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A dynamic programming approach to stochastic control with multiperiod constraints in portfolio optimization

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Abstract. The purpose of this paper is to study stochastic control problems with constraints on multiple time periods. The Bellman principle of optimality is the core of this study and will be used to derive subproblems which are of a standard stochastic control nature with constraints at the end time. A general Linear Quadratic (LQ) control problem with constraints on multiple time periods will also be solved.

As an application the paper deals with the financial problem of finding an optimal portfolio for an investment strategy where different risk levels are given during different periods of the investment horizon. This will be formulated as a mean-variance control problem with variance constraints on the given periods of time. Finally a closed form solution will be derived from the theory of LQ control.

Contents

Notations

$$
\begin{array}{rcl}
\triangleq & = & \text{defined to be...} \\
\mathbb{R}^n & = & \text{n-dimensional real Euclidean space.} \\
\mathbb{R}^{n\times m} & = & \text{set of all } n \times m \text{ matrices.} \\
\mathcal{S}^n & = & \text{set of all } n \times n \text{ symmetric matrices.} \\
tr(A) & = & \text{trace of the square matrix A.} \\
x' & = & \text{transpose of the vector or matrix } x. \\
\langle \cdot, \cdot \rangle & = & \text{inner product of a Hilbert space.} \\
C(s,T;\mathbb{R}^n) & = & \text{the set of all continuous funtions } f:[s,T] \to \mathbb{R}^n. \\
L^p(s,T;\mathbb{R}^n) & = & \text{the set of all Lebesgue measurable funtions } f:[s,T] \to \mathbb{R}^n \\
 & & \text{such that } \int_s^T \mid f(t) \mid^p dt < \infty. \\
L^\infty(s,T;\mathbb{R}^n) & = & \text{the set of all essentially bounded measurable funtions} \\
 & & f:[s,T] \to \mathbb{R}^n. \\
(\Omega, \mathcal{F}, \mathbb{P}) & = & \text{probability space.} \\
\{\mathcal{F}_t\}_{t\geq 0} & = & \text{filtration.} \\
(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}) & = & \text{filtered probability space.} \\
\mathbb{E}\{X\} & = & \text{expectation of the random variable X.} \\
\mathbb{E}\{X \mid \mathcal{G}\} & = & \text{conditional expectation of the random variable X given } \mathcal{G}. \\
\mathbb{V}\{X\} & = & \text{variance of the random variable X.} \\
L^p_{\mathcal{F}}(s,T;\mathbb{R}^n) & = & \text{the set of all } \{\mathcal{F}\}_{t\geq 0}\text{-adapted } \mathbb{R}^n\text{-valued processes } X(\cdot) \\
 & & \text{such that } \mathbb{E}\{\int_s^T \mid X(t) \mid^p dt\} < \infty. \\
L^p_{\mathcal{F}}(\Omega;\mathbb{R}^n) & = & \text{the set of bounded } \mathbb{R}^n\text{-valued } \mathcal{F}\text{-measurable random variables.} \\
\mathcal{U}[s,T] & = & \text{the set of all } \{\mathcal{F}\}_{t\geq 0}\text{-adapted processes } u:[s,T] \to U. \\
\mathcal{U}^w[s,T] & = & \text{the set of stochastic weak admissible controls.}
\end{array}
$$

1. Introduction and Preliminaries

The purpose of this paper is to solve a version of the trajectory planning problem subject to dynamics with a diffusion process. It is based on the works done by [3] where optimal trajectory trajectory planning under a multiplicative stochastic uncertainty is studied and [6] where the continuous time mean-variance portfolio selection problem is formulated and solved using LQ techniques.

The solution to our problem is derived by the Bellman optimal principle. The Bellman principle states that for any point on an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point [1]. This is the corner stone of dynamic programming, a technique developed by R. Bellman in the early 50's. This principle allows us to build up solutions by processing backwards in time which leads to a partial differential equation (PDE) called the Hamilton-Bellman-Jacobi (HJB) equation.

A consequence of this principle is that an optimal control problem can be solved recursively with terminal evolution which is often known. More precisely, suppose u_t is a control variable whose value is to be chosen at time t. Since this principle is instrumental in solving the problem at hand, we give a self-contained presentation here. For an easier exposition of the idea we illustrate it in discrete time.

1.1. Short Introduction to Dynamics Programing.

Let $U_{t-1} = (u_0, u_1, \dots, u_{t-1})$ denote the partial sequence of controls (or decisions) taken over the first t stages. Suppose the cost up to the time horizon N is given by

$$c = G(U_{N-1}) = G(u_0, u_1, \dots, u_{N-1}).$$

Then the principle of optimality is expressed as follows. Define the function

$$G(t, U_{t-1}) = \inf_{u_t, u_{t+1}, \dots, u_{N-1}} G(U_{N-1}).$$

Then these obey the recursion

$$G(t, U_{t-1}) = \inf_{u_t} G(t+1, U_t) \ , \ t < N,$$

with terminal value $G(U_{N-1}, N) = G(U_{N-1})$.

Note that the control variable u_t is chosen on the basis of knowing U_{t-1} (which determines everything else). But a more economical representation of the past history is often sufficient.

For example, we may not need to know the entire path that has been followed up to time t, but only the place to which it has taken us. The idea of a state variable $x \in \mathbb{R}^n$ is that its value at t, x_t, is computed from known quantities and obeys a dynamical equation

$$x_{t+1} = f(t, x_t, u_t)$$

where $f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$. We wish to minimize a cost function of the form, or cost-to-go function

$$c = \sum_{t=0}^{N-1} c(t, x_t, u_t) + c_N(x_N)$$

by choice of controls $\{u_0, \ldots, u_{N-1}\}$. To use the optimality principle, we define the cost from t onwards as

$$c_s = \sum_{t=s}^{N-1} c(t, x_t, u_t) + c_N(x_N) \tag{1.1}$$

and the minimal cost from time s onwards as an optimization over $\{u_s, \ldots, u_{N-1}\}$ conditional on $x_s = x$

$$V(s, x) = \inf_{u_s, \ldots, u_{N-1}} c_s,$$

which is the minimal future cost from time t onwards, given the state x at t. The $V(t, x)$ satisfies

$$V(t, x) = \inf_u [c(t, x, u) + V(t+1, f(t, x, u))] \ , \ t < N, \tag{1.2}$$

with $V(N, x) = c_N(x)$, $x = x_t$. Note that the minimizing u in (1.2) is the optimal control $u(t, x)$ and values of $x_0, \ldots, x_{t-1}$ are irrelevant.

The optimality equation (1.2) is also called **dynamic programming** (DP) equation or **Bellman equation**. The DP equation defines an optimal control problem in what is called feedback (or closed loop form) with $u_t = u(t, x)$. This is in contrast to the open loop formulation in which $\{u_s, \ldots, u_{N-1}\}$ are to be determined all at once at time 0. A policy (or strategy) is a rule for choosing the value of the control variable under all possible circumstances as a function of the perceived circumstances. Thus dynamic programming has the following properties:

- The optimal u_t is a function only of x_t and t, i.e. $u_t = u(t, x_t)$.
- The DP equation yields the optimal u_t in closed loop form. It is optimal what ever the past control policy may have been.
- The DP equation is a backward recursion in time (from which we get the optimum at $N-1$ then $N-2$, and so on). The later policy is decided first.

This is, in fact, much like the phylosophy "Life must be lived forward and understood backwards " (Kierkegaard).

For our purpose of stochastic optimal control, we shall briefly describe the DP approach in a stochastic setting. Let $X_t = (x_0, \ldots, x_t)$ and $U_t = (u_0, \ldots, u_t)$ denote the x and u histories at t. As above, state structure is defined by a dynamical system, that is, the evolution of the process is described by x, having value x_t at time t, with the following properties.

- **Markov Dynamics:**

$$\mathbb{P}(x_{t+1} \mid X_t, U_t) = \mathbb{P}(x_{t+1} \mid x_t, u_t).$$

- **Decomposable cost:** The cost function given by (1.1).
  These assumptions define the state structures.

For the moment we also require for simple exposition.

- **Perfect State Observation:**
  The current value of the state is observable. That is known at the time at which u_t must be chosen. So, letting W_t denote the observed history at time t, we assume $W_t(X_t, U_{t-1})$.

Note that c is determined by W_N, so we might write $c = c(W_N)$. These assumptions define what is known as a discrete-time Markov decision process. As above the cost from time t onwards given by (1.1). Denote the minimal expected cost for the time invariant problem from time t onwards by

$$V(W_t) = \inf_{\pi} \mathbb{E}_\pi[c_t \mid W_t]$$

where π denotes a policy, i.e. a rule for choosing the controls $u_0, \dots, u_{N-1}$. We assert the following theorem.

Theorem 1. *$V(W_t)$ is a function of x_t and t alone, say $V(t, x_t)$. It obeys the DP equation*

$$V(t, x_t) = \inf_{u_t}\{c(t, x_t, u_t) + \mathbb{E}[V(t+1, x_{t+1}) \mid x_t, u_t]\} \tag{1.3}$$

and $V(N, x_N) = c_N(x_N)$. Moreover, a minimizing value of u_t in (1.3)*(which is also only a function of x_t and t) is optimal.*

Proof: The value of $V(W_N)$ is $c_N(x_N)$, so the asserted reduction of V is valid at time N. Assume it is valid at time $t+1$. The DP equation is then

$$V(W_t) = \inf_{u_t}\{c(t, x_t, u_t) + \mathbb{E}[V(t+1, x_{t+1} \mid X_t, U_t]\}. \tag{1.4}$$

But by assumption the right hand side of (1.4) *reduces to the rigth hand side of* (1.3)*. All assertions then follow.* □

Now we turn to the DP equation for controlled diffusion process. The description here is for a scalar process, which will be extended to vector case later in the paper.

