Math 772 Topics in Financial Math

Project

Dr. Grasselli

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Mengmeng Ma

Student ID: 0140076

The Cox, Ingersoll, and Ross Model

The CIR model is an equilibrium asset pricing model for the term structure of interest rate. It is defined under the assumption:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}\,dZt$$

where k, θ, σ are constants. k represents the rates of mean reversion, θ represents the long run average. Assuming $2k\theta \geq \sigma^2$, then the interest rate is ensured to be positive ($2k\theta < \sigma^2$, it is possible for r to reach zero instantaneously, but it will never become negative).

Under the equivalent risk-neutral measure, it be comes

$$dr_t = (k\theta - (k+\lambda)r_t)dt + \sigma\sqrt{r_t}\,dZt$$

where λ represents the risk adjustments when moving to the risk-neutral distribution.

The CIR model is affine, and has a non-central chi-squared distribution (with degree of freedom $v = \dfrac{4k\theta}{\sigma^2}$). By the Ricatti equations, the nominal risk-free bond price P(t,T) with payment of $1 at time T is given by

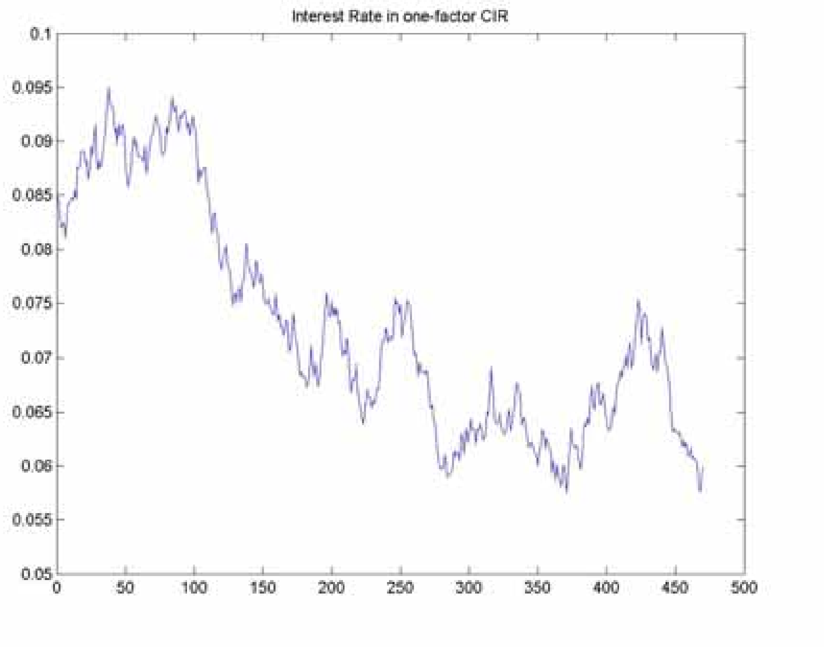
$$P(t,T) = E\left[\exp\left(-\int_t^T r_s ds\right) | \Im t\right] = A(t,T)\exp[(-B(t,T)x_t]$$

