Pricing of Interest Rate Swaps in the Aftermath of the Financial Crisis

Martin S. B. Laursen Meik Bruhs

MSc Finance Thesis

Supervisor:
David Skovmand
Department of Business Studies

Aarhus School of Business,
Aarhus University

August 2011

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The thesis has been typed with Computer Modern 12pt
Layout and typography is made by the authors using LaTeX

The authors wish to thank the following: Our supervisor David Skovmand for assistance along the way and for always encouraging us to spend additional time re-thinking critical problems. Martin D. Linderstrøm from Danske Bank and Graeme West for helpful comments. Our families for supporting us at all times. Meik's gratitude goes especially to Svenja for her affection and encouragement in the past two years. Finally, we would like to thank a small group of people whom throughout our thesis writing process contributed to a humorous and dynamic working environment in the ASB library.

Abstract

During the financial crisis the 3M Libor-OIS spread peaked at around 360 basis points which had devastating consequences for pricing interest rate derivatives. Likewise, other indicators for distress in the financial markets such as tenor basis spreads, cross-currency basis spreads and the gap between FRA rates and their replicated forward rates rose to levels never seen before. Inspired by the article from Linderstrøm and Rasmussen (2011) in Finans Invest and motivated by the actuality and importance of the topic, the thesis examines a new framework for pricing interest rate swaps that correctly incorporates basis spreads.

In the first part of the thesis, the traditional bootstrapping approach will be revisited. Here, the construction of the spot curve involves several steps: selecting liquid market instruments, interpolating key spot rates and including turn of year effects. Then, discounting and forward curves are derived from the bootstrapped spot curve to price swaps indifferent of their underlying tenor. In the second part of the thesis a historical analysis of the aforementioned spreads is conducted. Here, it becomes clear that each tenor contains its own credit and liquidity premia. Moreover, the no-arbitrage condition and the widening of spreads is reasoned applying a qualitative approach.

Finally, the main part of the thesis covers the theoretical framework and the practical implementation of 3D forward surfaces that enable the consistent determination of overnight index, interest rate, tenor and cross-currency swaps (CCS). The latter is of special interest since it requires the determination of a foreign discount curve as well as a foreign forward surface under the assumption of no arbitrage. Here, the difference between pricing constant notional and mark-to-market CCS is examined thoroughly. Furthermore, collateralization is a main topic addressed in the thesis.

Comparing the pricing of swaps before and after the financial crisis, it can be concluded that basis spreads might have a significant impact on swap rates, depending on the length of the contract and the change in the underlying interest rate. Moreover, factors such as thresholds, one-way credit support annexes (CSA) and other options regarding the posting of collateral considerably complicate the pricing framework due to model dependent parameters and may lead to varying swap rates. Setting up such a system becomes quite demanding when complicated calibration is needed for even plain vanilla instruments as well as the challenges w.r.t. implementing a sufficient noise reduction technique to recover the observed swap prices within reasonable calculation time. Whilst the thesis focuses on pricing swaps, the complete setup is also relevant to price other derivatives that depend on future interest rates.

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1

Introduction

The consequences of the financial crisis were spread throughout financial markets into the private and public sector affecting economies worldwide. Inevitably, companies and banks were facing the challenge to overcome liquidity and credit problems which are highly linked to their capability of trading financial products. In continuation hereof, correctly pricing financial products must incorporate liquidity and credit premia to reflect their true values. Otherwise, basing decisions on incorrect assumptions might have devastating consequences.

Typically, companies and banks manage their interest rate risk in the market for interest rate derivatives by using instruments such as caps, floors, collars, swaptions, swaps etc. Increasing liquidity and credit premia have had significant influence especially in the swap market, why theorists and practitioners draw further attention to this issue. Basis spreads between different tenors and currencies are indicators for distress in the financial markets and have been significantly different from zero during and after the financial crisis. This requires a new common pricing framework where instruments across different tenors and currencies can be valued to par without inconsistencies, allowing no opportunities for arbitrage (Chibane and Sheldon, 2009, p.1).

