial r}\frac{\partial P^{t-\Delta t}}{\partial r}+(\mu_s-\lambda\sigma_s)Q\frac{\partial P^{t-\Delta t}}{\partial r}-rQP^{t-\Delta t}\right)+\right.$$
$$\left.+\bar{\theta}\left(-\frac{\sigma_s^2}{2}\frac{\partial Q}{\partial r}\frac{\partial P^t}{\partial r}+(\mu_s-\lambda\sigma_s)Q\frac{\partial P^t}{\partial r}-rQP^t\right)\right]=0$$
$$\forall Q\in\mathcal{C}^1(\Omega) \qquad (5.3.1\#eq.3)$$

[48] http://www.lifelong-learners.com/pde

Discretize the interest rates, decomposing the solution into the weighted sum of finite elements roof-top functions suggested in (5.3.1#fig.1). Reassemble dependencies on r into overlap integrals of the form $< e_i(r)|r^k e'_j(r) >$, which can be calculated analytically for a homogeneous mesh. Assuming a constant volatility $\sigma_s^2 \equiv e$ in (3.5#eq.7) and drift of the form $\mu_s - \lambda\sigma_s = a(b-r) + \lambda\sigma\cos(2n\pi t/T)$, the second last term is conveniently re-written using the coefficient $d(t) = ab - \lambda\sigma\cos(2n\pi t/T)$ as

$$\int_{r_-}^{r_+} dr\bar{\theta}\left[\mu_s(r) - \lambda\sigma_s\right] e_i(r)\frac{\partial}{\partial r}\sum_{j=0}^{n-1} P_j^t e_j(r) = \sum_{j=0}^{n-1}\bar{\theta}P_j^t\left(-a\int_{r_-}^{r_+} dre_i re'_j - d\int_{r_-}^{r_+} dre_i e'_j\right)$$

Multiply by $-\Delta t$ and write all the unknown $P_j^{t-\Delta t}$ as a function of the known values P_j^t

$$\sum_{j=0}^{n-1}\left[< e_i|e_j > +\theta\Delta t\left(\frac{\sigma^2}{2} < e'_i|e'_j > +a < e_i|re'_j > +d < e_i|e'_j > + < e_i|re_j >\right)\right] P_j^{t-\Delta t} =$$
$$\sum_{j=0}^{n-1}\left[< e_i|e_j > -\bar{\theta}\Delta t\left(\frac{\sigma^2}{2} < e'_i|e'_j > +a < e_i|re'_j > +d < e_i|e'_j > + < e_i|re_j >\right)\right] P_j^t$$
$$\forall i = 0, ..., n-1 \quad (5.3.1\#eq.4)$$

To complete the formulation, the problem has to be supplemented with a terminal condition: from the definition of the discount function (2.2.2#eq.1) this is $P(T-t,T) \equiv 1$ when $t=0$. Boundary conditions have to be justified from no-arbitrage considerations: for simplicity, the yield in $P(0,t) = \exp[-Y(t,T)(T-t)]$ (2.2.2#eq.1) is here somewhat artificially associated with the spot rate and forced to zero with the Dirichlet condition $P(0,t) = 1, \forall t$. A similar reasoning justifies the Neuman condition $\partial_r P(r_+,t) \approx \partial_Y P(r_+,t) = -(T-t)P(t,T), \forall t$ and is here implemented for a homogeneous grid $r_j = jh$ using the second order finite difference approximation $\partial_r P(r_n) \approx [3P(r_n) - 4P(r_{n-1}) + P(r_{n-2})]/2h$ [1].

Because of the finite extension of finite element roof-top functions overlapping only with the nearest neighbors, the linear system of equations (5.3.1#eq.4) can be cast into

$$\sum_{j=1}^{n-1} a_{ij} f_j^{t-\Delta t} = \sum_{j=1}^{n-1} b_{ij} f_j^t \equiv c_i, \qquad i = 0, ..., n-1 \qquad (5.3.1\#eq.5)$$

The matrix a_{ij} is tridiagonal of the form $(l_i; d_i; r_i)$, except the last row, where an element created by the Neuman condition $a_{n,n-2} = 1$ has to be eliminated by hand (row $n-1$ minus l_{n-1} times row n) to preserve the structure of the matrix

$$\begin{array}{llll} l_{n-1}f_{n-2} & +d_{n-1}f_{n-1} & +r_{n-1}f_n & = c_{n-1} \\ f_{n-2} & -4\ \ f_{n-1} & +3\ \ f_n & = 2h\partial_r P(r_n) \end{array}$$

$$\Rightarrow \quad (d_{n-1} + 4l_{n-1})f_{n-1} \quad +(r_{n-1} - 3l_{n-1})f_n \quad = c_{n-1} - 2l_{n-1}h\partial_r P(r_n) \qquad (5.3.1\#eq.6)$$

After substituting the value for $\partial_r P(r_n)$ and replacing the last equation by (5.3.1#eq.6), the linear system is solved using standard LU factorization.

Quants: linear Galerkin finite element (FEM) discretization.
Decompose the solution P into a superposition of finite element roof-top functions

$$P(r) = \sum_{j=0}^{n-1} P_j e_j(r), \qquad \forall Q \in \{e_i(r)\}, \quad i = 0, ..., n-1 \tag{5.3.1\#eq.7}$$

$$e_j(x) = \begin{cases} (x - x_{j-1})/(x_j - x_{j-1}) & x \in [x_{j-1}; x_j] \\ (x_{j+1} - x)/(x_{j+1} - x_j) & x \in [x_j; x_{j+1}] \\ 0 & \text{else} \end{cases} \tag{5.3.1\#eq.8}$$

choosing the same elements for the test function Q to create as many equations as unknowns. Only the nearest neighbors contribute to the overlap integrals; these can be evaluated with

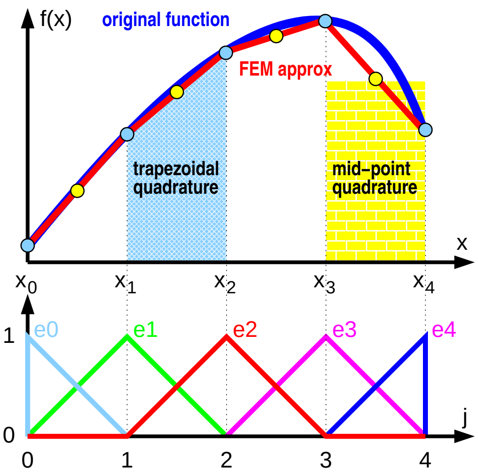

Figure 5.3.1#fig.1: Linear FEM approximation illustrated with a homogeneous mesh.

a combination of the trapezoidal and the mid-point rule

$$\int_{x_i}^{x_{i+1}} f(y)dy \approx (x_{i+1} - x_i) \left[\frac{p}{2} [f(x_i) + f(x_{i+1})] + (1-p) f\left(\frac{x_i + x_{i+1}}{2}\right) \right] \tag{5.3.1\#eq.9}$$

where a suitable choice of the *tunable integration parameter* reproduces piecewise constant ($p = 0$) or linear FEM approximations ($p = 1/3$) or the Crank-Nicholson method ($p = 1$). For a homogeneous mesh $x = x_{i+1} - x_i = ih$, the overlap integrals can be calculated analytically and the finite contributions yield

$$< e_i | x^k e_j > \equiv \int_{x_-}^{x_+} x^k \; e_i(x) e_j(x) dx \tag{5.3.1\#eq.10}$$

$$\begin{array}{rcl@{\qquad\qquad}rcl} < e_i | e_{i\mp 1} > & = & \frac{h}{4}(1-p) & < e_i | e_i > & = & \frac{h}{4}(1+p) \\ < e_i | e'_{i\mp 1} > & = & \mp\frac{1}{2} & < e_i | e'_i > & = & 0 \\ < e'_i | e'_{i\mp 1} > & = & -\frac{1}{h} & < e'_i | e'_i > & = & \frac{1}{h} \\ < e_i | x e_{i\mp 1} > & \approx & \frac{h^2}{8}(1-p)(2i \mp 1) & < e_i | x e_i > & \approx & \frac{h^2}{4}(1+p)i \\ < e_i | x e'_{i\mp 1} > & = & \frac{h}{4}(-2i \mp p + 1) & < e_i | x e'_i > & = & \frac{h}{2}(p-1) \end{array}$$