Where

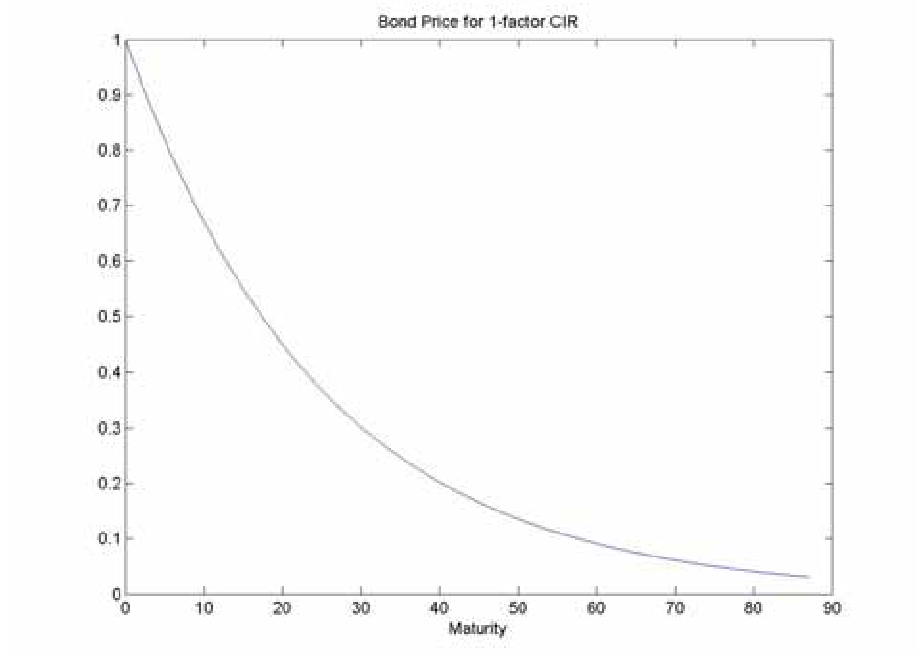
$$A(t,T) = \left[\frac{2\gamma e^{\frac{1}{2}(k+\lambda-\gamma)(T-t)}}{2\gamma e^{-\gamma(T-t)} + (k+\lambda+\gamma)(1-e^{-\gamma(T-t)})}\right]^{\frac{2k\theta}{\sigma^2}}$$

$$B(t,T) = \frac{2(1-e^{-\gamma(T-t)})}{2\gamma e^{-\gamma(T-t)} + (k+\lambda+\gamma)(1-e^{-\gamma(T-t)})}$$

$$\gamma = \sqrt{(k+\lambda)^2 + 2\sigma^2}$$



*all the interest rates are positive.



The continuously compounded yield is defined as:

$$Y(t,T) = \frac{-\ln(P(t,T))}{T-t}$$

As $T \to \infty$, it can be shown that

$$Y(t,\infty) \to \frac{2k\theta}{k+\lambda+\gamma}$$

Which is independent of r. If $r < \frac{2k\theta}{k+\lambda+\gamma}$, the term structure is upward sloping (figure 1); if $r > \frac{k\theta}{k+\lambda+\gamma}$, it is downward sloping (figure 2); $\frac{k\theta}{k+\lambda+\gamma} < r < \frac{2k\theta}{k+\lambda+\gamma}$, it will be upward first then downward sloping (figure 3).