Each tenor now incorporates its own liquidity and credit premia (Mercurio, 2009, p.4). Consequently, the pricing framework has changed from using *one forward curve* to an integrated *forward surface* consisting of several forward curves. Determining one single forward curve is no longer the question, now the challenge is to derive forward surfaces that account for quoted basis spreads. Moreover, the art of proper discounting must be revisited. The thesis strives to clarify an improved approach to price interest rate swaps using the most contemporary literature, supported by illustrations.

1.1 Literature review

The pricing of interest rate swaps was, before the financial crisis in 2007, a clear cut case with a framework that researchers agreed on. In non-credit related financial literature authors such as Ron (2000), Boenkost and Schmidt (2004) and Hull (2009) focus on bootstrapping the yield curve and determining a single forward curve such that at initiation, the present value of both legs in a swap contract equal each other. Interpolation techniques used to create smooth and continuous curves are covered in the papers of Hagan and West (2006), Andersen (2007) and Hagan and West (2008).

The impact of changing the discounting curve when pricing derivatives is discussed in Henrard (2007). Later, Henrard (2010) proposes a coherent valuation framework for derivatives based on different Libor tenors still using the traditional bootstrapping technique, though assuming the discounting curve as given. Ametrano and Bianchetti (2009) assume a segmentation of the market and bootstrap the swap rates within each tenor separately which makes their model subject to arbitrage. An extended version of this model is suggested in Bianchetti (2010) that uses the foreign exchange analogy to prevent arbitrage opportunities. Similar approaches have recently been followed by Chibane and Sheldon (2009) and Kijima, Tanaka, and Wong (2009). Extending the theory, Mercurio (2009) builds consistent interest rate curves by modeling the joint evolution of FRA rates and implied forward rates with an extended lognormal Libor Market Model. However, this paper lacks the discussion in a multi-currency situation. Additionally, Morini (2009) explains the market patterns of basis spreads by modeling them as options on the credit worthiness of the counterparty. Johannes and Sundaresan (2007) and Whittall (2010b) extend the multi-curve pricing framework by taking the effect of collateralization on swap rates into consideration.

Most recently, the work of Fujii, Shimada, and Takahashi (2009a), Fujii, Shimada, and Takahashi (2009b), Linderstrøm and Scavenius (2010) and Linderstrøm and Rasmussen (2011) provides a new consistent framework to construct a term structure in the presence of basis spreads and collateral in a multi-currency environment. Instead of building a yield curve by bootstrapping different liquid market instruments, now forward rates are backed out incorporating the effect of basis spreads.

1.2 Problem statement

Inspired by the previous mentioned literature, the thesis will try to demonstrate how to include basis spreads into pricing interest rate swaps. The main purpose of the thesis is:

to revisit the pricing of interest rate swaps by taking basis spreads into account

This involves the derivation of the forward surface and the determination of an appropriate discounting curve, which enables a common pricing framework for interest rate derivatives, leaving no space for arbitrage opportunities. Revisiting the pre-crisis pricing framework serves the purpose to underscore the differences in the two pricing methodologies. Obtaining the above will be done in a mostly descriptive matter. In addition, examples illustrating both frameworks will be presented to emphasize the importance of introducing a multi-curve framework.

1.3 Delimitations

The limited scope of the thesis requires to skip some otherwise interesting topics. Firstly, hedging plain vanilla interest rate swaps will not be part of the thesis. Hedging within the multi-curve framework becomes more complex since it implies that multiple bootstrapping and hedging instruments must be taken into account. Henrard (2010) and Bianchetti (2010) address the problem of delta hedging in a multi-curve setting.

Secondly, the thesis does not focus its attention to modeling issues as both the single- and the multi-curve framework will be explored in depth from a descriptive point of view. Nevertheless, model components will to some degree be part of the thesis since some of the applied methods simply require it. The alternative to bootstrapping a yield curve from market data which is regarded by Ametrano and Bianchetti (2009, p.4) *"more a matter of art than of science"*, is to assume that there exists a unique fundamental underlying short rate process able to model and explain the whole term structure of interest rates. Interested readers may find various literature that deal with short rate models, see among others Brigo and Mercurio (2006) and Björk (2009). To our knowledge, Fujii, Shimada, and Takahashi (2009a) were the first to present a framework of stochastic interest rate models with dynamic basis spreads.