Despite the rather sophisticated derivation, this finally yields a very elegant scheme that has been implemented in the VMARKET class FEMSolution.java as

```
double twopi    = 8.*Math.atan(1.);
double runTime  = runData.getParamValue("RunTime");
double timeStep = runData.getParamValue("TimeStep");
double sigma    = runData.getParamValue("Volatility");
double theta    = runData.getParamValue("TimeTheta");
double tune     = runData.getParamValue("TuneQuad");
double lambda   = runData.getParamValue("MktPriceRsk");
double t        = 1.-time/runTime;              // normalized time
double hump     = 1.7*(t-t*t*t*t*t*t);          // volatility shaping
double ca       = runData.getParamValue(runData.meanRevVeloNm);
double cb       = runData.getParamValue(runData.meanRevTargNm);
double cycles   = runData.getParamValue(runData.userDoubleNm);
double cd       = ca*cb-lambda*sigma*Math.cos(twopi*cycles*t);
double ce       = sigma*hump; ce=ce*ce;
                                                //--- CONSTRUCT MATRICES
BandMatrix a = new BandMatrix(3, f.length);     // Linear problem
BandMatrix b = new BandMatrix(3, f.length);     //  a*fp=b*f=c
double[]   c = new double[f.length];
                                                // Quadrature coeff
double h,h0,h0o,h1,h1m,h1p,h2,h2o;              //  independent of i
double t0,t0m,t0p,t1,t1m,t1p;                   //  depending on i
h= dx[0];
h0o= 0.25*h*(1-tune);            h0= 0.5*h*(1+tune);
h1m=-0.5; h1p=-h1m;              h1= 0.;
h2o=-1./h;                       h2= 2./h;
for (int i=0; i<=n; i++) {
  t0m=h*h*0.125*(1-tune)*(2*i-1);
  t0p=h*h*0.125*(1-tune)*(2*i+1);   t0=h*h*0.5*i*(tune+1);
  t1m=h*h*0.25*(-2*i-tune+1);
  t1p=h*h*0.25*(-2*i+tune+1);       t1=h*h*0.5*(tune-1);
  a.setL(i,h0o + theta   *timeStep*(t0m +0.5*ce*h2o +ca*t1m +cd*h1m));
  a.setD(i,h0  + theta   *timeStep*(t0  +0.5*ce*h2  +ca*t1  +cd*h1 ));
  a.setR(i,h0o + theta   *timeStep*(t0p +0.5*ce*h2o +ca*t1p +cd*h1p));
  b.setL(i,h0o +(theta-1)*timeStep*(t0m +0.5*ce*h2o +ca*t1m +cd*h1m));
  b.setD(i,h0  +(theta-1)*timeStep*(t0  +0.5*ce*h2  +ca*t1  +cd*h1 ));
  b.setR(i,h0o +(theta-1)*timeStep*(t0p +0.5*ce*h2o +ca*t1p +cd*h1p));
}
c=b.dot(f);
double dPdy0, dPdyn, c0, cn;                    //--- BC
a.setL(0, 0.);a.setD(0, 1.);a.setR(0, 0.);c[0]=1.;//left: Dirichlet
double a1n= a.getD(n-1) +4.*a.getL(n-1);        // right: Neuman
double ann= a.getR(n-1) -3.*a.getL(n-1);        //   O(h^2)
dPdyn=-time*f[n]; cn=c[n-1]-2*dx[0]*a.getL(n-1)*dPdyn;
a.setL(n,a1n);a.setD(n,ann);a.setR(n, 0.);c[n]=cn;
fp=a.solve3(c);                                 //--- SOLVE
for (int i=0; i<=n; i++) {                      //--- PLOT
  gp[i]=-Math.log(fp[i])/time;                  // yield(r)
  if (time<=timeStep) f0[i]=0.;
}
int i=(int)((time/runTime*n));                  // yield(t)
f0[i]=gp[n/4];
```

Two band matrices `a,b` and a vector `c` are created to first assemble the linear problem (5.3.1#eq.5) using the commands of the form `a.setL(j,*)`: they define matrix elements in row i either to the **L**eft, **R**ight or on the **D**iagonal of the matrices `a,b`. The right hand side vector is calculated with a product between the matrix `b` and the discount function `f` that is known from the previous time step. The solution `fp` is computed using LU factorization and the yield is defined from the discount function (2.2.2#eq.1) ready for plotting.
The VMARKET applet on-line shows the result obtained for a weakly implicit scheme (θ or `TimeTheta=0.55`) and a tunable integration parameter (p or `TuneQuad=1.`), which is equivalent to the popular Crank-Nicholson method used by the finite differences afficinados; to be financially meaningful, the solution has of course to be independent of the numerical method. The finite element formulation is slightly more complicated than the implicit finite differences but results in the same computational cost. The additional flexibility provided by a finite elements formulation is such that the Crank-Nicholson scheme should in fact be of little more than historical interest.

5.3.2 Extensions for derivatives ♠

Looking at a swap as being a bond paying a coupon $(r - X)dt$ over an infinitesimally short time interval, the differential change in the value of a bond (3.5#eq.6) can simply be supplemented with the corresponding increment. This immediately leads to the partial differential equation for a swap

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 P}{\partial r^2} + (\mu_s - \lambda\sigma_s)\frac{\partial P}{\partial r} = rP - (r - X) \qquad (5.3.2\#eq.1)$$

and can again be solved using the finite element method. The new term does not involve any unknown and therefore appears on the right hand side of the linear problem (5.3.1#eq.4) with a contribution that can be integrated analytically

$$\int_{r_-}^{r_+} dr e_i(r)\Delta t(r - X) = \Delta t h(ih - X) \qquad (5.3.2\#eq.2)$$

It has been coded into FEMSolution.java with an increment of the right hand side vector

```
                                                     // Construct the problem
c=b.dot(f);                                          //  A*fp=B*f=c as before
for (int i=1; i<n; i++)                              // Add swap source term
  c[i]+= timeStep*h*(i*h-X);
c0=(f[0]+(x[0]-X)*timeStep)*Math.exp(timeStep*X);// Bounday conditions
a.setL(0, 0.);a.setD(0, 1.);a.setR(0, 0.);c[0]=c0;//left: Dirichlet
dPdyn=(f[n]-f[n-1])/h; cn=c[n-1]-2*dx[0]*a.getL(n-1)*dPdyn;
a.setL(n,a1n);a.setD(n,ann);a.setR(n, 0.);c[n]=cn;//right:Neuman
```

The Dirichlet boundary condition has been modified to account for the compounded interest from the fixed swap rate $-X$, but the rest remains the same as for the pricing of a bond.

Similar considerations are valid for caplets and floorlets: viewed as a bond paying a coupon $\max(r - X, 0)dt$ for the caplet and $\max(X - r, 0)dt$ for the floorlet, this yields the modified Vasicek equation for a caplet

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 P}{\partial r^2} + (\mu_s - \lambda\sigma_s)\frac{\partial P}{\partial r} = rP - \max(r - X, 0) \qquad (5.3.2\#eq.3)$$

and, by replacing $\max(r - X, 0)$ with $\max(X - r, 0)$, the counterpart for the floorlet.

Both have been implemented into FEMSolution.java and the scheme for the caplet reads

```
                                            // Construct the problem
c=b.dot(f);                                 //  A*fp=B*f=c as before
for (int i=1; i<n; i++)
  if (i*h<X) c[i]+=0;
  else       c[i]+=timeStep*h*(i*h-X);
a.setL(0, 0.);a.setD(0, 1.);a.setR(0, 0.);c[0]=0.;//left: Dirichlet
dPdyn=0; cn=c[n-1]-2*dx[0]*a.getL(n-1)*dPdyn;
a.setL(n,a1n);a.setD(n,ann);a.setR(n, 0.);c[n]=cn;//right:Neuman
```

Having calculated the fair price for a bond, a swap, cap or floor, it is relatively easy to calculate the value of derivatives such as bond options, swaptions, captions and floortions: their value depends on the same random variable and therefore satisfies the same equation as the underlying. For example, after calculating the value of the bond by solving the Vasicek equation (3.5#eq.6) backwards in time $T_B \to T$, the terminal bond option payoff (2.2.4#eq.1) can be used to integrate backwards further $T \to t$ until the present value of the bond option is found.

5.4 Computer quiz

1. A half bell-shaped discount function shows that
   (a) investors are risk averse (negative market price of risk)
   (b) investors are risk seeking (positive market price of risk)
   (c) the volatility is relatively low

2. A negative market price of risk shows that the investors expect
   (a) a rize of the interest rates
   (b) a drop of the interest rates
   (c) a flattening of the interest rates

3. For the owner of a swap, a downward drift in the spot rate
   (a) is good news, since the swap becomes an asset the holder can sell further
   (b) is bad news, but cannot say if earning or losing money
   (c) is bad news, since the swap becomes a liability the holder is obliged to pay

4. A negative price for a cap is obtained
   (a) when the spot rate rises above the cap rate
   (b) when the spot rate falls below the cap rate
   (c) by mistake, since a cap does not carry any obligation

5. A sudden rise in the volatility makes the caplet
   (a) cheaper
   (b) more expensive
   (c) cannot say

6. In a good FEM calculation, the parameter θ (`TimeTheta` in the applet) should ♠
   (a) be as small as possible in the interval [0.5;1]
   (b) be as large as possible in the interval [-1;1]
   (c) not affect the solution

5.5 Exercises

5.01 Yield curve modeling. To develop your intuition for the forecasting of interest rates and use an example from a market, find a set of input parameters that reproduce the yield curves (in 2.2.2#fig.1) using a simulation with the VMARKET applet. To which extent is your parametrization unique?