The Wiener process W_t is a scalar process for which $W_0 = 0$, the increments in W over disjoint time intervals are independent and W_t is normally distributed with zero mean and variance t. (W is often called Brownian motion). This specification is internally consistent since

$$W_t = W_{t_1} + (W_t - W_{t_1})$$

and for $0 \le t_1 \le t$ the two terms on the right hand side are independent normal variable of zero mean and with variance t_1 and $t - t_1$ respectively. If δW is the increment of W in a time interval of length δt, then

$$\mathbb{E}[\delta W] = 0\ ,\ \mathbb{E}[(\delta W)^2] = \delta t$$

where the expectation is conditional on the past of the process. Note that since

$$\mathbb{E}[(\frac{\delta W}{\delta t})^2] = O(\delta t^{-1}) \to \infty$$

the formal derivative $\epsilon = \frac{\delta W}{\delta t}$(continuous time "white noise") does not exist in a mean-square sense, but expectations such as

$$\mathbb{E}[(\int \alpha(t)\epsilon(t)dt)^2] = \mathbb{E}[(\int \alpha(t)dW(t))^2] = \int \alpha^2(t)dt$$

make sense if the integral is convergent.

Now consider a stochastic differential equation

$$\delta x = a(x,u)\delta t + g(x,u)\delta W$$

which we formally write as

$$\dot{x} = a(x,u) + g(x,u)\epsilon.$$

This, as a Markov process, has an infinitesimal generator with action

$$\mathcal{A}\phi(x) = \lim_{\delta t \searrow 0} \mathbb{E}[\frac{\phi(x(t+\delta t)) - \phi(x(t))}{\delta t} \mid x(t) = x, u(t) = u] = \phi_x a + \frac{1}{2}\phi_{xx} g^2.$$

The DP equation is thus

$$\inf_u [c + V_t + V_x a + \frac{1}{2} g^2 V_{xx}] = 0. \tag{1.5}$$

1.2. **Main contribution to Stochastic Optimal Control.** We shall in this paper apply DP to stochastic optimal control problems with multiple point constraint. Assume the time interval $I = [t_0, t_N]$ is divided into $t_0 < t_1 < \cdots < t_N$, where t_i are constrained points. For each interval $I_k = [t_{k-1}, t_k]$ we can formulate an optimization problem which can be solved either by DP or by Pontryagin's maximum principle. The key idea for dealing with the intervals is to use dynamic programming, see [3]. That is, we first solve an optimal control problem on I_{N-1} with terminal value from I_N, and so on. In other words, we solve the optimal control problem at hand by solving N suboptimal problems backwards in time intervals. The object in our current paper are stochastic dynamical systems where some technical treatments are required. These treatments are based on standard stochastic optimal control techniques, presented e.g. in [8].

The idea of solving subproblems can heuristically be depicted by supposing that a control problem is subject to constraints on n different points of the time interval on which it is defined. By a subproblem we mean the control problem defined on the interval of time between each constraint. Hence each subproblem can be solved using traditional methods of stochastic controls.

It turns out that the trajectory planning with stochastic dynamics studied in this paper is a suitable model in a financial application. We assume an investor is able to trade in continuous time and wishes to find an investment strategy that enables him/her to specify different risk levels at different times of the investment horizon.

If the investor measures his/her risk level by the variance of the portfolio then the investor optimal trading strategy will be the solution of the mean-variance portfolio selection problem with variance constraints on different times.

H. Markowitz [2] studied the mean-variance portfolio optimization problem in a single period. It awarded him the Nobel prize in economy 1997. It has become a basis for mathematical finance. The mean-variance portfolio problem that will be studied in this paper has some differences from the original Markowitz problem. First, the possibility of continuous trading is assumed instead of one period trading. Second, the portfolios in this formulation are not relatively weighted and a riskless asset must be included in the strategy.

Since we assume continuous time trading the natural mathematical description for the portfolio dynamics are continuous semimartingales. This means that the market will be represented by a diffusion model. More specifically the diffusion that will be used is linear and can be derived from the Black-Scholes model consisting of one riskless asset and an arbitrary number of risky assets. The riskless asset is normally thought of as a bank account while the risky assets are stocks.

From [6] we see that the mean-variance portfolio selection problem can be formulated as a stochastic LQ control problem. Here we demonstrate how the risk constraints on the different time periods can be represented by simply adding variance constraints to the formulation. Consequently the solution of the financial problem can be derived in a straight forward way from the LQ multi period constraint framework which in turn is derived from the general stochastic control theory with multi period constraints developed in the paper.

The rest of the paper is organized as follows. We give some very brief mathematical preliminaries essential for the development of the theory. Basically the dynamic programming principle for stochastic controls will be described and some fundamental results are stated. Then the problem formulation for trajectory planning with diffusion process is given and the solution of the mathematical problem is described. Finally we turn to the financial problem.

2. Mathematical Preliminaries

In this section we give a short review of stochastic control theory that will be used in the sequel.

2.1. General Stochastic Control by dynamic programming.

There are two versions of the stochastic control problem called the strong and weak formulations. In the strong formulation there is a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ where $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration generated by an m-dimensional standard Brownian motion $W(t)$ that satisfies the usual conditions of right continuity and completeness. Let the dynamics of the problem be continuous $\{\mathcal{F}_t\}$-adapted, right continuous with left limits (also known as RCLL or CADLAG in french) and strictly positive semimartingales represented by the nonlinear stochastic differential equations

$$(2.1) \qquad \begin{cases} dx(t) &= b(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dW(t) \quad t \in [0,T] \\ x(0) &= x_0 \end{cases}$$

along with the cost function

$$(2.2) \qquad J(u(\cdot)) = \mathbb{E}\Big\{ \int_0^T f(\tau, x(\tau), u(\tau))d\tau + h(x(T)) \Big\}$$

and the space of controls:

$$\mathcal{U}[s,T] \triangleq \{u(\cdot) : [0,T] \to U \ \mid \ u(\cdot) \text{ is measurable and } \{\mathcal{F}_t\}_{t\geq 0}\text{-adapted}\}$$

where $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(s,T;\mathbb{R})$, $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$ and $u(\cdot) \in \mathcal{U}[s,T]$.

Mainly the task of the strong problem can be formulated as:

Problem 1. *Minimize* (2.2) *subject to* (2.1) *over* $\mathcal{U}[0,T]$.

Definition 1. *We call* $(\bar{x}(t), \bar{u}(t))$ *an optimal pair if the trajectory* $\bar{x}(t)$ *given by the control* $\bar{u}(t)$ *solves Problem 1.*

For the weak formulation the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ can vary along with the Brownian motion $W(t)$ that generates the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. This formulation gives a useful model that is applied in the solution of the strong formulation when using the dynamic programming approach.

More precisely, if $x(\cdot)$ is state trajectory starting from x_0 at time 0 in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ along with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ then for any $t > 0$, $x(t)$ is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$ rather than a deterministic point in $\mathbb{R}^n$. However, a feasible control $u(\cdot)$ is $\{\mathcal{F}_t\}_t$-adapeted, i.e. at any time instant t the control knows about all the relevant information of the system up to time t (as specified by $\{\mathcal{F}_t\}_t$) and in particular about $x(t)$.

This implies that $x(t)$ is actually not uncertain for the control at time t. That is, $x(t)$ is almost surely deterministic under a different probability space $\mathbb{P}(\cdot \mid \mathcal{F}_t)$. Thus the above idea requires us to vary the probability space as well in order to apply dynamic programming. It is the reason we use the weak formulation of the stochastic contorl as an auxiliary formulation. For $T > 0$, a metric space U, and any $(s,y) \in [0,T] \times \mathbb{R}^n$ consider the state equation

$$\begin{cases} dx(t) &= b(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dW(t) \quad t \in [s,T] \\ x(0) &= x_0 \end{cases}$$

along with the cost functional

$$J(s,y,u(\cdot)) = \mathbb{E}\Big\{ \int_0^T f(\tau, x(\tau), u(\tau))d\tau + h(x(T)) \Big\}.$$

We fix $s \in [0,T]$ and define the weak feasible control space $\mathcal{U}^w[s,T]$ as all 5-tuples $(\Omega, \mathcal{F}, \mathbb{P}, W(\cdot), u(\cdot))$ satisfying

- $(\Omega, \mathcal{F}, \mathbb{P})$ is complete.
- $W(\cdot)$ is an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ over $[0, T]$ (with W(0) = 0 a.s.), $\mathcal{F}_t^s = \sigma\{W(r) : s \le r \le t\}$ augmented by all the P-null sets in $\mathcal{F}$.
- $u : [0, T] \times \Omega \to U$ is an $\mathcal{F}_t$-adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$.
- under $u(\cdot)$, for any $y \in \mathbb{R}^n$ the dynamics above admits a unique solution $x(\cdot)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge s}, \mathbb{P})$.
- $f(\cdot, x(\cdot), u(\cdot)) \in L_F^1(0, T; \mathbb{R}^n)$ and $h(x(T)) \in L_{F_T}^1(\Omega; \mathbb{R}^n)$ are defined on the given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge s}, \mathbb{P})$ associated with the given 5-tuple.