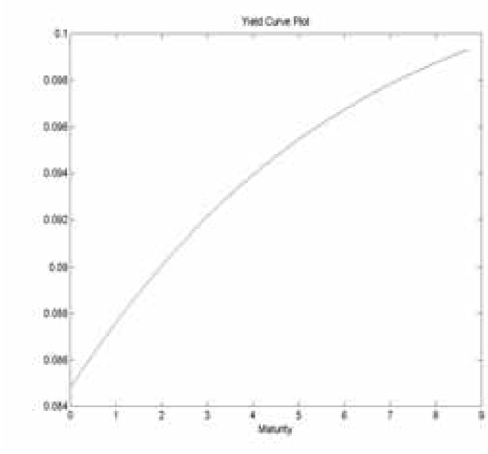


Figure 1

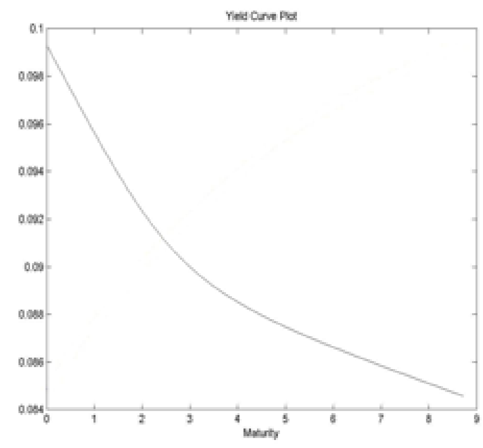


Figure 2

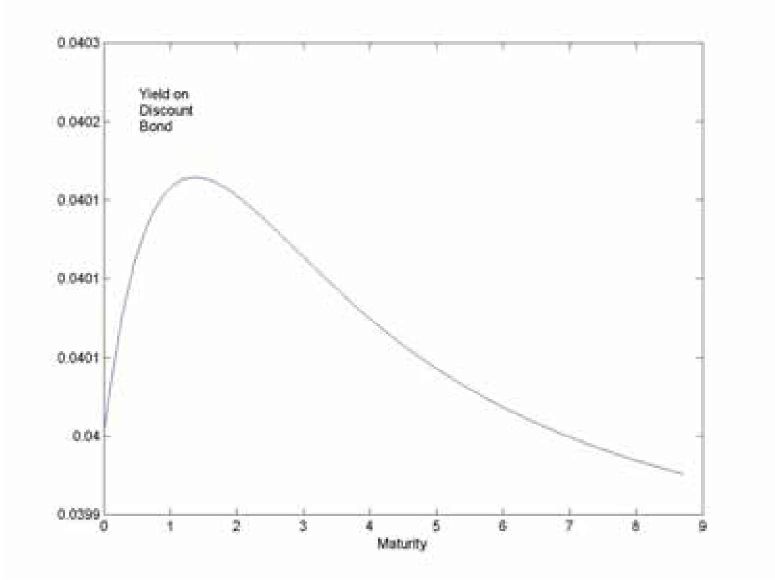


Figure 3

MULTIFACTOR CIR MODEL

The CIR model is frequently presented as a one-factor model; in Cox, Ingersoll, and Ross(1985), they showed how to extend the multiple factors. The multifactor CIR model is set up under the assumptions:

$$dx_{jt} = k_j(\theta_j - x_{jt})dt + \sigma_j \sqrt{x_{jt}} dZjt\ , \quad j = 1, \ldots, n$$

$x_j, j = 1, \ldots, n$ are independent, and Brownian motions Zjt are mutually independent. Also here we assume $2k\theta \geq \sigma^2$ to ensure the positivity. Similarly, under the risk-neutral measure, the model becomes:

$$dx_{jt} = (k_j\theta_j - (k_j + \lambda_j)x_{jt})dt + \sigma_j \sqrt{x_{jt}} dZjt \quad for \quad j = 1, \ldots, n$$

Here we assumed the market price of risk λ_j as a fixed parameter. The instantaneous default-free short interest rate and default intensity are assumed to be the positive linear combinations of *n* factors x_{jt} under weights w_j and $\overline{w_j}$