5.02 Forecasting interest rates. From historical data, determine a set of parameters to forecast the interest rate and calculate the yield of a bond in an economic cycle of your country. Describe your simulation in financial terms.

5.03 Price of a collar.$^\diamond$ Combine the payoff from a caplet and a floorlet to estimate the fair price of a 10 years one-period *collar*, which guarantees that the floating mortgage rate from a loan remains between 2-4% when the spot rate is currently at 3%. For simplicity, assume a bond-like evolution of the volatility peaking at 5%, a market price of risk that is neutral and long term interest rates that are expected to stay at the spot rate level. Compare with the value of a swap struck at the spot rate.

5.04 Equilibrium swap rate.$^\diamond$ Calculate the equilibrium swap rate for a market with a volatility $\sigma = 0.01$ and a market price of risk $\lambda = 0.3$. Compare the values you obtain from a direct simulation of the swap with a calculation involving the discount function to define a weighted average of the forward rates in (2.2.3#eq.7).

5.05 Model for a coupon paying bond.$^\spadesuit$ Show that a bond paying a coupon $X(r,t)$ satisfies the bond pricing equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 P}{\partial r^2} + (\mu_s - \lambda\sigma_s)\frac{\partial P}{\partial r} = rP - X$$

Implement the new term into VMARKET and discuss how it affects the yield curve.

5.06 Modeling a capped bond.$^\spadesuit$ Show that a portfolio holding both a bond and a caplet can be modeled as a bond paying a continuously compounded coupon $\min(r, X)dt$. Start by writing a differential equation and discuss the boundary and terminal conditions. Implement your model using finite elements to solve for the payoff from a capped bond using the VMARKET applet. Compare your result by adding the values obtained for a bond and a caplet separately.

5.07 Forecasting volatility.$^\spadesuit$ Extend the finite elements scheme for a swap, using a volatility forecast that can be function of both the interest rates and time of the form $\sigma_s(r,t) = \sqrt{c(t)r + e(t)}$. Use a tunable integration to approximate the overlap integral and implement your model in the VMARKET applet. Choose your functions $c(t) \propto$ `UserDouble` and $\sqrt{e} \propto$ `Volatility` reproducing a volatility starting at 7%, reaching a peak of 10% after 3 years and decaying to around 5% after 10 years.

5.08 Hull and White model for a bond.$^\spadesuit$ Assume that the market forward rates can simply be parametrized with a quadratic polynomial $F(0,0,x) = f_0 + f_1 x + f_2 x^2$ using the factors $f_0 = 0.02$, $f_1 = 0.12$,$f_2 = -0.08$ and a time $x = t/T$ that has been normalized to the $T = 10$ years lifetime of a bond. Implement the Hull and White model (5.1.1#eq.4) to forecast interest rates and calculate the yield of a bond in a market with 2% volatility.

5.6 Further reading and links

- **Option pricing.**
  Hull$^\diamondsuit$[11], Wilmott$^\spadesuit$[24], Rebonato$^{\spadesuit\spadesuit}$ [19].

- **Numerial methods.**
  Finite elements: Jaun[49] [12]$^\spadesuit$ and references therein.

5.7 Quick intermediate evaluation form

Your opinion is precious. Please fill in the anonymous evaluation form in the web edition. Thank you very much in advance for your collaboration.

[49]http://www.lifelong-learners.com/pde

6 AMERICAN OPTION PAYOFF DYNAMICS

6.1 American stock options

The *American exercise style* differs from its *European* counterpart by the time when the contract comes into life: the European option can be exercised only on the expiry date, whereas the American option can be exercised any time up to this expiry date. The additional right granted to the holder of an American contract has of course a value and affects the price of American options before they expire.

The first part of this chapter examines the payoff from American options, using stock market derivatives to illustrate a feature that can also be found in credit derivatives and indeed any type of financial derivatives where the price of an underlying good is is obtained from a consensus between offer and demand.

6.1.1 The American Black-Scholes model for dummies

Since an American option confers its holder the right to buy or sell an underlying share S any time up to the expiry date, the option price $V(S,t)$ can never drop below the *intrinsic value* that is equal to the terminal payoff $\Lambda(S)$ from chapter 2. Indeed, if the price got lower, arbitragers would immediately seize the opportunity and buy a large amount of options only to exercise them immediately for a risk less profit $\Lambda(S) - V(S,t)$. The VMARKET applet on-line illustrates this with an American put option, where the price never drops below the intrinsic value even in the presence of a finite interest rate.

Virtual market experiments: American options

1. Compare the payoff from both the `American` and `European` exercise styles; for which value of the underlying is the difference largest?
2. Switch to `Call`, `VSpread` and `SuperShr` to study how the exercise style affects the price of both vanilla and binary options.
3. Compare the true American payoff with an approximation obtained by taking the larger of the European payoff and the intrinsic value (6.1.1#eq.1).

Since the payoff necessarily exceeds the *intrinsic value*, American options never develop the *negative time value* $V(S,t) - \Lambda(S)$ previously observed in the case of European options. In fact, the experiments above suggest that a crude estimate for the value of an American option can be obtained simply by choosing whichever is larger, the European Black-Scholes formula or the intrinsic value

$$V_{\mathrm{AMR}}(S,t) \approx \max(V_{\mathrm{EUR}}(S,t), \Lambda(S)) \qquad (6.1.1\#\text{eq}.1)$$

Discontinuities where the European payoff intersects the intrinsic value are in contradiction with the efficient market hypothesis: indeed, delta-hedging strategies exist from which risk-free profits can be made and arbitragers quickly smooth out the transition. This shows that the value (6.1.1#eq.1) is not sufficient for the general pricing of American options; nevertheless, the approximation can be useful when an explicit formula is required instead of a more elaborate numerical solution.

6.1.2 Parameters illustrated with VMARKET experiments

Although the pararameters of American options are the same as the European discussed in sect.4.1.2, the option payoff is here altered by the possibility of an early exercise. Starting with the simplest situation without drift (setting `SpotRate=Dividend=0`) the VMARKET applet on-line shows that the European and the American payoff is sometimes identical.

Virtual market experiments: volatility

1. Repeat the experiments comparing the `American` and `European` payoff using vanilla `Call` and also binary options such as `VSpread`, `SuperShr`.
2. Under which circumstance do you get the same American & European payoff?

In absence of drifts, the experiments with vanilla call and put options show that American and European exercise styles yield the same payoff. Dramatic differences do however appear for super-share and other binary options for which the terminal payoff $\Lambda(S)$ is convex.

To study how the American exercise style affects the drifts, let us perform a second series of experiments setting the volatility to zero and increasing the spot rate and the dividend yield parameters to unrealistically large values.

Virtual market experiments: spot rate and dividend yield

1. Select `Put` option and compare the `American` and `European` exercise styles; explain what you observe.
2. Switch back to `American` and `Call` option, and after modifying the spot rate `SpotRate=0-0.6` and the dividend yield `Dividend=0-0.6`, determine under which circumstances the payoff is made of three (rather than two) segments.
3. Try to list all the contracts where the American and the European exercise style results in the same payoff.

Rather than repeating conclusions similar to those that have been obtained from experiments with European options, we encourage the reader to review sect.4.1.2 and develop a qualitative understanding for how the volatility, the spot rate and the dividend affects the payoff from both European and American options.

6.1.3 Application

With an intuition for the parameters describing the price of an American option, we are ready to use the VMARKET applet and compare the numerical solution with market prices. Take the American put derived from shares in Cisco and that expired on Jan 18, 2003 with a strike at USD 20. About half a year before the option expiry (data from Aug 12, 2002, i.e. $159/360 = 0.44$ year before), the Cisco share was trading for USD 13.12 with a market volatility around 60% . Under reasonable assumptions of a 3% spot rate from the US Treasury and no dividend payed that share, the VMARKET applet on-line calculates the fair price using the Black-Scholes model with an American exercise style. After interpolation, the value obtained (USD 7.104) is very close to the value that was quoted on the Chicago Board of exchange CBOE (USD 7.10).

6.2 Methods for American options: finite elements (FEM) ♠

What appears to be a minor extension of the European contract results in considerable mathematical difficulties: so much in fact, that an explicit and exact pricing formula cannot anymore be found analytically. Fortunately, the variational calculus introduced in the previous chapter in a combination with the finite element method[50] now provides a solid theoretical framework that can be extended to obtain a numerical solution using only minor changes to the code for the European option.