The notation $u(\cdot) \in \mathcal{U}^w[s, T]$ will be used instead of $(\Omega, \mathcal{F}, \mathbb{P}, W(\cdot), u(\cdot)) \in \mathcal{U}^w[s, T]$ if no confusion exists. Note also that we restrict $\{\mathcal{F}_t\}_{t \ge s}$ to be generated by the Brownian motion above.

Problem 2. *For given $(s, y) \in [0, T] \times \mathbb{R}^n$ find a 5-tuple $\bar{u}(\cdot) = (\bar{\Omega}, \mathcal{F}, \mathbb{P}, \bar{W}(\cdot), \bar{u}(\cdot)) \in \mathcal{U}^w[s, T]$ such that*

$$J(s, y; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}^w[s,T]} J(s, y; u(\cdot)).$$

Further assumptions must be made in order to ensure both the solvability and the uniqueness of solution to the dynamics above, and the well-definedness of cost functional.

(S1) (U,d) is a polish space (a separable metric space) and $T > 0$.
(S2) There exists a constant $L > 0$ such that

$$\begin{cases} \mid \phi(t, x, u) - \phi(t, \hat{x}, u) \mid & \le L \mid x - \hat{x} \mid \quad , \ \forall\, t \in [0, T], x, \hat{x} \in \mathbb{R}, u \in U \\ \mid \phi(t, 0, u) \mid & \le L \quad , \ (t, u) \in [0, T] \times U \end{cases}$$

where the function $\phi(t, x, u)$ is either of the uniformly continuous functions

$$b(t, x, u), \sigma(t, x, u), f(t, x, u), h(x).$$

Note that the existence of weak solutions does not imply the existence of strong solutions and weak uniqueness does not imply pathwise uniqueness not strong uniqueness. Relations between the strong and weak solutions is that strong existence and uniqueness are equivalent to weak existence and pathwise uniqueness which imply weak uniqueness.

The following lemma will be used quite frequently in the derivation of subproblems defined on time intervals related to the constraints. Basically what the lemma says is how the cost function is defined on any time between the start and end. Notice that the expectation changes to a conditional expectation over the filtration from s to $\hat{s}$ where $\hat{s} \in [s, T]$.

Lemma 1. *Let* $(s, y) \in [0, T) \times \mathbb{R}^n$ *and* $(\Omega, \mathcal{F}, \mathbb{P}, W(\cdot), u(\cdot)) \in \mathcal{U}^w[s, T]$*. Then, for any* $\hat{s} \in [s, T)$ *and* $\mathcal{F}^s_{\hat{s}}$*-measurable random variable* ξ*,*

$$J(\hat{s}, \xi(\omega); u(\cdot)) = \mathbb{E}\Big\{ \int_{\hat{s}}^T f(\tau, x(\tau; s, y, u(\cdot)), u(\tau))d\tau + h(x(T; \hat{s}, \xi, u(\cdot))) \mid \mathcal{F}^s_{\hat{s}} \Big\}(\omega) \ , \ \mathbb{P} - a.s.\omega. \tag{2.3}$$

As pointed out earlier an essential concept in dynamic programming is the value function which can be viewed as the optimal cost, and is defined as:

$$\begin{cases} V(s, y) & \triangleq \inf_{u(\cdot) \in \mathcal{U}^w[s,T]} J(s, y; u(\cdot)), & \forall\ (s, y) \in [0, T) \times \mathbb{R}^n, \\ V(T, y) & = h(y), & \forall\ y \in \mathbb{R}^n. \end{cases} \tag{2.4}$$

The stochastic version of Bellman's principle of optimality is stated below. It's proof together with all of the other proofs of this section can be found in [8].

Theorem 2 (Bellman's Principle of optimality)**.** *Let (S1) and (S2) hold. Then for* $\forall\ (s, y) \in [0, T] \times \mathbb{R}^n$*,*

$$V(s, y) = \inf_{u(\cdot) \in \mathcal{U}^w[s,T]} \mathbb{E}\Big\{ \int_s^{\hat{s}} f(\tau, x(\tau; s, y, u(\cdot)), u(\tau))d\tau + V(\hat{s}, x(\hat{s}; s, y, u(\cdot))) \Big\}, \qquad \forall\ 0 < s \leq \hat{s} \leq T. \tag{2.5}$$

This equation is the DP equation in vector case. The following theorem will be used extensively in the proofs of the multiperiod constraint results.

Theorem 3. *Let (S1) and (S2) hold. If* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is optimal for Problem 2, then*

$$V(t, \bar{x}(t)) = \mathbb{E}\Big\{ \int_t^T f(\tau, \bar{x}(\tau), \bar{u}(\tau))d\tau + h(\bar{x}(T)) \mid \mathcal{F}^s_t \Big\}, \ \mathbb{P}\text{-}a.s, \ \ t \in [s, T]. \tag{2.6}$$

Hence dynamic programming solves the original control problem by solving the DP equation, a PDE also called the Hamilton-Jacobi-Bellman equation. The solution of the HJB is the value function defined above. From this solution both the optimal control and trajectory can be found. For historical comments on this equation see e.g.[8].

In order to gain some insight of the above theorem, we proceed a heuristic derivation of the HJB equations.

Heuristics 1 (Intuitive view on the derivation of the HJB equation)**.**
Suppose we are in the end time of an optimal trajectory, then by taking an infinitesimal step backwards in time Bellman's principle tells that we would still be on the optimal trajectory. Defining this end point by

$$V(T, \bar{x}(T)) = J(T, \bar{x}(T), \bar{u}(T)),$$

assuming differentiability of V and using the taylor formula, an infinitesimal step back in time would place us on

$$V(T-\delta,\bar{x}(T-\delta)) = J(T-\delta, x(T-\delta), \bar{u}(\cdot)) = f(t,\bar{x}(t),\bar{u}(t)) + V(T,\bar{x}(T)).$$

A simple change of variables $t = T-\delta$ results in

$$V(t,\bar{x}(t)) = f(t,\bar{x}(t),\bar{u}(t)) + V(t+\delta,\bar{x}(t+\delta)) = \inf_u\{f(t,x(t),u(\cdot)) + V(t+\delta, x(t+\delta))\}.$$

Now we come to the crucial part, since $x(t)$ is a diffusion we can use the infinitesimal generator to find a better expression for $V(t+\delta,\bar{x}(t+\delta))$. From the study of Markov processes in the introduction it is clear that the infinitesimal generator $\mathcal{A}$ can be defined by:

$$\begin{aligned} \mathcal{A}f(x(t)) &= \lim_{h\to 0} \frac{\mathbb{E}^x\left\{f(x(t+h)) - f(x(t))\right\}}{h} \\ &= \sum_i b_i(t,x(t),u(t))\partial_{x_i} f(x(t)) + \tfrac{1}{2}\sum_{i,j}\sigma_{i,j}(t,x(t),u(t))\partial_{x_i x_j} f(x(t)). \end{aligned}$$

So remembering that we are in a heuristic context we can write

$$V(t+\delta,\bar{x}(t+\delta)) = V(t,\bar{x}(t)) + \partial_t V(t,\bar{x}(t)) + \mathcal{A}V(t,\bar{x}(t)).$$

This means that

$$V(t,\bar{x}(t)) = \inf_u\{f(t,x(t),u(\cdot)) + V(t,\bar{x}(t)) + \partial_t V(t,\bar{x}(t)) + \mathcal{A}V(t,\bar{x}(t))\}.$$

Consequently we arrive at the HJB partial differential equation:

$$\begin{cases} \partial_t V(t,x(t)) + \inf_u\{\mathcal{A}V(t,x(t)) + f(t,x(t),u(\cdot))\} = 0 & ,\ t\in[0,T), \\ V(T,x(T)) = h(x(T)). \end{cases}$$

The following proposition states the HJB in a formal way.

Proposition 1 (Hamilton-Jacobi-Bellman PDE)**.** *Let (S1) and (S2) hold. Then $V \in C^{1,2}([0,T]\times\mathbb{R}^n)$,continuously differentiable on t and twice continuously differentiable on x, is a solution to the following terminal value problem:*

$$\text{(2.7)}\qquad \begin{cases} -\partial_t v + \sup_{u\in U} G(t,x,u,-\partial_x v, -\partial_{x^2} v) = 0 & ,\ (t,x)\in[0,T)\times\mathbb{R}^n, \\ v\mid_{t=T} = h(x) & ,\ x\in\mathbb{R}^n, \end{cases}$$

where

$$\text{(2.8)}\quad G(t,x,u,p,P) \triangleq \frac{1}{2}tr(\sigma'(t,x,u)P\sigma(t,x,u)) + \langle p, b(t,x,u)\rangle - f(t,x,u)\rangle,$$
$$\forall\ (t,x,u,p)\in[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n.$$

Note that

$$G(t,x,u,-\partial_x v,-\partial_{x^2} v) = -\mathcal{A}V(t,x(t)) - f(t,x(t),u(\cdot))$$

where $P\in\mathcal{S}^n$.