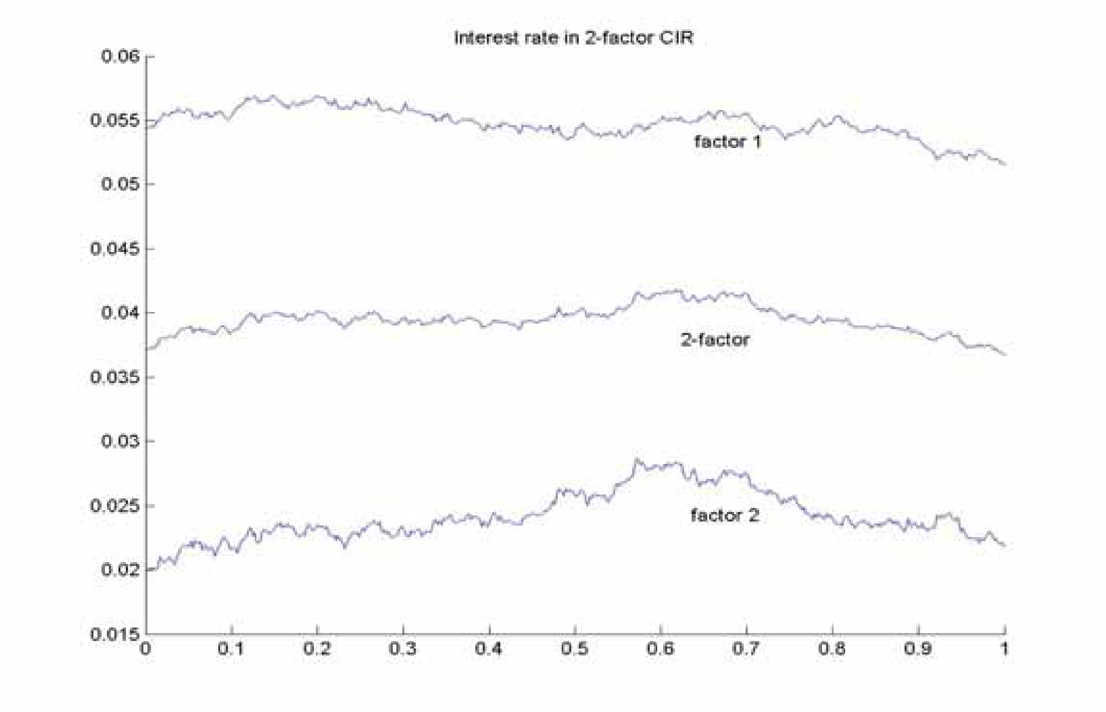
$$r = \sum_{j=1}^{n} w_j x_j$$

$$\lambda = \sum_{j=1}^{n} \overline{w_j} x_j$$

(follows from historical data in Chen, Scott 1995):

Estimates from Weekly Data, 1980-1988, T=470

	k	θ	σ	λ
Factor 1	0.13974	0.08480	0.10001	-0.07132
Factor 2	0.7298	0.04013	0.16885	-0.01731



The nominal risk-free bond price P(t,T) with payment of $1 at time T is given by

$$P(t,T) = E\left[\exp\left(-\int_t^T r_s ds\right) \mid \Im t\right] = E\left[\exp\left(-\int_t^T \sum_j^n w_j x_{js} ds\right) \mid \Im t\right] = E\left[\exp\left(-\sum_j^n \int_t^T w_j x_{js} ds\right) \mid \Im t\right]$$

$$= E\left[\prod_j^n \exp\left(-\int_t^T w_j x_{js} ds\right) \mid \Im t\right] = \prod_j^n E\left[\exp\left(-\int_t^T w_j x_{js} ds\right) \mid \Im t\right]$$
$$= A_1(t,T,c_1)\cdots A_n(t,T,c_n)\exp[(-B_1(t,T,c_1)w_1x_{1t}\cdots(-B_n(t,T,c_n)w_nx_{nt}]$$

Since the factors are independent by the assumption, and $A_j(t,T)$ and $B_j(t,T)$ have the forms given by

$$A_j(t,T,c) = \left[\frac{2\gamma_j e^{\frac{1}{2}(k_j+\lambda_j-\gamma_j)(T-t)}}{2\gamma_j e^{-\gamma_j(T-t)} + (k_j+\lambda_j+\gamma_j)(1-e^{-\gamma_j(T-t)})}\right]^{\frac{2k_j\theta_j}{\sigma_j^2}}$$

$$B_j(t,T,c) = \frac{2(1-e^{-\gamma_j(T-t)})}{2\gamma_j e^{-\gamma_j(T-t)} + (k_j+\lambda_j+\gamma_j)(1-e^{-\gamma_j(T-t)})}$$

and $\gamma_j = \sqrt{(k_j + \lambda_j)^2 + 2c_j \sigma_j{}^2}$

Then the defaultable bond prices under recovery rate q are given by:

$$\overline{P}(t,T) = E[\exp(-\int_t^T r(s) + q\lambda(s)ds) \mid \Im t]$$
$$= A_1(t,T,w_1 + q\overline{w_1}) \cdots A_n(t,T,w_n + q\overline{w}_n)$$
$$\exp[(-B_1(t,T,w_1 + q\overline{w_1})(w_1 + q\overline{w_1})x_{1t} \cdots (-B_n(t,T,w_n + q\overline{w_n})(w_n + q\overline{w_n})x_{nt}]$$

The continuous compounded yield for the bond follows by:

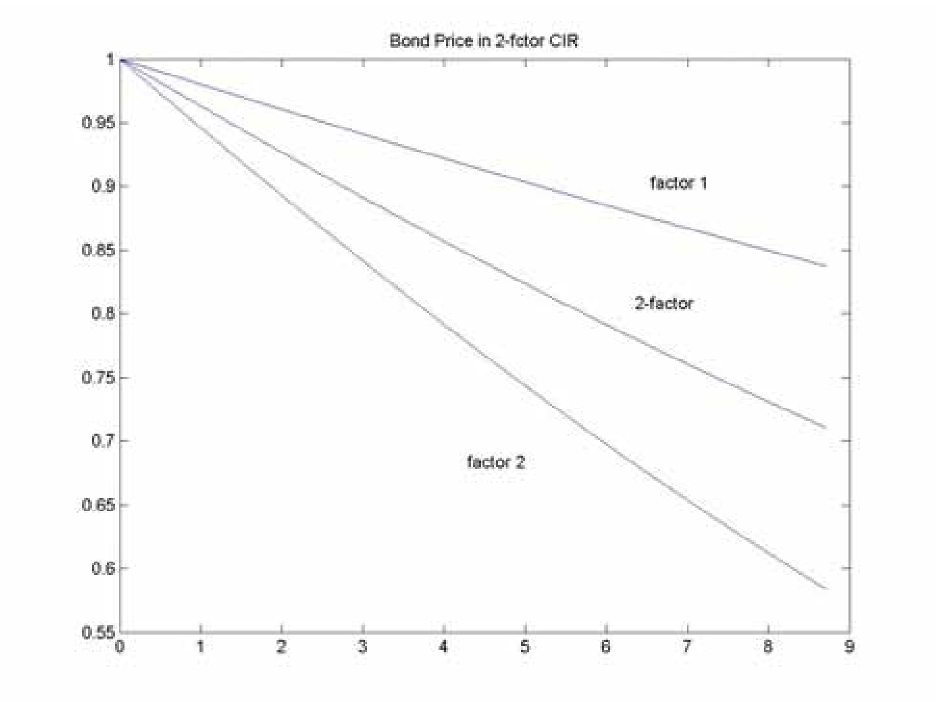
$$Y_j(t,T) = \frac{-\ln(P_j(t,T))}{T-t}$$

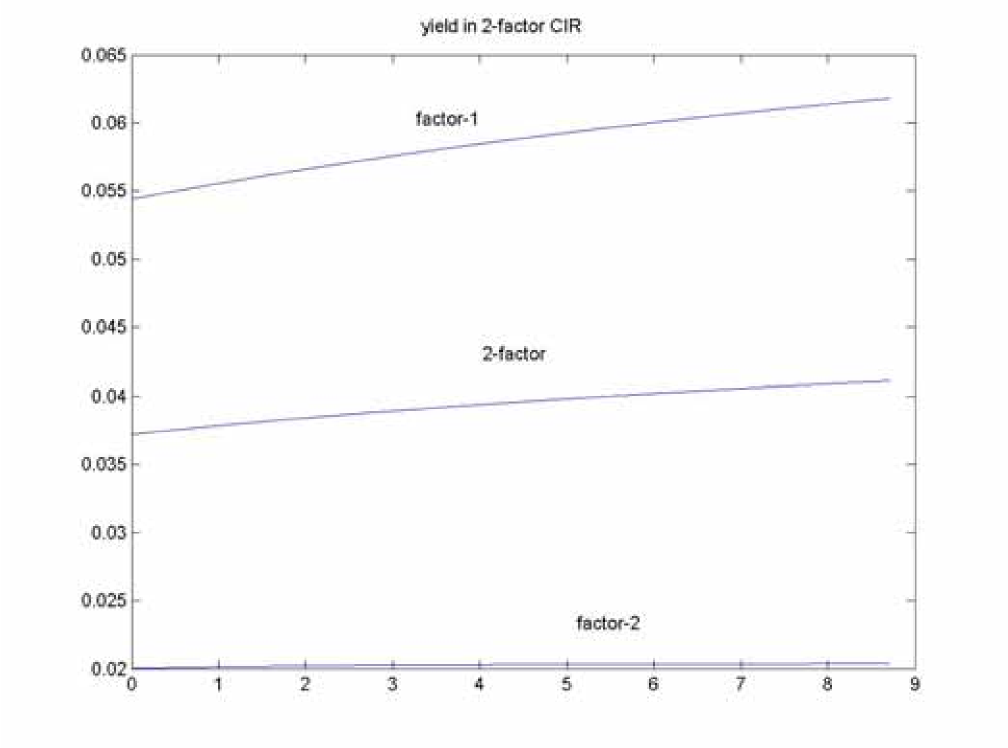
$$= -\frac{\ln[A_1(t,T,w_1)\cdots A_K(t,T,w_n)] - B_1(t,T)w_1 x_{1t} \cdots - B_K(t,T)w_n x_{Kt}}{T-t}$$