6.2.1 The Black-Scholes equation for American options

Following the same procedure as in chapter 3, the Black-Scholes model is here extended to account for the possibility of exercising the American option anytime up to the expiry. Allowing moreover for a continuous dividend payment at a rate D_0, the random walk for the underlying price increment (3.3.1#eq.1) is modified according to

$$\frac{dS}{S} = (\mu - D_0)dt + \sigma dW(t) \tag{6.2.1\#eq.1}$$

Create a portfolio and combine an American option with a number $-\Delta$ of the underlying shares. The initial value and the incremental change are given by

$$\Pi = V - \Delta S \tag{6.2.1\#eq.2}$$

$$d\Pi = dV - \Delta(dS + D_0 S dt) \tag{6.2.1\#eq.3}$$

Using Itô's lemma (3.3.2#eq.2) to calculate the stochastic increment in the option value dV as a function of the underlying, the random component is again eliminated by continuously re-hedging the portfolio with a number $\Delta = \partial V/\partial S$ of shares. No arbitrage arguments show that without taking any risk, the portfolio can at most earn the risk-free return of the spot rate. Because an American option can be exercised any time until it expires, the incremental change in the portfolio value satisfies the *inequality*

$$d\Pi = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - D_0 S \frac{\partial V}{\partial S}\right] dt \leq r\Pi dt \tag{6.2.1\#eq.4}$$

which leads directly to *Black-Scholes* equation for American options

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV \leq 0 \tag{6.2.1\#eq.5}$$

In addition to the usual boundary and terminal condition $V(S,T) = \Lambda(S)$, this inequality (known in mathematics as an obstacle problem) must be supplemented by the so-called *free boundary condition* $V(S,t) \geq \Lambda(S), \forall t$. Apart from that, the Black-Scholes equation is the same for European options paying a dividend (exercise 3.03) with a strict equality replaced by an inequality. The same change of variables (4.3.1#eq.1,4.3.1#eq.4) can therefore be used to transform the problem to log-normal variables $u(x,\tau)$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0 \tag{6.2.1\#eq.6}$$

keeping in mind that the solution has to satisfy the corresponding free-boundary condition of the form $u(x,\tau) \geq c(x,\tau)$.

[50] http://www.lifelong-learners.com/pde

6.2.2 Solution of the variational inequality using finite elements

To satisfy the Black-Scholes and the free-boundary inequalities simultaneously, we follow the same method that is generally used when dealing with obstacle problems. Both inequalities are first cast into a *complementary problem*

$$(u-c)\left(\frac{\partial u}{\partial \tau}-\frac{\partial^2 u}{\partial x^2}\right)\geq 0 \qquad (6.2.2\#eq.1)$$

which has to be satisfied subject the free-boundary condition $(u-c)\geq 0$. Following the spirit of sect.5.3 and using Galerkin's method to solve an equivalent variational principle, choose a test function v that is "sufficiently general" and that satisfies the same constraints as the solution u. By construction, it satisfies the inequality

$$(v-c)\left(\frac{\partial u}{\partial \tau}-\frac{\partial^2 u}{\partial x^2}\right)\geq 0 \qquad (6.2.2\#eq.2)$$

Integrate both inequalities over the domain $\Omega=[r_-;r_+]$ where a solution is sought and, subtracting (6.2.2#eq.1) from (6.2.2#eq.2), formulate an equivalent variational principle

$$\int_{x-}^{x+} dx\ (v-u)\left(\frac{\partial u}{\partial \tau}-\frac{\partial^2 u}{\partial x^2}\right)\geq 0 \qquad (6.2.2\#eq.3)$$

which depends only implicitly on the conditions $u\geq c$ through the choice of the test function. Integrate by parts the second order derivative with a vanishing surface term

$$\int_{x-}^{x+} dx\ \left[(v-u)\frac{\partial u}{\partial \tau}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}\right]\geq 0 \qquad (6.2.2\#eq.4)$$

Decompose the solution and test functions in a series of finite element roof-tops

$$u(x)=\sum_{j=0}^{n-1}u_je_j(x), \qquad v(x)=\sum_{i=0}^{n-1}v_ie_i(x) \qquad (6.2.2\#eq.5)$$

$$\int_{x-}^{x+} dx\ \left[\left(\sum_{i=0}^{n-1}(v_i-u_i)e_i(x)\right)\left(\sum_{j=0}^{n-1}\frac{\partial u_j}{\partial \tau}e_j(x)\right)+\left(\sum_{i=0}^{n-1}(v_i-u_i)e_i'(x)\right)\left(\sum_{j=0}^{n-1}(u_ie_j'(x)\right)\right]\geq 0$$
$$(6.2.2\#eq.6)$$

Using the notation for the overlap integrals in (5.3.1#eq.10), this yields the inequality

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(v_i-u_i)\left[\frac{\partial u_j}{\partial \tau}<e_i|e_j>+u_i<e_i'|e_j'>\right]\geq 0 \qquad (6.2.2\#eq.7)$$

Discretize time with small steps using a forward difference for the first term and a partly implicit evaluation for the second, writing $u=\theta u^{\tau+\Delta\tau}+\bar{\theta}u^{\tau}$ and $\theta=1-\bar{\theta}\in[1/2;1]$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(v_i-u_i^{\tau+\Delta\tau})\left[\frac{u_j^{\tau+\Delta\tau}-u_j^{\tau}}{\Delta\tau}<e_i|e_j>+\theta u_i^{\tau+\Delta\tau}<e_i'|e_j'>+\bar{\theta}u_i^{\tau}<e_i'|e_j'>\right]\geq 0$$
$$(6.2.2\#eq.8)$$

which finally yields

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(v_i - u_i^{\tau+\Delta\tau}) \times \\ \left(\left[< e_i|e_j > +\Delta\tau\theta < e_i'|e_j' >\right] u_j^{\tau+\Delta\tau} - \left[< e_i|e_j > -\Delta\tau\bar{\theta} < e_i'|e_j' >\right] u_j^{\tau}\right) \geq 0 \qquad (6.2.2\#\text{eq.}9)$$

and remains subject to the free-boundary condition $(u - c) \geq 0$. As for obstacle problems in general, it is possible to show that both conditions can only be satisfied if the inequality (6.2.2#eq.9) satisfies the equality

$$\sum_{j=0}^{n-1}\left(\left[< e_i|e_j > +\Delta\tau\theta < e_i'|e_j' >\right] u_j^{\tau+\Delta\tau} - \left[< e_i|e_j > -\Delta\tau\bar{\theta} < e_i'|e_j' >\right] u_j^{\tau}\right) = 0 \\ \forall i = 1, ..., n \qquad (6.2.2\#\text{eq.}10)$$

where the solution satisfies the free-boundary condition $(u - c) \geq 0$. This problem can now finally be solved using an iterative method called projected successive over-relaxation (SSOR), where the solution is sucessively improved from an initial guess, in a manner that guarantees that it always satisfies the free-boundary condition $(u - c) \geq 0$. Note the strong resemblance with the finite element scheme for European options, which uses exactly to the same matrices. The European and the American problems can therefore both be implemented into the same program, changing only the boundary conditions and the SSOR solver to account for the different exercise styles.

Using the same header and footer as in sect.4.4.2 to transform between financial and log-normal variables, the entire scheme has been implemented in FEMSolution.java as

```
                                                      //--- CONSTRUCT MATRICES
    BandMatrix a = new BandMatrix(3, f.length);      // Linear problem
    BandMatrix b = new BandMatrix(3, f.length);      //  a*fp=b*f=c
    double[]   c = new double[f.length];
    double htm = dxx*(1-tune)/4;                      // Quadrature coeff
    double htp = dxx*(1+tune)/4;
    double halpha = dtau/dxx;                         // PDE coefficient
    for (int i=0; i<=n; i++) {
      a.setL(i,   htm -halpha* theta    );
      a.setD(i,2*(htp +halpha* theta   ));
      a.setR(i,   htm -halpha* theta    );
      b.setL(i,   htm -halpha*(theta-1) );
      b.setD(i,2*(htp +halpha*(theta-1)));
      b.setR(i,   htm -halpha*(theta-1) );
    }
    c=b.dot(fm);
    a.setL(0,0.);a.setD(0,1.);a.setR(0,0.);c[0]=0;   // First equation idle

                                                      //--- BC + SOLVE

    if (scheme.equals(vmarket.EUNORM)) {              // European option
      a.setL(1, 0.);a.setD(1,1.);a.setR(1,0.);       //   in-money: Dirichlet
      c[1]=Math.exp(0.5*k2m1*xx0+0.25*k2m1*k2m1*tau);
      a.setL(n,-1.);a.setD(n,1.);a.setR(n,0.);c[n]=0.; // out-money: Neuman
      f=a.ssor3(c,fm);
      f0=strike*Math.exp(-rate*time);

    } else if (scheme.equals(vmarket.AMNORM)) {       // American option
      double[] min = new double[f.length];            // Obstacle
      double[] max = new double[f.length];
      for (int i=0; i<=n; i++) {
        xxi=xx0+(i-1)*dxx;
        min[i]=Math.exp((0.25*k2m1*k2m1 +k1)*tau) *
               Math.max(0., Math.exp(0.5*k2m1*xxi) -
                            Math.exp(0.5*k2p1*xxi)  );
        max[i]=Double.POSITIVE_INFINITY;
        fm[i]=Math.max(min[i],f[i]);                  //   IC interp error
      }
      a.setL(1, 0.);a.setD(1,1.);a.setR(1,0.);       // In-money: Dirichlet
      c[1]=Math.max(min[1],Math.exp(0.5*k2m1*xx0+0.25*k2m1*k2m1*tau));
      a.setL(n,-1.);a.setD(n,1.);a.setR(n,0.);c[n]=0.; //out-money: Neuman
      double precision = strike*Math.pow(10.,-6);    // relativ.to strike
      int maxIter = 30;
      double w = 1.2;                                 // relaxation parameter
      f=a.ssor3(c,fm,min,max,precision,w,maxIter);   // projected-SSOR
      fp[0]=strike;
    }
```

The code has intentionally been restricted here to the case of an American put option, leaving the complete implementation with a call as exercise 6.06. Also note that the terminal and boundary conditions have been specified here in log-normal variables (4.4.2#eq.4).