2.2. **Linear Quadratic Control.**
Consider now the linear stochastic differential equation

$$\text{(2.9)} \qquad \begin{cases} dx(t) &= [A(t)x(t) + B(t)u(t)]dt + \sigma(t)u(t)dW(t) \quad t \in [s,T] \\ x(s) &= y \end{cases}$$

along with the quadratic cost function

$$\text{(2.10)} \qquad J(s,y;u(\cdot)) = \mathbb{E}\Big\{\int_s^T \frac{1}{2}[x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt + \frac{1}{2}x'(T)\hat{P}x(T) + x'(T)\hat{g} + \hat{r}\Big\}$$

where $0 \leq s \leq T$, $A(\cdot) \in L^\infty(s,T;\mathbb{R}^{n\times n})$, $B(\cdot),\sigma(\cdot) \in L^\infty(s,T;\mathbb{R}^{n\times m})$, $Q(\cdot) \in L^\infty(s,T;\mathcal{S}^n)$, $R(\cdot) \in L^\infty(s,T;\mathcal{S}^m)$, $\hat{P} \in \mathcal{S}^n$, $\hat{g} \in \mathbb{R}^n$ and $\hat{r} \in \mathbb{R}$.
The formulation of the LQ control problem is:

Problem 3. *Minimize* (2.10) *subject to* (2.9).

Using dynamic programming we derive the Hamilton-Jacobi-Bellman equation

$$\text{(2.11)} \qquad \begin{cases} 0 &= \partial_t V + \sum_i[(Ax+Bu)_i\partial_{x_i}V] + \frac{1}{2}\sum_{ij}(u'\sigma)_{ij}\partial_{x_ix_j}V + \frac{1}{2}(x'Qx + u'Ru) \\ V(T,x(T)) &= \frac{1}{2}x'(T)\hat{P}x(T) + x'(T)\hat{g} + \hat{r}. \end{cases}$$

It can be proved that the solution is

$$\text{(2.12)} \qquad V(t,x(t)) = \frac{1}{2}x'(t)P(t)x(t) + x'(t)g(t) + r(t).$$

By substitution of (2.12) into (2.11) the optimal control is found to be

$$\text{(2.13)} \qquad \bar{u}(t) = -K(t)^{-1}B'(t)(P(t)\bar{x}(t) + g(t))$$

where $K(t) = \sigma(t)P(t)\sigma'(t) + R(t)$. We see that the Hamilton-Jacobi-Bellman equation becomes

$$\begin{aligned} 0 = \;& \tfrac{1}{2}x'\dot{P}x + x'\dot{g} + \dot{r} + x'A'Px + x'A'g - x'P'BK^{-1}B'Px - 2x'P'BK^{-1}B'g - g'BK^{-1}B'g \\ &+\tfrac{1}{2}(x'P'BK^{-1}\sigma P\sigma'K^{-1}B'Px + 2x'P'BK^{-1}\sigma P\sigma'K^{-1}B'g + g'BK^{-1}\sigma P\sigma'K^{-1} + B'g) \\ &+\tfrac{1}{2}(x'Mx + x'P'BK^{-1}NK^{-1}B'Px + 2x'P'BK^{-1}NK^{-1}B'g + g'BK^{-1}NK^{-1}B'g), \end{aligned}$$

where t has been suppressed from all terms for notational convenience. A solution to this equation can be found by solving the following ordinary differential equations matrix:

$$\text{(2.14)} \qquad \begin{array}{ll} \dot{P}(t) + A'(t)P(t) + P(t)A(t) - P'(t)B(t)K(t)^{-1}B'(t)P(t) + Q(t) = 0 & ,\; P(T) = \hat{P}, \\ \dot{g}(t) + A'(t)g(t) - P'(t)B(t)K(t)^{-1}B'(t)g(t) = 0 & ,\; g(T) = \hat{g}, \\ \dot{r}(t) - \frac{1}{2}g'(t)B(t)K(t)^{-1}B'(t)g(t) = 0 & ,\; r(T) = \hat{r}, \end{array}$$

The first equation is called the (stochastic) Riccati equation. since we now have the optimal control $\bar{u}(\cdot)$ substituting it into the dynamics (2.9) yields the optimal dynamics

$$\text{(2.15)} \qquad \begin{cases} d\bar{x}(t) &= [A(t)\bar{x}(t) - B(t)(K(t)^{-1}B'(t)(P(t)\bar{x}(t) + g(t)))]dt \\ &\quad -(K(t)^{-1}B'(t)(P(t)\bar{x}(t) + g(t)))'\sigma(t)dW(t), \qquad t \in [s,T], \\ \bar{x}(s) &= y \end{cases}$$

The main result is that if the Riccati equation has a solution then there is an optimal solution. In the derivation above, it only required the inverse of $K(t)$, $\forall\ t \in [s, T]$. Nevertheless, the constraint $K > 0$ is needed for the original LQ problem to have an optimal control, see details in [8]. Local solvability of Riccati equations is always true if $K > 0$ for all t. However it is not very useful. In case $Q \geq 0, \hat{P} \geq 0$ we can prove that there is a unique solution to the Riccati equation over $[0, T]$ assuming some regularity on parameters R, σ. It should also be noted that the solvability of the Riccati equation $P(t)$ in general remains an open question. Theory of Riccati equations is itself an importnat issue. Even though $P(t)$ might have a solution the chances of having an analytic expression are pretty slim so the use of a numerical method can be applied in a straight forward manner. An example of such a method is the backward Euler method. In case of one dimensional state we are able to give a close form solution which is the case in our financial application since we can indeed find an analytic solution to its corresponding Riccati equation.

3. Trajectory planning with diffusion process

3.1. **Problem formulation.**

Given the same setting as above let the dynamics be

$$\begin{cases} dx(t) &= b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t) \qquad t \in [0, T] \\ x(s) &= y \\ \mathbb{E}\Big\{ c_k(x(t_{k+1})) \mid \mathcal{F}^s_{t_k} \Big\} &= \alpha_{k+1} \in \mathbb{R}^p \qquad k \in \{1, 2, .., N-1\} \end{cases} \tag{3.1}$$

where $0 \leq s = t_0 < \cdots < t_k < t_{k+1} \cdots < t_N = T$.

We formulate the main problem of this paper by

Problem 4. *Minimize* (2.2) *subject to* (3.1) *over* $\mathcal{U}[0, T]$.

The main difference between Problem 1 and Problem 4 are the constraints

$$\mathbb{E}\Big\{ c_k(x(t_{k+1})) \mid \mathcal{F}^s_{t_k} \Big\} = \alpha_{k+1}.$$

Note that each $c_k(\cdot)$ is a given function, meaning that if

$$c_k(a) \triangleq (a - \mathbb{E}\{a\})^2$$

then the constraints are variance constraints.

Explicitly we wish to solve the problem

$$\min_{u(\cdot) \in \mathcal{U}[0,T]} J(u(\cdot)) = \mathbb{E}\Big\{ \int_0^T f(\tau, x(\tau), u(\tau))d\tau + h(x(T)) \Big\}$$

subject to (3.1).

In the spirit of both the LQ and the general multiperiod constraint problem suitable for dynamics programming it is natural to define the LQ multiperiod constraint problem in the following way. Given the dynamics

$$(3.2)\qquad \begin{cases} dx(t) &= [A(t)x(t) + B(t)u(t)]dt + \sigma(t)u(t)dW(t) \qquad t \in [0,T] \\ x(s) &= y \\ \mathbb{E}\Big\{ C_k x(t_{k+1}) \mid \mathcal{F}^s_{t_k} \Big\} &= \alpha_{k+1} \in \mathbb{R}^p \qquad k \in \{0,1,2,..,N\} \end{cases}$$

where $0 < s = t_0 < \cdots < t_k < t_{k+1} \cdots < t_N = T$.
Then the LQ control with multiperid constraints is

Problem 5. *Minimize* (2.10) *subject to* (3.2) *over* $\mathcal{U}^w[s,T]$.

Explicitly we wish to solve the problem

$$\min_{u(\cdot)\in\mathcal{U}^w[s,T]} J(u(\cdot)) = \mathbb{E}\Big\{ \int_s^T \frac{1}{2}[x'(t)Q(t)x(t)+u'(t)R(t)u(t)]dt+\frac{1}{2}x'(T)\hat{P}x(T)+x'(T)\hat{g}+\hat{r} \Big\}$$

subject to (3.2).

3.2. **Solutions to trajectory planning problems.** Control problems with boundary constraints are defined in this paper as control problems with an additional constraint on the terminal time. So given the dynamics

$$(3.3)\qquad \begin{cases} dx(t) &= b(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dW(t) \qquad t \in [s,T] \\ x(s) &= y \\ \mathbb{E}\Big\{ c(x(T)) \Big\} &= \alpha_T \end{cases}$$

the optimization problem is

Problem 6. *Minimize* (2.2) *subject to* (3.3) *over* $\mathcal{U}^w[s,T]$.

From standard optimization theory we know that by using a lagrange multiplier λ on problem 6 we can lift up the constraints into the cost function so (2.2) takes the form

$$(3.4)\qquad J(s,y;u(\cdot);\lambda) \triangleq \mathbb{E}\Big\{ \int_s^T f(\tau,x(\tau),u(\tau))d\tau + h(x(T)) + \lambda'(c(x(T)) - \alpha_T) \Big\}$$

and the dynamics (3.3) take the form of (2.1). So Problem 6 is transformed into

Problem 7. *Minimize over* $u(\cdot)$ *and maximize over* $\lambda \in \mathbb{R}^p$ (3.4) *subject to* (2.1) *over* $\mathcal{U}^w[s,T]$.