$$= \frac{\sum_{j=1}^{n} -\ln A_j(t,T,w_j) + B_j(t,T)w_j x_{jt}}{T-t}$$

We can observe that the term structure turns out to be a linear function of the unobservable state variables.

Here we show the curve of two-factor CIR bond prices and yields





Default digital payoffs

The price for a default digital put with maturity T (protection leg of a default digital swap) in CIR model is:

$$D = \int_0^T E\left[\lambda(t) e^{-\int_0^t (\lambda(s)+r(s)ds)} \right] dt$$

$$= \int_0^T \left(\left(\sum_{i=1}^n \overline{w}_j (w_j + \overline{w}_j) \right) \left(k_j \theta_j B_j(0,t,w_j + \overline{w}_j) + \frac{\partial B(0,t,w_j + \overline{w}_j)}{\partial t} x_j(0) \right) \right) \prod_{j=1}^n \overline{M}_{j0}(0,t) dt$$

Where $\overline{M}_{j0}(0,t) = A_j(0,t,w_j + \overline{w}_j) \exp[-B_j(0,t,w_j + \overline{w}_j)(w_j + \overline{w}_j) x_j]$

Proof:

First, the equation in the expectation operator can be simplified:

$$\left[\lambda(t) e^{-\int_0^t (\lambda(s)+r(s)ds)} \right] = \left(\sum_i^n \overline{w_i} x_{it} \right) \exp\left[\sum_j^n - \int_=^t (w_j + \overline{w}_j) x_{js} ds \right]$$

$$= \sum_{i}^{n} \overline{w_i} x_{it} \exp\left[\sum_{j}^{n} - \int_{=}^{t} (w_j + \overline{w}_j) x_{js} ds\right]$$

Then for each fixed *i*,

$$E\left[\overline{w_i} x_{it} \exp\left[\sum_{j}^{n} - \int_{=}^{t} (w_j + \overline{w}_j) x_{js} ds\right]\right]$$
$$= E\left[\overline{w_i} x_{it} \exp\left(-\int_{=}^{t} (w_i + \overline{w}_i) x_{is} ds\right)\right] \times \prod_{j \neq i} E\left[-\int_{=}^{t} (w_j + \overline{w}_j) x_{js} ds\right]$$

by the independence of factors

$$= \overline{w_i} E\left[x_{it} \exp\left(-\int_{=}^{t} (w_i + \overline{w}_i) x_{is} ds\right)\right] \times \prod_{j \neq i} \overline{M}_{j0}(0, t)$$

Where $\overline{M}_{j0}(0,t)$ is defined as above.