6.3 Computer quiz

1. Comparing American and European exercise styles
   (a) American options are always more expensive
   (b) American options are always less expensive
   (c) American put options are more expensive and call options are cheaper

2. American options are known to have
   (a) no intrinsic value
   (b) a larger intrinsic value than their European counterparts
   (c) no time value
   (d) a larger time value than their European counterparts

3. The price of an American call in-the-money is always
   (a) smaller than the European call if both have no dividend and a positive spot rate
   (b) equal to the European call if both have no dividend and a positive spot rate
   (c) larger than the European call if both have no dividend and a positive spot rate
   (d) equal to the European call if both have a positive dividend and zero spot rate

4. An approximation $V_{\mathrm{AMR}}(S,t) \approx \max(V_{\mathrm{EUR}}(S,t), \Lambda(S))$ based on a European contract
   (a) always underestimates the fair price of the American contract
   (b) always overestimates the fair price of the American contract
   (c) cannot say

5. American and European options have boundary (BC) and terminal conditions (TC)
   (a) with same BC, same TC
   (b) with same BC, different TC
   (c) with different BC, same TC
   (d) with different BC, different TC

6.4 Exercises

6.01 Price of an American call option. Calculate the price of an American vanilla call option nine months before it expires with a strike at EUR 12, if the underlying share is now trading for EUR 20 and pays a 4% dividend in a market with 50% volatility and a 5% spot rate. Preset the default parameters in the VMARKET applet ready to perform a finite element calculation. Compare the price you obtain with the value of a European exercise style. Show that the difference between both prices is within the numerical accuracy that you can expect from your simulations.

6.02 Time value of European and American options. Compare the time value of a super-share options with European and American exercise styles, for a terminal payoff of USD 10 when the underlying trades in the interval USD 12-18 and zero otherwise. Assume that the underlying does not pay any dividend, in a market with a 30% volatility and 4% interest rates.

6.03 Hedging the shares of your portfolio. Propose a practical hedging strategy using American options to limit the downside risk from stock in your model portfolio (exercise 1.01) to 30% of the original investment. Explain your risk management strategy, illustrating how the portfolio value changes with different plausible scenarios; estimate the cost of your strategy in terms of the reduced return on your capital investment.

6.04 Deterministic interest rates for American options. Modify the Black-Scholes equation to account for a deterministic evolution of the interest rates in a cycle of the form $r(t,T) = r_0(1 + 0.5\sin(2\pi t/T))$. Implement this in the VMARKET applet to calculate the price of an American put option and illustrate your solution with a choice of parameters that highlight the effect from the varying interest rates.

6.05 American options using FEM with financial variables. Formulate a finite element scheme solving the Black-Scholes equation for American options directly in financial variables–without transforming into log-normal variables. Validate your solution with the one that is already provided in the VMARKET applet by selecting *Finite elements*, *Stock option* and *American*. Discuss the advantages / drawbacks of such an implementation.

6.06 American call using FEM with log-normal variables. Extend the finite element scheme in log-normal variables (which has intentionally been limited to the case of an American put option) to allow for calculation featuring American call options.

All these problems can be edited and submitted for correction directly from your web browser, selecting *WORK:assignments* from the course main page.

6.5 Further reading and links

- **Option pricing.**
  Wilmott♠[24].
- **Finite elements, obstacle problem.**
  Jaun [12]♠ and references therein.

6.6 Quick intermediate evaluation form

Your opinion is precious. Please fill in the anonymous evaluation form in the web edition. Thank you very much in advance for your collaboration.

7 EXTREMAL EVENTS

This is only a place holder for a chapter that will be developed in the future.

7.1 Basel accord on banking

7.2 Value at risk (VaR)

7.3 Copulas for risk management

8 MULTI-FACTOR MODELS

This is only a place holder for a chapter that will be developed in the future.

8.1 Principal components analysis

8.2 Martingales and measures

8.3 Two factor models

9 LEARNING LABORATORY ENVIRONEMENT

This chapter gives a short introduction, advice and links to the further documentation for the tools that are used to run this course in a virtual university environement. Most of the tables can be consulted directly when needed, by following the link above the input windows and using the browser Back button to recover input. They are here only given for reference.

9.1 Typesetting with TEX

The text input in the first window is typeset using the TEXlanguage and is translated into HTML with the tth compiler installed on our server. You have to view documents using the Western character set ISO-8859, which is generally set by default in recent browsers. If this page doesn't display the symbols correctly, please refer to the frequently asked questions `FAQ` link on the course main page.

TEXbasics.

Normal ASCII input is interpreted in *text mode* and TEXcommands starting with the backslash character \ are used for formatting. Mathematical symbols are typed in *math mode* delimited by two dollar signs (`$\partial_t f$` yields $\partial_t f$) or in an equation:

```
\begin{equation}\label{advection}
\frac{d}{dt}f \equiv
\frac{\partial f}{\partial t}
  +u\frac{\partial f}{\partial x}=0
\end{equation}
```

yields

$$\frac{d}{dt}f \equiv \frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x} = 0 \quad (9.1.0\#eq.1)$$

where `(\ref{advection})` appears in the text as (9.1.0#eq.1) and can be used to reference your equations within the document. You can also add links and HTML inserts using

```
\href{http://address}{text} create a link from text to http://address
\special{html:stuff} inserts HTML stuff
```

Character type and size.

Rom \textrm{}	*Ital* \textit{}	**Bold** \textbf{}	`Type` \texttt{}
Rom \mathrm{}	*Ital* \mathit{}	**Bold** \mathbf{}	`Type` \mathtt{}
small \small{}	normal \normalsize{}	large \large{}	Large \Large{}

Special characters and accents (text mode).

$ \$	& \&	% \%	# \#	{ \{	} \}	- \-
é \'{e}	è \`{e}	ê \^{e}	ë \"{e}	ç \c{c}		
† \dag	‡ \ddag	§ \S	¶ \P	© \copyright	£ \pounds	

Greek letters (math mode).

α \alpha	β \beta	γ \gamma	δ \delta	ϵ \epsilon	ε \varepsilon
ζ \zeta	η \eta	θ \theta	ϑ \vartheta	ι \iota	κ \kappa
λ \lambda	μ \mu	ν \nu	ξ \xi	o o	π \pi
ϖ \varpi	ρ \rho	ϱ \varrho	σ \sigma	ς \varsigma	τ \tau
υ \upsilon	ϕ \phi	φ \varphi	χ \chi	ψ \psi	ω \omega
Γ \Gamma	Δ \Delta	Θ \Theta	Λ \Lambda	Ξ \Xi	Π \Pi
Σ \Sigma	Υ \Upsilon	Φ \Phi	Ψ \Psi	Ω \Omega	

Binary operation and relation symbols (math mode).

$\pm$ \pm	$\mp$ \mp	$\times$ \times	$\div$ \div	$\ast$ \ast	$\circ$ \circ
$\bullet$ \bullet	$\cdot$ \cdot	$\cap$ \cap	$\cup$ \cup	$\dagger$ \dagger	$\ddagger$ \ddagger
$\leq$ \leq	$\geq$ \geq	$\ll$ \ll	$\gg$ \gg	$\subset$ \subset	$\supset$ \supset
$\subseteq$ \subseteq	$\supseteq$ \supseteq	$\in$ \in	$\ni$ \ni	$\equiv$ \equiv	$\approx$ \approx
$\sim$ \sim	$\simeq$ \simeq	$\neq$ \neq	$\propto$ \propto	$\perp$ \perp	$\mid$ \mid
$\parallel$ \parallel					

Arrows and miscellaneous symbols (math mode).