Let $\bar{\lambda}$ be such that $\sup_\lambda J(s,y;u(\cdot);\lambda)$ is found and define the value function by:

$$(3.5)\qquad \begin{cases} V(s,y) &= \inf_{u(\cdot)\in\mathcal{U}^w[s,T]} J(s,y;u(\cdot);\bar{\lambda}) \quad \forall\ (s,y) \in [0,T) \times \mathbb{R}^n \\ V(T,y) &= h(y) + \bar{\lambda}'(c(y) - \alpha_T) \qquad \forall\ y \in \mathbb{R}^n. \end{cases}$$

Definition 2. *The triple* $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{\lambda})$ *is called optimal if it is a solution to Problem 7.*

The following corollary to the Bellman principle theorem is simple but essential for the multiperiod constraint theory.

Corollary 1. *Let (S1) and (S2) hold. If* $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{\lambda})$ *is optimal for Problem 7, then*

(3.6)

$$V(t, \bar{x}(t)) = \mathbb{E}\left\{ \int_t^T f(\tau, \bar{x}(\tau), \bar{u}(\tau)) d\tau + h(\bar{x}(T)) + \bar{\lambda}'(c(\bar{x}(T)) - \bar{\alpha}_T) \mid \mathcal{F}_t^s \right\}, \ \mathbb{P}\text{-}a.s, \ t \in [s, T]$$

Proof: Let $h(\bar{x}(T); \bar{\lambda}) \triangleq h(\bar{x}(T)) + \bar{\lambda}'(c(\bar{x}(T)) - \bar{\alpha}_T)$. Then the cost function is

$$J(s, y; u(\cdot); \lambda) \triangleq \mathbb{E}\left\{ \int_s^T f(\tau, x(\tau), u(\tau)) d\tau + h(x(T); \bar{\lambda}) \right\}$$

Hence the corollary follows from theorem 3. □

Finally we see that the Hamilton-Jacobi-Bellman equation for the boundary constraint problem becomes

$$\text{(3.7)} \qquad \begin{cases} -\partial_t v + \sup_{u \in U} G(t, x, u, -\partial_x v, -\partial_{x^2} v) = 0 & , \ (t, x) \in [0, T) \times \mathbb{R}^n \\ v \mid_{t=T} = h(x) + \bar{\lambda}'(c(x) - \alpha_T) & , \ x \in \mathbb{R}^n. \end{cases}$$

We now turn to the discussion of the main results in this paper. Defining a control for each subinterval $[t_k, t_{k+1}]$ by

$$u_k(t) \triangleq u(t) \ , \ t \in [t_k, t_{k+1}).$$

Notice that the interval $[t_k, t_{k+1})$ is half open, the reason for this is best explained by considering what happens at the terminal time T. At T there is no decision to be made, but at time $T - \delta$, when $\delta > 0$, there is. Hence, decisions are made during the time interval $[0, T)$. Since we consider t_{k+1} as the terminal time of interval $[t_k, t_{k+1}]$ the last decision taken during this interval is really taken during the half open interval $[t_k, t_{k+1})$, i.e. $u(t_{k+1})$ is the first decision of the interval $[t_{k+1}, t_{k+2}]$ and not the last of $[t_k, t_{k+1}]$.

Assume

(S2')

$$\begin{cases} \mid \phi(t, x, u_k) - \phi(t, \hat{x}, u_k) \mid \ \leq L \mid x - \hat{x} \mid & , \ \forall \ t \in [t_k, t_{k+1}), x, \hat{x} \in \mathbb{R}, u_k \in U \\ \mid \phi(t, 0, u_k) \mid \ \leq L & , \ (t, u_k) \in [t_k, t_{k+1}) \times U \end{cases}$$

hold (instead of (S2)). The only difference from the theory already studied is the definition of the time interval.

The main idea is to apply dynamic programming on the intervals $[t_k, t_{k+1}], k = 0, 1, \ldots, N-1$. To solve the problem, let us define the cost-to-go functions, for any

$s \in [t_k, T]$

$$\begin{aligned} J_k(t_k, y, u(\cdot)) &= \mathbb{E}\Big\{ \int_{t_k}^{T} f(\tau, x(\tau; x(\tau, t_k, y, u(\cdot))), u(\tau)) d\tau + h(x(T; t_k, y, u(\cdot))) \mid \mathcal{F}_{t_k}^s \Big\} \\ &= \mathbb{E}\Big\{ \sum_{i=k}^{N-1} \int_{t_i}^{t_{i+1}} f(\tau, x(\tau), u(\tau)) d\tau + h(x(T; t_k, x, u(\cdot))) \mid \mathcal{F}_{t_k}^s \Big\} \end{aligned}$$

and the minimal cost-to-go function

$$V_k(t_k, y) = \inf_{u(\cdot) \in \mathcal{U}^w[t_k, T]} J_k(t_k, y, u(\cdot)) \tag{3.8}$$

subject to

$$\begin{cases} dx(t) &= b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t) \quad t \in [t_i, t_{i+1}] \\ x(t_k) &= y \\ \mathbb{E}\Big\{ c(x(T)) \mid \mathcal{F}_{t_{i+1}}^{t_i} \Big\} &= \alpha_{i+1} \\ J_N(T, y) &= h(y) \\ & i = k, k+1, \ldots, N-1. \end{cases}$$

Now we shall derive the dynamic programming recursion. Note that

$$\begin{aligned} J_k(t_k, x, u(\cdot)) &= \mathbb{E}\Big\{ \int_{t_k}^{T} f(\tau, x(\tau), u(\tau)) d\tau + h(x(T; t_k, y, u(\cdot))) \mid \mathcal{F}_{t_k}^s \Big\} \\ &= \mathbb{E}\Big\{ \int_{t_k}^{t_{k+1}} f(\tau, x(\tau), u(\tau)) d\tau \\ &\quad + \mathbb{E}\Big[\int_{t_{k+1}}^{T} f(\tau, x(\tau), u(\tau)) d\tau + h(x(T; t_k, y, u(\cdot))) \mid \mathcal{F}_{t_{k+1}}^{t_k} \Big] \mid \mathcal{F}_{t_k}^s \Big\} \\ &= \mathbb{E}\Big\{ \int_{t_k}^{t_{k+1}} f(\tau, x(\tau), u(\tau)) d\tau + J_{k+1}(t_{k+1}, x(t_{k+1}; t_k, x, u(\cdot)); u(\cdot)) \mid \mathcal{F}_{t_k}^s \Big\}. \end{aligned}$$

Next we state as a proposition the dynamic programming recursion

Proposition 2. *The optimal cost-to-go functions satisfies the recursion*

$$V_k(t_k, x_k) = \inf_{u(\cdot) \in \mathcal{U}^w[t_k, T]} \mathbb{E}\Big\{ \int_{t_k}^{t_{k+1}} f(\tau, x(\tau; t_k, x_k, u(\cdot)), u(\tau)) d\tau + V_{k+1}(t_{k+1}, x(t_{k+1}; t_k, x_k, u(\cdot))) \mid \mathcal{F}_{t_k}^s \Big\}.$$

Proof: Clearly the first iteration

$$\begin{aligned} V_{N-1}(t_{N-1}, x) = \ & \inf_{u(\cdot) \in \mathcal{U}^w[t_{N-1}, T]} \mathbb{E}\Big\{ \int_{t_{N-1}}^{t_N} f(\tau, x(\tau; t_0, x_0, u(\cdot)); u(\cdot)), u(\tau)) d\tau \\ & + V_N(t_N, x(t_N; t_{N-1}, x, u(\cdot))) \mid \mathcal{F}_{t_{N-1}}^s \Big\} \end{aligned}$$

holds by the definition of the value function. Note that $V_N(t_N, x(t_{N-1}; t_{N-1}, x, u(\cdot)) = h(x(T))$*. Further more the second iteration*

$$\begin{aligned} V_{N-2}(t_{N-2}, x) = \ & \inf_{u(\cdot)\in\mathcal{U}^w[t_{N-2},T]} \mathbb{E}\Big\{ \int_{t_{N-2}}^{t_{N-1}} f(\tau, x(\tau; t_0, x_0, u(\cdot)); u(\cdot)), u(\tau)) d\tau \\ & + V_{N-1}(t_{N-1}, x(t_{N-1}; t_{N-2}, x, u(\cdot))) \mid \mathcal{F}^s_{t_{N-2}} \Big\} \end{aligned}$$

holds by the Bellman Principle of optimality. By induction the proposition follows. □

By this proposition, we can solve N optimal control problem of type Problem 6 on the intervals $[t_k, t_{k+1}], k = 0, 1, \ldots, N-1$. Thus the solutions of Problem 4 can be found by solving the minimal cost-to-go function backwards in time periods $[t_{N-1}, T], [t_{N-2}, t_{N-1}], \ldots, [t_0, t_1]$. Note that the solutions Problem 6 can be found by Lagrange relaxation. We shall show later how to find an explicit Lagrange multiplier for a special type of optimal control problem, LQ control.

3.3. **Solutions of the Linear Quadratic Control problem.**
Consider now the linear system with a terminal constraint

$$\begin{cases} dx(t) &= [A(t)x(t) + B(t)u(t)]dt + \sigma(t)u(t)dW(t) \quad t \in [s, T] \\ x(s) &= y \\ \mathbb{E}\Big\{ x(T) \Big\} &= \alpha_T \end{cases} \tag{3.9}$$

we wish to

Problem 8. *Minimize* (2.10) *subject to* (3.9).