Then we do the change of measure to $\tilde{P}_c$ whose restriction on $\Im_t$ is under the associated Radon-Nikodym density w.r.t *P*, and the process $\tilde{W}_t^c$ is a $\tilde{P}_c$-Brownian motion:

$$dWt = d\tilde{W}_t^c - B(t, T, c) c\sigma\sqrt{x} dt$$

We set $c = w_i + \overline{w}_i$, then

$$E\left[x_{it} \exp\left(-\int_{=}^{t} (w_i + \overline{w}_i) x_{is} ds\right)\right] = \overline{M}_{i0}(0, t) E^{\tilde{P}c}[x_i(t)]$$

On the other hand, under $\tilde{P}_c$, the CIR interest rate *x* has the dynamics:

$$dx = [k\theta - (k + B(t, T, c) c\sigma^2) x] dt + \sigma\sqrt{x} d\tilde{W}^c$$

And *x(T)* given *x(t)* is non-central chi-square random variable distributed under $\tilde{P}_c$ with weight:

$$\eta = \frac{c\sigma^2}{4} B(t, T, c)$$

Degrees of freedom:

$$v = \frac{4k\theta}{\sigma^2}$$

Non-centrality:

$$\tilde{\lambda} = \frac{4x(t)}{\sigma^2}\frac{\partial/\partial T[B(t,T,c)]}{B(t,T,c)}$$

Therefore, the expectation of $x_i(t)$ under $\tilde{P}_c$ is:

$$E^{\tilde{P}_c}[x_i(t)] = \eta_t(v_t + \tilde{\lambda}_t)$$

$$= \frac{c\sigma_i^2}{4}B_i(0,t,c)\left(\frac{4k_i\theta_i}{\sigma_i^2} + \frac{4\frac{\partial}{\partial T}B_i(0,t,c)}{B_i(0,t,c)}x_i(0)\right)$$

$$= ck_i\theta_i B_i(0,t,c) + c\frac{\partial}{\partial T}B_i(0,t,c)x_i(0)$$

$\Rightarrow$

$$E\left[\lambda(t)e^{-\int_0^t(\lambda(s)+r(s)ds)}\right] = \sum_i^n \overline{w}_i\overline{M}_{i0}(0,t)E^{\tilde{P}_c}[x_i(t)]\times\prod_{j\neq i}\overline{M}_{j0}(0,t)$$

$$= \sum_i^n \overline{w}_i E^{\tilde{P}_c}[x_i(t)]\times\prod_j\overline{M}_{j0}(0,t)$$

$$= \left(\sum_i^n \overline{w}_i\left(w_i + \overline{w}_i\right)\left(k_i\theta_i B_i(0,t,w_i+\overline{w}_i) + \frac{\partial}{\partial T}B_i(0,t,w_i+\overline{w}_i)x_i(0)\right)\right)\times\prod_j\overline{M}_{j0}(0,t)$$

And finally combining with the integral, the equation for the price for a default digital put is proved.

Numerical Estimation of Multifactor CIR Model

We estimate the CIR model by the discrete method in Chen, Scott (1995)

The individual interest rate are performed in a discrete representation by

$$x_{jt} = \theta_j(1-e^{-k_j\Delta t}) + e^{-k_j\Delta t}x_{j,t-1} + v_{jt},\ for\quad j = 1,\ldots,n$$

$$Y(t,x_i) = \frac{\sum_{j=1}^n -\ln A_j(t,T_i) + B_j(t,T_i)x_{kt}}{T_i - t}, i = 1,\ldots,N$$

where Δt represents the length of the time interval; v_{jt} represents the error term in x_{jt}, it has a conditional expectation zero and conditional variance which is:

$$\sigma_j^2\left(\frac{1-e^{-k_j\Delta t}}{k_j}\right)\left(\frac{1}{2}\theta_j(1-e^{-k_j\Delta t})+e^{-k_t\Delta t}x_{j,t-1}\right).$$

The expectation for x_{jt} with information till *(t-1)* is given by

$$E[x_{jt} \mid \Im_{t-1}] = \theta_j(1-e^{-k_j\Delta t}) + e^{-k_j\Delta t}x_{j,t-1};$$

Conversely, the estimation of unobservable variables can be computed by nonlinear Kalman filter.

***Reference**s:*

Chen, Ren-Raw, and Louis Scott (1995). *Multifactor Cox-Ingersoll-Ross Models of the Term Structure: Estimates and Tests from a Kalman Filter Model*:2-3

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