$\leftarrow$ \leftarrow	$\rightarrow$ \rightarrow	$\Leftarrow$ \Leftarrow	$\Rightarrow$ \Rightarrow
$\leftrightarrow$ \leftrightarrow	$\Leftrightarrow$ \Leftrightarrow	$\uparrow$ \uparrow	$\downarrow$ \downarrow
$\Uparrow$ \Uparrow	$\Downarrow$ \Downarrow	$\mapsto$ \mapsto	$\aleph$ \aleph
$\hbar$ \hbar	$\imath$ \imath	ℓ \ell	$\wp$ \wp
$\Re$ \Re	$\Im$ \Im	$\prime$ \prime	$\emptyset$ \emptyset
∇ \nabla	$\surd$ \surd	$\|$ \—	$\angle$ \angle
$\forall$ \forall	$\exists$ \exists	$\backslash$ \backslash	∂ \partial
∞ \infty	$\clubsuit$ \clubsuit	$\diamondsuit$ \diamondsuit	$\heartsuit$ \heartsuit
$\spadesuit$ \spadesuit			

Operations and functions (math mode).

$\sum$ \sum	$\prod$ \prod	$\int$ \int	$\oint$ \oint	$\sqrt{a}$ \sqrt{a}
a^b a^{b}	a_{ij} a_{ij}	$\sinh$ \sinh	$\arccos$ \arccos	$\cos$ \cos
$\arcsin$ \arcsin	$\sin$ \sin	$\arctan$ \arctan	$\tan$ \tan	$\arg$ \arg
$\cot$ \cot	$\cosh$ \cosh	$\det$ \det	$\dim$ \dim	$\exp$ \exp
$\lim$ \lim	$\ln$ \ln	$\log$ \log	$\max$ \max	$\min$ \min
$\tanh$ \tanh	$\frac{a}{b}$ \frac{a}{b}			

Format, list and equations.

\begin{quote} \end{quote}	\begin{itemize} \item \end{itemize}
\begin{quotation} \end{quotation}	\begin{enumerate} \item \end{enumerate}
\begin{center} \end{center}	\begin{description} \item \end{description}
\begin{verse} \end{verse}	\begin{equation} \label{key} \end{equation}
\begin{verbatim} \end{verbatim}	\begin{equation*} \end{equation*}

Tables (text mode) and arrays (math mode).

```
\begin{tabular}{|llc|}
\multicolumn{2}{c}{ITEM} &
  \multicolumn{1}{c}{PRICE} \\
gnat & (dozen) & 3.24 \\
gnu  & (each)  & 24.00
\end{tabular} \\
```

yields

ITEM		PRICE
gnat	(dozen)	3.24
gnu	(each)	24.00

```
\begin{equation*}
\begin{array}{clcr}
a+b+c & uv    & x-y & 27 \\
a+b   & u+v   & z   & 134 \\
a     & 3u+vw & xyz & 2,978
\end{array}
\end{equation*}
```

yields

$$\begin{array}{clcr} a+b+c & uv & x-y & 27 \\ a+b & u+v & z & 134 \\ a & 3u+vw & xyz & 2,978 \end{array}$$

```
\begin{eqnarray}
\lefteqn{a+b+c=} \nonumber \\
& & c+d+e+f+g+h  \nonumber
x & < & y
\end{eqnarray}
```

yields

$$a+b+c= \\ c+d+e+f+g+h \\ x \quad < \quad y \qquad (9.1.0\#eq.2)$$

9.2 Programming in JAVA

The numerical schemes submitted from the Java window are automatically inserted in the VMARKET source code (e.g. solution 2.01) and compiled on our server before you can download them for execution locally in your browser. This section introduces a limited number of Java commands that will be useful when you carry out your assignments. More details concerning the VMARKET applet can be found in the program tree, the name index and finally in the program listing in the course environement. For a complete tutorial in Java programming, consult the excellent course[51] from Sun Microsystems.

VMARKET = Virtual MARKET applet.

The VMARKET applet is a wrapper to perform a stepwise evolution of functions, i.e. solving time-dependent differential equations using algorithms that can schematically be written as

1. At `time=0` use the well known terminal payoff of a contract to initialize functions `f0[i],f[i],g[i]`, the exact definition of which depend on the specific problem.
2. Plot `f` (black curve), `f0` (grey), and `g` (blue).
3. Use the Java-code submitted in an assignment to calculate the new values for `fp` after a small time step in terms of present `f` and sometimes past values `fm`.
4. Shift the time levels `time=time+timeStep` and the solution arrays `fm=f; f=fp`.
5. Goto 2 until finished.

Only the third step can be modified by the user in the Java window; for example, a simple loop

```
double scale = runData.getParamValue("UserDouble");
for (int i=0; i<=n; i++) {
  fp[i]=scale*f[i];
}
```

executes an artificial evolution, where the next value `fp` is obtained from a simple scaling of the present value `f`. Remember that you have to force your browser to reload the applet after each modification, or prevent your browser from using the older version that is often stored in the browser data cache.

VMARKET variables.

From the list of run parameters (a Java object called `runData`), the VMARKET applet first defines local variables (double = 16 digits precision real, int = up to 9 digits

[51] http://java.sun.com/docs/books/tutorial/

signed integer) using statements

vanilla call option	`boolean isCall=ic.equals(vmarket.CALL);`
vanilla put option	`boolean isPut =ic.equals(vmarket.PUT);`
total run time	`double rTime=runData.getParamValue("RunTime");`
drift	`double mu   =runData.getParamValue("Drift");`
volatility	`double sigma=runData.getParamValue("Volatility");`
distribution	`double kappa=runData.getParamValue("LogNkappa");`
interest rate	`double rate =runData.getParamValue("SpotRate");`
dividend	`double divid=runData.getParamValue("Dividend");`
exercise price	`double strke=runData.getParamValue("StrikePrice");`
relative barrier	`double bar  =runData.getParamValue("Barrier");`
market price of risk	`double lamb =runData.getParamValue("MktPriceRsk");`
mean reversion value	`double ca   =runData.getParamValue("MeanRevTarg");`
mean reversion speed	`double cb   =runData.getParamValue("MeanRevVelo");`
initial condition	`double ic0  =runData.getParamValue("Shape0");`
lower value x-axis	`double x0   =runData.getParamValue("MeshLeft");`
length of x-axis	`double len  =runData.getParamValue("MeshLength");`
number mesh points	`int    n    =runData.getParamValueInt("MeshPoints");`
number realizations	`int    N    =runData.getParamValueInt("Walkers");`
time step	`double step =runData.getParamValue("TimeStep")();`
implicit time param	`double theta=runData.getParamValue("TimeTheta");`
tunable quadrature	`double tune =runData.getParamValue("TuneQuad");`
user defined integer	`int    myInt=runData.getParamValueInt("UserInt");`
user defined real	`double myDbl=runData.getParamValue("UserDouble");`

The evolution is then computed with the help of predefined arrays containing the solution (an object called `solution`)

time in the simulation	`double time;`
last index of solution	`int n = x.length-1;`
mesh points, intervals	`double[] x,dx;`
solution functions	`double[] f0,f,g;`
old, present future	`double[] fm,f,fp;`
derivatives	`double[] dfm,df,dfp;`
current realization	`double[][] currentState;`

Note that in Java (as in C and C++), the index of arrays starts with zero (`x[0]`) and finishes with an index lower by one element less than its size (`x[x.length-1]`).

Debugging.
Having corrected all the compiler errors does unfortunately not mean that your scheme immediately behaves the way you want... You may have to monitor the value of the quantities you defined, using statements of the form

```
/* the mistake dividing by zero has been commented out for debugging
   double error = fp[n]/0;
*/
   System.out.println("Value fp["+i+"] = "+fp[i]);
```

This example will print the values of the array `fp` to the *Java Console* of your browser (with Netscape select *Communicator* + *Tools* + *Java Console*, with Explorer first select *Tool + Internet Options + Advanced + Java console enabled* and then *View + Java Console*). From the values that are printed after a single step, it is possible to track down most of the mistakes. Another debuging strategy is to temporarily des-activate a portion of your program, using the `\* Java comment delimiters *\` that can extend over several lines.

Common errors.
Avoid the most common difficulties when you start programming for your assignments

- Every new variable that is not explicitly listed in the variable index) has to be declared; in particular, memory must be allocated for arrays and objects using the command `new`

```
int i = 3;                   // Declare i as an integer
double[] c;                  // Declare c[] array 16 digits nbrs
c = new double[i];           // Memory for c[0], c[1], c[2]
BandMatrix  A;               // Declare A as a BandMatrix object
A = new BandMatrix(3,10); // Memory for 3 bands with 10 doubles
```

- Accessing the element `c[0]` before the memory has been attributed with a `new` statement leads to the infamous `java.lang.NullPointerException` error; using `c[3]` throws an `java.lang.ArrayIndexOutOfBoundsException:3`, since the array is accessed outside its valid range 0,1,2.
- In Java, the assigning equal sign is denoted by as single `=` whereas the comparing equal sign by a double `==`

```
int a = 42;
if (a == 17) System.out.println("a is equal to 17");
if (a != 17) System.out.println("a is not equal to 17");
```

will print the text "`a is not equal to 17`" to the *Java Console.*

9.3 VMarket parameters and preset in HTML

Problem defining selectors.
The selectors appear on the top of VMARKET plot window and allow you to define the type of problem you want to solve. Careful, white spaces here do count!