Lagrange relaxation gives a cost function of the form

$$J(s, y; u(\cdot)) = \mathbb{E}\Big\{ \int_s^T \frac{1}{2}[x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt + \frac{1}{2}x'(T)\hat{P}x(T) + x'(T)\hat{g} + \hat{r} + \lambda'(Cx(T) - \alpha_T) \Big\}.$$

This is equal to

$$J(s, y; u(\cdot)) = \mathbb{E}\Big\{ \int_s^T \frac{1}{2}[x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt + \frac{1}{2}x'(T)\hat{P}x(T) + x'(T)(\hat{g} + C'\lambda) + \hat{r} - \lambda'\alpha_T \Big\}. \tag{3.10}$$

Now problem 8 can be reformulated as

Problem 9. *Minimize* $u(\cdot)$ *and Maximize* λ *on* (2.10) *subject to* (2.9).

Our aim is to find a closed form solution to this problem. The equations (2.14) become

$$\begin{aligned} &\dot{P}(t) = -A'(t)P(t) - P(t)A(t) + P'(t)B(t)K(t)^{-1}B'(t)P(t) - M(t) &&, \ P(T) = \hat{P} \\ &\dot{g}(t) = -A'(t)g(t) + P'(t)B(t)K(t)^{-1}B'(t)g(t) &&, \ g(T) = \hat{g} + C'\lambda \\ &\dot{r}(t) = \tfrac{1}{2}g'(t)B(t)K(t)^{-1}B'(t)g(t) &&, \ r(T) = \hat{r} - \alpha_T'\lambda \end{aligned} \tag{3.11}$$

and notice that $\dot{g}(t)$ is a linear homogeneous ordinary differential equation so we can use its state-transition matrix to express both $g(t)$ and $r(t)$ as:

$$\begin{aligned}
g(t) \quad &= \Phi'(T,t)g(T) = \Phi'(T,t)(\hat{g} + C'\lambda) = \Phi'(T,t)\hat{g} + \Phi'(T,t)C'\lambda \\
r(t) \quad &= \hat{r} - \alpha_T'\lambda - \textstyle\int_t^T g'(\tau)B(\tau)K(\tau)^{-1}B'(\tau)g(\tau)d\tau \\
&= \hat{r} - \alpha_T'\lambda - \textstyle\int_t^T g'(T)\Phi(T,\tau)B(\tau)K(\tau)^{-1}B'(\tau)\Phi'(T,\tau)g(T)d\tau \\
&= \hat{r} - \alpha_T'\lambda - g'(T)[\textstyle\int_t^T \Phi(T,\tau)B(\tau)K(\tau)^{-1}B'(\tau)\Phi'(T,\tau)d\tau]g(T) \\
&\Psi(T,t) = 2\textstyle\int_t^T \Phi(T,\tau)B(\tau)K(\tau)^{-1}B'(\tau)\Phi'(T,\tau)d\tau \\
r(t) \quad &= \hat{r} - \alpha_T'\lambda - \tfrac{1}{2}g'(T)\Psi(T,t)g(T) \\
&= \hat{r} - \alpha_T'\lambda - \tfrac{1}{2}(\hat{g} + C'\lambda)'\Psi(T,t)(\hat{g} + C'\lambda).
\end{aligned}$$

Note that the state-transition matrix is given by the expression

$$\Phi(T,t) = e^{\int_t^T (A'(s)g(s) - P'(s)B(s)K(s)^{-1}B'(s))ds}.$$

Inserting these into the value function (2.12) which is equal to the value function of this problem we get

$$V(s,x(s)) = \frac{1}{2}x'(s)P(s)x(s) + x'(s)\Phi'(T,s)(\hat{g} + C'\lambda) + \hat{r} - \alpha_T'\lambda - \frac{1}{2}(\hat{g} + C'\lambda)'\Psi(T,s)(\hat{g} + C'\lambda). \tag{3.12}$$

Now we can optimize λ from (3.12)

$$\begin{aligned}
0 &= \partial_\lambda(x'(s)\Phi'(T,s)C'\lambda - \alpha_T'\lambda - \hat{g}'\Psi(T,s)C'\lambda - \frac{1}{2}\lambda'C\Psi(T,s)C'\lambda) \\
&= C\Phi(T,s)x(s) - \alpha_T - C\Psi(T,s)\hat{g} - \frac{1}{2}(C\Psi(T,s)C' + C\Psi'(T,s)C')\lambda
\end{aligned}$$

to get the a suitable λ for the lagrange relaxation

$$\bar{\lambda} = (C\Psi(T,s)C')^{-1}(C(\Phi(T,s)y - \Psi(T,s)\hat{g}) - \alpha_T). \tag{3.13}$$

The optimal control derived from 2.13 in terms of $\bar{\lambda}$ is

$$\bar{u}(t) = -K(t)^{-1}B'(t)[P(t)x(t) + \Phi'(T,t)(\hat{g} + C'(C\Psi(T,s)C')^{-1}(C(\Phi(T,s)y - \Psi(T,s)\hat{g}) - \alpha_T))] \tag{3.14}$$

meaning that we now have an optimal control for a problem defined on a time interval where both end points are subject to constraints. In conclusion, we state above discussion as a theorem.

Theorem 4. *The optimal control for Problem 8 is given by* (3.14). □

In case of multiperiod constraints, we apply the above theorem on each $[t_k, t_{k+1}]$. Just as before we find the Riccati equations:

$$\begin{array}{ll} \dot{P}_k(t) = -A'(t)P_k(t) - P_k(t)A(t) + P_k'(t)B(t)K_k(t)^{-1}B'(t)P_k(t) - M(t) & , \ P_k(T_k) = \hat{P}_k \\ \dot{g_k}(t) = -A'(t)g_k(t) + P_k'(t)B(t)K_k(t)^{-1}B'(t)g_k(t) & , \ g_k(T_k) = \hat{g}_k + C_k'\lambda_k \\ \dot{r_k}(t) = \frac{1}{2}g_k'(t)B(t)K_k(t)^{-1}B'(t)g_k(t) & , \ r_k(T_k) = \hat{r}_k - \alpha_k'\lambda_k \end{array}$$

where

$$\hat{P}_k = P_{k+1}(t_{k+1}) \ , \hat{g}_k = g_{k+1}(t_{k+1}) \ , \hat{r}_k = r_{k+1}(t_{k+1})$$

if $k \neq n$ and

$$\hat{P}_n = \hat{P} \ , \hat{g}_n = \hat{g} \ , \hat{r}_n = \hat{r}.$$

The optimal λ_k from the value function is

$$\bar{\lambda}_k = (C_k\Psi(T_k, t_k)C_k')^{-1}(C_k(\Phi(T_k, t_k)\bar{x}(t_k) - \Psi(T_k, t_k)\hat{g}_k) - \alpha_k) \tag{3.15}$$

and the optimal control depending on $\bar{\lambda}_k$ is

$$\begin{aligned} \bar{u}_k(t) = &-K_k(t)^{-1}B'(t)[P_k(t)\bar{x}(t) + \\ &\Phi'(T_k, t_k)(\hat{g}_k + C_k'(C_k\Psi(T_k, t_k)C_k')^{-1}(C_k(\Phi(T_k, t_k)\bar{x}(t_k) - \Psi(T_k, t_k)\hat{g}_k) - \alpha_k))]. \end{aligned} \tag{3.16}$$

When solving a problem with multiperiod constraints one is to insert the optimal control $\bar{u}_k(\cdot)$ into the dynamics (2.9) during the time interval $[t_k, t_{k+1}]$. When the trajectory has passed this interval into the next $[t_{k+1}, t_{k+2}]$ simply use the next control $\bar{u}_{k+1}(\cdot)$ and the trajectory will keep its optimality property.

4. Financial Preliminaries

4.1. **Market Model.**

Let $\mathcal{M}$ denote a financial market consisting of $n+1$ assets that are traded continuously. The first of these is a riskless asset called a bond whose price process has the following equation

$$\begin{cases} dB(t) & = a(t)B(t)dt \\ B(s) & = 1 \end{cases} \tag{4.1}$$

where $a(t)$ can be interpreted as the interest rate of the bond.
The remaining n assest are subject ro risk, we shall call them stocks and model asset i with the linear stochastic differential equation

$$\begin{cases} dP_i(t) & = P_i(t)[\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)] \\ P_i(s) & = p_i. \end{cases} \tag{4.2}$$

$\mathcal{M}$'s uncertainty is modelled by the independent components of a n-dimentional Brownian motion $W(t)$ meaning that the market that we are dealing with is complete. This gives the coefficient $\sigma_{ij}(t)$ of (4.2) the financial interpretation of the

instantaneous intensity with which the Brownian motion influences the stock. The coefficient $\mu_i(t)$ becomes then the expected rate of return of the stock.
Some assumptions made on the coefficients $a(t)$, $\mu(t)$ and $\sigma(t)$ are that $\sigma(t)$ be invertible $\forall\, t$, that they should be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and satisfy the condition

$$\int_s^T (\mid a(t) \mid + \mid\mid \mu(t) \mid\mid + \mid\mid \sigma(t) \mid\mid^2) dt < \infty, \text{a.s.}$$

These assumption preclude anticipation of the future and allow for dependence on the past of the driving brownian motion.

Let $x(t)$ denote the wealth of a portfolio consisting of n stocks and the bond at time t. Call this the wealth-process and model it with

$$x(t) = u_0(t)B(t) + \sum_{i=1}^{n} u_i(t)P_i(t)$$

where $u_i(t)$ are the number of the ith asset held in the portfolio. In differential form this is equal to

$$\left\{ \begin{array}{ll} dx(t) & = [a(t)x(t) + b(t)u(t)]dt + u'(t)\sigma(t)dW(t) \quad\quad t \in [s, T] \\ x(s) & = y \end{array} \right. \tag{4.3}$$

where $b(t) = \mu'(t) - a(t)$ and y is the initial capital invested in the portfolio.

It should me noted that the portfolios must always be held by small investors, by this we mean that the decisions he/she makes will not influence the market.

4.2. **Portfolio Optimization.**

The mean-variance portfolio optimization problem can be stated as a multi-objective optimization problem with two criteria in conflict as show in [6]. From a control theoretic perspective we can denote the multi-objective cost function

$$(J_1(s, y; u(\cdot), J_2(s, y; u(\cdot)) = (-\mathbb{E}\{x(T)\}, \mathbb{V}\{x(T)\}) \tag{4.4}$$

and formulate the mean-variance portfolio selection problem as

Problem 10 (Mean-Variance Optimization)**.** *Minimize* (4.4) *subject to* (4.3)*.*

Using the auxiliary method from [6] it can be shown that the solution of the following problem is equal to the solution of problem 10. Let the cost function be

$$J(s, y; u(\cdot) = \mathbb{E}\{\hat{\mu}x(T)^2 - \eta x(T)\} \tag{4.5}$$

and

Problem 11 (LQ Mean-Variance Optimization)**.** *Minimize* (4.5) *subject to* (4.3)*.*

With the following variable substitutions $A(t) = a(t)$, $B(t) = b(t) = \hat{b}'(t) - a(t)$, $C = 1$, $f(t) = 0$, $M(t) = N(t) = \hat{r} = 0$, $\hat{P} = \mu = 2\hat{\mu}$ and $\hat{g} = -\eta$ we see that (4.5) is a special case of (2.10). So we proceed in the normal fashion and derive the solution from LQ control theory.

From (2.14) the riccati equations to this problem become

$$\begin{array}{lll} \dot{P}(t) & = (\rho(t) - 2a(t))P(t) & , P(T) = \mu \\ \dot{g}(t) & = (\rho(t) - a(t))g(t) & , g(T) = -\eta \\ \dot{r}(t) & = \frac{1}{2}\rho(t)\frac{g(t)^2}{P(t)} & , r(T) = 0 \end{array}$$

where $\rho(t) = b(t)(\sigma(t)\sigma'(t))^{-1}b'(t)$. Since they are homogeneous linear ordinary differential equations we can solve them and get

$$\begin{array}{ll} P(t) & = \mu e^{-\int_t^T (\rho(\tau) - 2a(\tau))d\tau} \\ g(t) & = -\eta e^{-\int_t^T (\rho(\tau) - a(\tau))d\tau}. \end{array}$$

Substitution of $P(t)$ and $g(t)$ into (2.13) gives us the optimal control:

$$\begin{array}{ll} \bar{u}(t) & = -(\sigma(t)\sigma'(t))^{-1}b'(t)(\bar{x}(t) + \frac{g(t)}{P(t)}) \\ h(t) & \triangleq \frac{g(t)}{P(t)} = -\frac{\eta}{\mu}e^{-\int_t^T a(\tau)d\tau} \\ \gamma & \triangleq \frac{\eta}{\mu} \end{array}$$

$$\bar{u}(t) = -(\sigma(t)\sigma'(t))^{-1}b'(t)(\bar{x}(t) - \gamma e^{-\int_t^T a(\tau)d\tau}). \tag{4.6}$$

Lets proceed now with the derivation of the efficiet frontier for the mean-variance problem 11. The wealth process (4.3) under the optimal control (4.6) follows the equation

$$\left\{ \begin{array}{rl} d\bar{x}(t) = & [(a(t) - \rho(t))\bar{x}(t) + \gamma\rho(t)e^{-\int_t^T a(s)ds}]dt \\ & +b(t)\sigma(t)^{-1}(\gamma e^{-\int_t^T a(s)ds} - \bar{x}(t))]dW_t \\ \bar{x}(s) = & y. \end{array} \right. \tag{4.7}$$

Moreover, using the Ito's formula on $x(t)^2$ gives

$$\left\{ \begin{array}{rl} d\bar{x}(t)^2 = & [(2a(t) - \rho(t))\bar{x}(t)^2 + \gamma^2\rho(t)e^{-2\int_t^T a(s)ds}]dt \\ & +2\bar{x}(t)b(t)\sigma(t)^{-1}(\gamma e^{-\int_t^T a(s)ds} - \bar{x}(t))]dW_t \\ \bar{x}^2(s) = & y^2. \end{array} \right. \tag{4.8}$$

Taking the expectation of both (4.7) and (4.8) results in the nonhomogeneous linear ordinary differential equations

$$\left\{ \begin{array}{rl} d\mathbb{E}\{\bar{x}(t)\} = & [(a(t) - \rho(t))\mathbb{E}\{\bar{x}(t)\} + \gamma\rho(t)e^{-\int_t^T a(s)ds}]dt \\ \mathbb{E}\bar{x}(s) = & y \end{array} \right. \tag{4.9}$$

and

$$(4.10)\qquad \begin{cases} d\mathbb{E}\{\bar{x}(t)^2\} = & [(2a(t)-\rho(t))\mathbb{E}\{\bar{x}(t)^2\} + \gamma^2\rho(t)e^{-2\int_t^T a(s)ds}]dt \\ \mathbb{E}\bar{x}^2(s) = & y^2. \end{cases}$$

Solving these we get

$$\mathbb{E}\{\bar{x}(T)\} = \kappa y + \beta\gamma = \alpha_T$$
$$\mathbb{E}\{\bar{x}(T)^2\} = \delta y^2 + \beta\gamma^2$$

where

$$\kappa \triangleq e^{\int_s^T (a(t)-\rho(t))dt}$$
$$\beta \triangleq 1 - e^{-\int_s^T \rho(t)dt}$$
$$\delta \triangleq e^{\int_s^T (2a(t)-\rho(t))dt}$$

$$(4.11)\qquad \gamma = \frac{\mathbb{E}\{\bar{x}(T)\} - \kappa y}{\beta}.$$

By finding an expression of γ we have now completed the solution of problem 11. The interseting property of γ is that it remains constant over the whole investment horizon. We also notice that the only arbitrary components that γ depends on are the initial wealth y and the terminal expected wealth $\mathbb{E}\{\bar{x}(T)\}$.
The well known equality $\mathbb{V}\{\bar{x}(T)\} = \mathbb{E}\{\bar{x}(T)^2\} - \mathbb{E}\{\bar{x}(T)\}^2$ is crucial for the formulation of the main financial problem studied in this paper. The following calculation gives us the possibility of expressing the variance of the wealth process in forms of an expected value combined with the given parameters $y, a(t)$ and $\rho(t)$.

$$\begin{aligned} \mathbb{V}\{\bar{x}(T)\} &= \mathbb{E}\{\bar{x}(T)^2\} - \mathbb{E}\{\bar{x}(T)\}^2 \\ &= \beta(1-\beta)\gamma^2 - 2\kappa\beta\gamma y + (\delta - \kappa^2)\alpha_s^2 \\ &= \frac{e^{-\int_s^T \rho(t)dt}}{1 - e^{-\int_s^T \rho(t)dt}}(E\{x(T)\} - ye^{\int_s^T a(t)dt})^2 \end{aligned}$$

$$(4.12)\qquad \mathbb{E}\{\bar{x}(T)\} = ye^{\int_s^T a(t)dt} + \sqrt{\frac{1 - e^{\int_s^T \rho(t)dt}}{e^{-\int_s^T \rho(t)dt}}\mathbb{V}\{\bar{x}(T)\}}.$$

5. Financial Problem Formulation

Consider a small investor with an initial amount of money which he/she wishes to invest on the market. After specifying an investment horizon say, that the he/she wants to specify different risk levels on the portfolio during different period.

An example could be a pension plan where the capital is invested on the market. One idea could be to minimize the risk as the person gets older. One way to do so is to give different risk levels for each five year period of the persons life after a certain age. Say that during the ages 50-55 the risk level should be 5 while during the ages 55-60 the risk level should be 4 and so on.

Another example could be a company that has much of its capital invested on the market. Suppose that the share holders are very concerned with the companies budget. Then the company might want to minimize its risk exposure a number of times before the yearly report is due.

If the investor measures the risk exposure of his/her portfilio with the variance this problem can be formulated in the following way. Given the wealth process of the portfolio

$$\begin{cases} dx(t) &= [a(t)x(t) + b(t)u(t)]dt + u'(t)\sigma(t)dW(t) \quad t \in [s,T] \\ x(s) &= y \\ \mathbb{V}\{x(T_k)\} &= \vartheta_k \quad k \in \{1,2,..,n\} \end{cases} \tag{5.1}$$

$$0 < s = t_1 < \cdots < t_k < t_{k+1} \cdots < t_N = T.$$

Problem 12. *Minimize* (4.5) *subject to* (5.1).

Note that the constraints $\mathbb{V}\{x(T_k)\} = \vartheta_k$ can be reformulated using (4.12) to $\mathbb{E}\{x(T_k)\} = \alpha_k$ giving us the usual LQ multiconstraint form.