- **topic** selects the type of the financial product. Choices include "StckOption", "ZeroCpBond", "BondOption", "IRateSwap", "CreditModel", "RandomWalk", "Exercise".
- **method** selects the numerical method. Choices include "FinDifferen", "FinElements", "Monte-Carlo", "Monte-Carlo*", "DistribFct", "DistribFct*".
- **scheme** selects the flavour of a given product and method. Choices include "European", "European logn", "American", "American logn", "inBarrier", "outBarrier", "particles", "Binom tree", "Exercise 1.01".
- **ic** selects the type of initial or terminal condition. Choices include "Put", "Call", "VSpread", "SuperShr", "Floorlet", "Caplet", "Gaussian".

Editable run parameters.

The following list of parameters are first preset to default values, then modified according to the *TAG* parameters (below) and can be modified at run time by double-clicking the name appearing on the left of the applet:

- **RunTime** the run time [years], e.g. 0.25 for 3 months
- **Drift** the relative total drift of the underlying [1/year], e.g. 0.05 for 5%
- **Volatility** the relative volatility of the undelying [1/year], e.g. 0.5 for 50%
- **LogNkappa** exponent in the random walk, e.g. 1=log- / 0=normal
- **SpotRate** the present short term interest (spot) rate [1/year], e.g. 0.03 for 3%
- **Dividend** dividend yield [1/years], e.g. 0.04 for 4%
- **StrikePrice** the option exercise price at expiry [currency], e.g. 10 for EUR 10
- **Barrier** relative to underlying, e.g. -0.1 for a barrier 10% below the spot price
- **MktPriceRsk** market price of risk, e.g. -0.5 for risk averse market
- **MeanRevTarg** target of mean reversion rate, e.g. 0.05 for 5%
- **MeanRevVelo** speed of mean reversion process, e.g. 2 for 1/2 year
- **Shape0** shape parameter or the yield curve amplitude
- **Shape1** shape parameter or the yield curve slope
- **Shape2** shape parameter or the yield curve convexity
- **MeshLeft** the lower end of the price range [currency]
- **MeshLength** the price range [currency]
- **MeshPoint** the number of mesh points
- **Walkers** the number of random walkers
- **TimeStep** the step [1/year], e.g. 0.00274 for one day, 0.01923 for one week
- **TimeTheta** the implicity parameter for time integration
- **TuneQuad** the tunable quadrature parameter for FEM
- **UserInteger** the user defined integer value
- **UserDouble** the user defined double value

Only the parameters specified in the applet *TAG* are initially displayed; switch from **Double-click below:** to **Show all parameters** to get a complete list

Applet TAG modifiers.

The VMARKET applet is included an HTML document with a specific header: the first couple of lines specify the path name of the executable, the position and the size of the window where the applet will appear. The *TAG* modifiers that follow defines the list of parameters that will be displayed and attributes default values when the applet is first initialized

```
<applet codebase="$user_dir/applet/VM" code=$applet
        align=center width=780 height=420>
       <param name=topic                value="Exercise">
       <param name=scheme               value="Exercise 4.07">
       <param name=ic                   value="Put">
       <param name=method               value="FinDifferen">
       <param name=RunTime              value= 0.5>
       <param name=Drift                value= 0.>
       <param name=Volatility           value= 0.4>
       <param name=LogNkappa            value= 1.>
       <param name=SpotRate             value= 0.1>
       <param name=Dividend             value= 0.>
       <param name=StrikePrice          value= 10.>
       <param name=Barrier              value= 0.>
       <param name=MktPriceRsk          value= 0.>
       <param name=MeanRevTarg          value= 0.05>
       <param name=MeanRevVelo          value= 0.>
       <param name=Shape0               value= 1.>
       <param name=Shape1               value= 0.001>
       <param name=Shape2               value=-0.05>
       <param name=MeshLeft             value= 0.>
       <param name=MeshLength           value= 20.>
       <param name=MeshPoints           value= 21>
       <param name=Walkers              value= 300>
       <param name=TimeStep             value= 0.00397>
       <param name=TimeTheta            value= 0.7>
       <param name=TuneQuad             value= 0.333>
       <param name=UserInteger          value= 0>
       <param name=UserDouble           value= 0.>
</applet>
```

These parameters can always be recovered simply by refreshing the webpage.

9.4 Quick intermediate evaluation form

Your opinion is precious. Please fill in the anonymous evaluation form in the web edition. Thank you very much in advance for your collaboration.

10 APPENDIX

10.1 Glossary of keywords

Accrual period Time interval between the payment of bond coupons or swap settlements.

American exercise style Contract that can be exercised at any time during its life.

Arbitrage Trading strategy taking advantage of the price difference of two or more securities to make an immediate (risk free) profit.

Arbitrageur Person who uses arbitrage as a trading strategy to make immediate (risk free) profits.

Asset Something of value, a property owned by a person or a company.

Bear Animal used as a symbol when the market prices are falling in an economic downturn.

Bid price The price a potential buyer is willing to pay for a security.

Boundary conditions (natural / essential) Conditions (usually justified by no-arbitrage arguments) that are imposed on the boundary of the domain where the solution of a differential equation is sought. Natural conditions are usually imposed through a surface term after partial integration; essential conditions are imposed explicitly in the linear system by replacing an equation by the condition.

Broker Individual who buys and sells goods for other persons.

Bull Animal used as a symbol when the market prices are rising in an economic upturn.

Call option Security giving its holder the right and no obligation to buy an underlying asset.

Cap Collection of *caplets* maturing at different times.

Caplet Interest rate option providing for an upper limit on the interest rate; a caplet entitles the holder to the difference between the spot rate and the strike if this is positive, or zero otherwise.

Capital Amount of money that is invested or used to start a business.

Capital gain Amount of cash raised by the original owners who sell shares in the initial public offering (IPO) of a company.

Capital asset pricing model (CAPM) Linear fit measuring the relative performance of a portfolio in comparison with a market index average.

Capitalism Economic system in which a country's business and industry is controlled and run for profit by private owners.

Clearing margin A margin deposit by a member of a clearing house (e.g. a broker) that guarantees the performance of all the parties in a financial transaction.

Collar Interest rate option combining a *cap* and a *floor* used as an insurance to guarantee that the interest rates remain within a certain interval.

Commission Part of a cost that is proportional to the total value of a trade.

Contingent claims A demand that can be made only if one or more specified outcomes occur.

Coupon Predetermined amount of cash payed as an interest during the life of a bond.

Covered A written option is covered if the writer also has an opposing market position on a share-for-share basis in the underlying security.

Delivery date The date when a *forward* or *futures* contract ends with an amount of cash payed in exchange for the *underlying asset*.

Delivery price Amount of cash in a *forward contract* that will be payed on the *maturity date* in exchange of the *underlying asset*.

Derivative Financial instrument whose value is derived from another asset.

Deterministic Which can be predicted with certainty from the past.

Discount Amount below the par value; difference between a bond principal and the present value.

Discount bond Zero coupon bond.

Discount factor Present value of EUR 1 received some time in the future.

Discount function Function measuring the present value of one unit due at a later time.

Distribution (log-normal) Function measuring the probability density of an event using a log-normal law; incremental changes of stock prices are almost log-normally distributed.

Distribution (normal) Function measuring the probability density of an event using the famous bell-shaped curve; incremental changes of bond prices are sometimes normally distributed.

Diversification Dividing the investment into a variety of securities.

Dividend Portion of a company's profits payed out in cash to the shareholders.

Drift Slow systematic movement in the same direction.

Efficient market Economy in which prices immediately and fully reflect all relevant information.

Efficient frontier In the modern portfolio theory, it is the locus of all the portfolios where the highest possible return is achieved after reducing the specific risk through diversification.

Entrepreneur Person who tries to make money by starting or running a business, especially when this involves taking a financial risk.

Equilibrium swap rate Fixed coupon making the swap worthless when it is initially issued.

European exercise style Contract that can be exercised only at the expiry date.

Exercise an option Use the right to exchange the underlying for a fixed amount of cash.

Exercise (or strike) price The price at which the underlying may be bought or sold.

Exotic option Option that is not plain vanilla and is generally not traded on an exchange.

Expected value Average value obtained by weighting *possible realizations* by their *probabilities.*

Expiry time Date when an option contract ends.

Face (or principal) value Amount of cash an issuer (borrower) agrees to pay at the maturity.

Fannie Mae US government-sponsored federal national mortgage association.

Fee Part of a cost that does not depend on the total value of a trade.

Fixed income instruments Bonds and preferred stock that pay a predetermined amount of cash.

Fixed interest rate Predetermined return on investment of a bond, which remains independent of the market *spot rate.*

Fixed leg (of a swap) Part of the contract involving payments that are predetermined (risk-free) and remain independent of the spot rate.

Floating leg (of a swap) Part of the contract involving payments that depend on the spot rate and therefore carry a financial risk.