5.1. **Financial Solutions.**

5.1.1. *Boundary Constraints.*

The following results will be stated in a very short and simple manner since they are derived directly from the LQ boundary constraint theory. Let the dynamics be the same as in the previous section with an additional constraint at the end point

$$\begin{cases} dx(t) &= [a(t)x(t) + b(t)u(t)]dt + u'(t)\sigma(t)dW(t) \quad t \in [s,T] \\ x(s) &= y \\ \mathbb{E}\{x(T)\} &= \alpha_T \end{cases} \tag{5.2}$$

and let the problem be

Problem 13. *Minimize* (4.5) *subject to* (5.2).

We derive the riccati equations from (3.11)

$$\begin{aligned} \dot{P}(t) &= (\rho(t) - 2a(t))P(t) \quad &, P(T) = \mu \\ \dot{g}(t) &= (\rho(t) - a(t))g(t) \quad &, g(T) = \lambda - \eta \\ \dot{r}(t) &= \tfrac{1}{2}\rho(t)\frac{g(t)^2}{P(t)} \quad &, r(T) = -\lambda\alpha_T. \end{aligned}$$

The solutions to these equations are

$$\begin{aligned} P(t) &= \mu e^{-\int_t^T (\rho(\tau)-2a(\tau))d\tau} \\ g(t) &= (\lambda-\eta)\Phi(T,t) \\ r(t) &= -\lambda\alpha_T - (\tfrac{1}{2}\lambda^2 - \lambda\eta + \tfrac{1}{2}\eta^2)\Psi(T,t) \end{aligned}$$

where

$$\Phi(T,t) = e^{-\int_t^T (\rho(\tau)-a(\tau))d\tau}$$

$$\Psi(T,t) = \int_t^T \rho(\tau)\frac{\Phi(T,\tau)^2}{P(\tau)}d\tau.$$

From (3.13) the optimal λ becomes

$$\bar{\lambda} = \frac{\Phi(T,s)y - \alpha_T}{\Psi(T,s)} + \eta \tag{5.3}$$

and finally we arrive at the optimal control

$$\begin{aligned} \bar{u}(t) &= -(\sigma(t)\sigma'(t))^{-1}b'(t)(\bar{x}(t) + \frac{g(t)}{P(t)}) \\ h(t) &\triangleq \frac{g(t)}{P(t)} \\ &= \frac{\bar{\lambda}-\eta}{\mu}e^{-\int_t^T a(\tau)d\tau} \\ &= \frac{\Phi(T,s)y - \alpha_T}{\Psi(T,s)\mu}e^{-\int_t^T a(\tau)d\tau} \\ \gamma &\triangleq -\frac{\Phi(T,s)y - \alpha_T}{\Psi(T,s)\mu} \end{aligned}$$

$$\bar{u}(t) = -(\sigma(t)\sigma'(t))^{-1}b'(t)(\bar{x}(t) - \gamma e^{-\int_t^T a(\tau)d\tau}). \tag{5.4}$$

The γ in this last equation can be computed using the expression from (4.11). Notice that both the γs from the initial and boundary constraint problem are equal. This is natural since the contextual interpretion of both problem is the same.

5.1.2. *Multiperiod Constraints.*
We have now arrived at the final and main result from a financial perspective of this paper. Fortunately all of the necessery calculation have been done in the previous sections. So the only thing we need to keep in mind is that we have to separate the time intervals on which the constraints are defined and simply apply the results from boundary constraint section.

For each period $t \in [t_k, T_k]$ we have:

Riccati Equations:

$$\begin{array}{lll} \dot{P}_k(t) & = (\rho(t) - 2a(t))P_k(t) & , P_k(T_k) = \hat{P}_k \\ \dot{g}_k(t) & = (\rho(t) - a(t))g_k(t) & , g_k(T_k) = \hat{g}_k + \lambda_k \\ \dot{r}_k(t) & = \frac{1}{2}\rho(t)\frac{g_k(t)^2}{P_k(t)} & , r_k(T_k) = \hat{r}_k - \alpha_k\lambda_k \end{array}$$

where

$$\hat{P}_k = P_{k+1}(t_{k+1}) \ , \hat{g}_k = g_{k+1}(t_{k+1}) \ , \hat{r}_k = r_{k+1}(t_{k+1})$$

if $k \neq n$ and

$$\hat{P}_n = \mu \ , \hat{g}_n = -\eta \ , \hat{r}_n = 0.$$

Solutions to the riccati equations:

$$\begin{array}{ll} P_k(t) & = \hat{P}_k e^{-\int_t^{T_k}(\rho(\tau)-2a(\tau))d\tau} \\ g_k(t) & = (\hat{g}_k + \lambda_k)e^{-\int_t^{T_k}(\rho(\tau)-a(\tau))d\tau} \end{array}$$

where

$$\Phi(T_k, t_k) = e^{-\int_{t_k}^{T_k}(\rho(\tau)-a(\tau))d\tau}$$

$$\Psi(T_k, t_k) = \int_{t_k}^{T_k} \rho(\tau)\frac{\Phi(T_k,\tau)^2}{P_k(\tau)}d\tau.$$

The optimal λ_k

(5.5) $$\bar{\lambda}_k = \frac{\Phi(T_k, t_k)x(t_k) - \alpha_k}{\Psi(T_k, t_k)} - \hat{g}_k$$

and there by we obtain the optimal control

$$\bar{u}_k(t) = -(\sigma(t)\sigma'(t))^{-1}b'(t)(\bar{x}(t) + \frac{g_k(t)}{P_k(t)})$$

$$\begin{aligned} h_k(t) &\triangleq \frac{g_k(t)}{P_k(t)} \\ &= \frac{\hat{g}_k + \bar{\lambda}_k}{\hat{P}_k}e^{-\int_{t_k}^{T_k} a(\tau)d\tau} \\ &= \frac{\Phi(T_k, t_k)x(t_k) - \alpha_k}{\Psi(T_k, t_k)\hat{P}_k}e^{-\int_{t_k}^{T_k} a(\tau)d\tau} \end{aligned}$$

$$\gamma_k \triangleq -\frac{\Phi(T_k, t_k)x(t_k) - \alpha_k}{\Psi(T_k, t_k)\hat{P}_k}$$

(5.6) $$\bar{u}_k(t) = -(\sigma(t)\sigma'(t))^{-1}b'(t)(\bar{x}(t) - \gamma_k e^{-\int_{t_k}^{T_k} a(\tau)d\tau}).$$

From the efficient frontier of the initial wealth portfolio optimization section we derive γ_k from (4.11)

$$\gamma_k = \frac{\alpha_k - \kappa_k x(t_k)}{\beta_k} \tag{5.7}$$

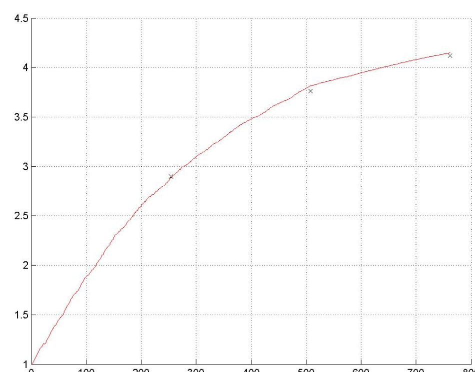
where

$$\kappa_k \triangleq e^{\int_{t_k}^{T_k}(a(t)-\rho(t))dt}$$

$$\beta_k \triangleq 1 - e^{-\int_{t_k}^{T_k}\rho(t)dt}.$$

Hence we have arrived at an analytic expression for the portfolio controls of each period.

5.2. **An example.** Lets consider an example where an investor decides he/she wants to invest on two assets. The first asset is a bond with a nominal anual interest rate of 1 percent. The second asset is a stock with a nominal appreciation rate of 12 percent and a standard deviation of 15 percent. The investor seeks an investment strategy consisting of both assets where he/she is able to specify three different risk levels. The first,second and third risk levels have standard deviations of 500,100,10 percent respectively. These risk levels are absurdly aggressive but give a good illustration of the interpolation of the portfolio with the different risklevels. The figure shows the expected trajectory of the portfolio through the different riskleves in 2000 simulations.



When the initial capital is 1 million dollars the investor needs to invest 21.95 million on the stock for the first risk level, meaning that the investor needs to short the bond for an amount of 20.95 million and invest on the stock. For the second and third risk levels the investor needs to rebalance his portfolio so that 9.82 and

3.1 million dollars respectively are invested on the stock.

6. Conclusion

This paper studies stochastic optimal control problems with multiperiod constraints from a dynamic programming perspective. First the general problem is formulated, investigated and some results on the separation into subproblems using lagrange multipliers are presented. Then, a linear quadratic formulation of the general case is solved by deriving an explicit and analytic expression for the control based on a pair of ordinary differential equations called the Riccati equations.

A financial portfolio optimization problem with different risk levels for their respective time periods is formulated and solved using the results of the linear quadratic case in this paper. The solution is analytic and is based on a pair of simple and completely solvable linear and homogeneous ordinary differential equations.

Finally an example is presented where an investor seeks the optimal trading strategy of investing capital on both a bond and a stock. In this example the investor also wants be able to specify different risklevels for different time periods. A figure representing the expected trajectory of the investors portfolio is shown, giving some what of an intuitive feeling of the results presented in this paper.

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