Floor Collection of *floorlets* maturing at different times.

Floorlet Interest rate option providing for a lower limit on the interest rate; it entitles the holder to the difference between the strike and the spot rate if this is positive, or zero otherwise.

Forward rate Interest $F(t, T_1, T_2)$ payed today at time t for a loan with a specified maturity T_2 and starting at some point in the future T_1, where $t < T_1 < T_2$.

Forward rate agreement (FRA) Agreement to borrow or lend an amount of cash some time in the future at an interest rate that is fixed today.

Forward contracts Agreement between a buyer and a seller to exchange certain goods for a fixed price some time in the future.

Forward price The delivery price in a forward contract chosen so as to make it worthless.

Freddie Mac US government-sponsored federal home mortgage loan corporation.

Futures contract Special type of standardized *forward contract* enabling anonymous trades on an exchange with a protection against defaults through a *clearing margin*.

Gearing Strategy increasing the return of portfolio by increasing the investment risk.

Gross domestic product (GDP) Total value of goods and services produced by a country.

Go long Purchase an asset in exchange of cash.

Go (or sell) short Sell an asset that has been borrowed from another investor.

Hedger Person who buys securities to reduce the investment risk in a portfolio.

Hedging Strategy reducing the investment risk of a portfolio at the expense of smaller returns.

Holder The purchaser of an option.

Implied volatility *Volatility* of an underlying asset, as measured from the price of derivatives assuming a standard pricing model.

In-the-money Subset of an option series that has a finite *intrinsic value* that is payed out to the holder at the expiry.

Initial / Terminal conditions Value of a function of time that is known at the beginning of a calculation.

Initial public offering (IPO) First sale of a company's shares to the public.

Intrinsic value The value of an option if it would expire with the underlying at its current price.

Investor Person or organization that buys property in the hope of making a profit.

Investment Money used to realize a project in the hope of making a profit.

Itô's lemma Mathematical formula relating the differential of a stochastic function to differential of its stochastic arguments.

London inter bank offered rate (LIBOR) The rate of interest that major international banks in London charge each other for borrowings.

Market Occasion when people buy and sell goods.

Market maker Person who's job it is to determine a fair price of a certain asset and to help buyers and sellers exchange that over-the-counter outside the market.

Market price of risk Parameter measuring how much the investors are risk seeking or risk averse.

Market value *Spot* price obtained via offer and demand from sellers and buyers on the market.

Markov process Stochastic process where consecutive increments are independent from the past.

Martingales

Maturity date The end of the life of a contract.

Maximum likelyhood estimation Statistical method built so as to maximize the chance that a model fits a given dataset.

Mean reversion Tendency of a quantity to evolve towards a long term average.

Modern Portfolio Theory (MPT) Description of rational investment choices based on risk-return trade-offs and efficient diversification.

Monte-Carlo simulation Computer calculation performed with a crowd of random walkers to statistically sample the evolution of market prices by adding small increments.

Net asset Value (NAV) Total value of the fund's investment.

Notional principal The amount of cash used to calculate the payments in an interest rate swap; the principal is "notional" because it is never really exchanged.

Numeraire asset Arbitrary asset chosen to measure the relative performance of an investment in dimensionless units.

Offer price The price a seller is asking in exchange for a security.

Open-end fund Mutual fund where the holdings are continually reinvested and new shares are created on demand.

Option Security giving its holder a right, but not the obligation, to buy or sell an asset at a set price on or before a given date.

Option class Options having the same underlying and the same type of contract (put, call, etc).

Option series Option from the same *class* having the same exercise price and expiry date.

Out-of-the-money Subset of an option series that has no *intrinsic value* and expires worthless.

Over-the-counter (OTC) Non-standard exchange of goods carried out between two parties outside the market, generally without disclosing the price to the public.

Par Equal to the keywd:principal or face value of a security.

Parameters: financial / numerical Financial parameters entirely specify the problem; numerical parameters only serve to control the calculation and should never affect the result.

Path dependent option Option with a payoff depending on the price history of the underlying.

Portfolio Set of shares and financial instruments held by a person or an organization.

Possible realizations Outcomes of a random variable that have a finite probability to occur.

Premium Amount that is in excess of the par value, i.e. the positive difference between the present value and the nominal principal value.

Principal (or face) value Amount of cash an issuer (borrower) agrees to pay at the maturity.

Principal components Partly uncorrelated random varialbles that can explain most of the statistical observations from the markets.

Probability Measure of the likelihood that something will occur.

Put-call parity Relation between the price of vanilla options with the same strike price and expiry.

Put option Security giving its holder the right and no obligation to sell an underlying asset.

Random Which cannot be predicted with certainty from the past.

Redemption date Date when a debt security is has to be payed back, marking the end of the lifetime of a bond.

Random walk Unpredictable motion resulting from increments that are generally assumed to be independent of the past (Markov property).

Reset and payment times Beginning and end of the time interval (*accrual period*) between the payment of coupons in a bond or the exchange of interest payments in a swap.

Risk The possibility of something bad happening sometime in the future.

Risk premium Reward the investors ask for taking a larger investment risk.

Security Document proving that somebody is the owner of certain goods or has a right to acquire them in the future.

Smile Graph with a minimum *implied volatility* for an underlying at-the-money.

Specific risk The uncertain outcome of an investment can be divided in specific and non-specific risks. Specific risk can entirely be eliminated through combination of anti-correlated assets and diversification; non-specific risk affects the entire market.

Speculator Person who buys and sells goods in the hope of making a profit from his view on the evolution of the market.

Split When a growing company emits new shares to reduce the price quoted in the market. In a 2-for-1 split, 2 new shares are exchanged for every share that was previously owned.

Spot The value for immediate delivery.

Stochastic Something that includes an unpredictable random component.

Strike (or exercise) price The price at which the underlying may be bought or sold.

Suzerain In the Middle Ages, the suzerain was a person who owned the right over another (called the *vassal*) who promised to fight and be loyal in return for being given land to live on.

Swap (of interest rates, currency exchange rates, etc) Contract whereby two parties agree to exchange, at known dates in the future, a fixed for a floating set of rates without ever exchanging the principal.

Tenor of a bond Time interval between the payment of consecutive coupons.

Term structure of interest rates Interest rates calculated for bonds of different maturities.

Time value Difference between the *intrinsic value and the value of an option before it expires.*

Transaction costs *The cost of carrying out a trade (fees, commissions, plus the difference between the price obtained and the middle of the bid-offer prices quoted on the market).*

Treasury rate *Interest rate payed by the central bank responsible for a given currency.*

Tree *Method to approximate a dynamical system by recursively adding / subtracting a fixed number of increments to all the possible outcomes.*

Underlying *Security that parties agree to exchange under conditions in a derivative contract.*

Vanilla *Simplest form of a contract.*

Venture capital *High risk investment given in return for a participation in the control and the future earnings of a start-up company that develops a new product.*

Volatility *A measure of the uncertainty of the price of an asset.*

Wiener process *Markov process where the increments are normally distributed with zero mean and a variance proportional to the time step.*

Writer *The seller of an option, usually a large financial institution.*

Zero-coupon bond *A bond without coupon, where the principal and the interest are paid at the maturity date.*

Zero-sum game *Game where the earning from one player exactly equals the loss from another.*

10.2 Notation and symbols

A	value of a fixed amount of cash.
B	barrier of an option, resistance / support level in a market.
$Bnd(t, \{t_i\}, T)$	value at time t of a bond paying coupons at times $\{t_i\}$ up to the maturity date T.
C	plain vanilla call option.
D	fixed dividend per share.
E	earnings from an asset in the P/E ratio.
G	annual growth rate from an asset, similar to R.
N, n	numbers counting random walkers, days, intervals, etc.
K	strike price / rate of an stock / credit option, or swap rate.
P	price of an asset in the P/E ratio or plain vanilla put option.
$P(t, T)$	value at time t of a discount bond with a maturity T.
R	continuously compounded interest rate.
R_s	simply compounded interest rate.
R_l	discretely compounded (every l years) interest rate.
r, dr	spot rate and its increments for an interest rate.
$t, \Delta t$	time and time interval.
S, dS	spot price and its increment for a share.
S_x	Sharpe ratio for a portfolio x.
X, dX	unknown and its differential in general.
X_i	pre-determined value of a bond coupon payed at time $\{t_i\}$.
Y	annual yield measuring an exponential growth $A\exp(Yn)$.
α	CAPM parameter measuring the performance from arbitrage and costs.
β	CAPM parameter measuring the performance from taking risk.
ϵ	relative error ($\propto \sqrt{N}$ in Monte-Carlo simulations.
$E(S)$	expected value from a random variable S.
$\Pi(t)$	value at time t of a portfolio.
σ	volatility, standard deviation of expected returns.
ζ	random number

10.3 Final interactive evaluation form

Your anonymous opinion is precious. Please fill in the **final evaluation form** in the web edition. Thank you very much in advance!

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