Finally, it is assumed that the reader is familiar with the application of interest rate swaps and has a basic mathematical knowledge on a master's level. Naturally, more advanced concepts will be examined thoroughly.

1.4 Structure

In addressing the problem statement the thesis consists of several chapters that in continuation of each other strive to give the reader a clear understanding of why the post-crisis pricing framework is needed as well as understanding the basic concepts being applied.

Chapter 2 gives an introduction to the interest rate derivatives market and its importance to the financial industry. In a steadily increasing interest rate swap market, corporations, banks and other financial institutions are highly dependent on a correct pricing framework.

Chapter 3 addresses the pre-crisis pricing framework of interest rate swaps. Here, the chapter will focus on how to bootstrap the spot curve using different financial instruments as well as analyzing different interpolation techniques used to determine a smooth, continuous yield curve. Turn effects will likewise be taken into consideration in the pricing framework. Understanding the methodology used in pricing interest rate swaps before the financial crisis is the main purpose of this chapter.

Chapter 4 seeks to analyze the evolution of basis spreads and investigates the impact from the distressed financial markets. This involves an analysis of the divergence between FRA rates and forward rates implied by deposits, OIS-Libor spreads, tenor basis spreads and cross-currency basis spreads.

Chapter 5 addresses the key challenge of the thesis i.e. to present a theoretical and practical framework for the pricing of interest rate swaps in the aftermath of the financial crisis. Here, a review of the no-arbitrage condition as well as pricing swaps with respect to collateralized and uncollateralized swaps will be examined. Revisiting the theoretical framework from chapter 3, there exists a platform for deriving the forward surface and discounting curve for each currency to correctly price interest rate swaps even in distressed financial markets. Finally, this chapter seeks to emphasize the impact of basis spreads and collateralization on swap pricing.

Chapter 6 reflects on the content presented in the previous chapters. This includes a section on critiques as well as the applicability of the post-crisis pricing framework in practice. Lastly, suggestions to further research ideas are proposed.

Chapter 7 summarizes the main points and concludes.

2

Market overview

The market for over-the-counter (OTC) traded derivatives has increased significantly during the last decade as a result of a higher demand for customized products that deal with financial risks. Figure 2.1 clearly illustrates the significant rise in the asset class *interest rate (IR) derivatives* since 1998, underscoring its importance to the financial markets. Interestingly, the traded volume of *credit default swaps (CDS)* experienced a steep increase up until the financial crisis broke out in the beginning of 2008 whereas the other asset classes roughly remained at their respective levels.

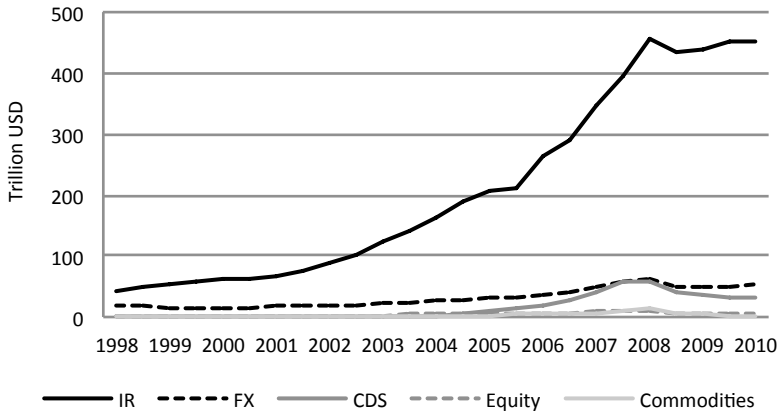


Figure 2.1: OTC derivatives by asset class 1998 - 2010

Source: Bank for International Settlements, semiannual OTC derivatives statistics

Table 2.1 presents the notional amount outstanding of different asset classes as of June 30th 2010. Here, amounting to 451.8 trillion US Dollar, which is equivalent to 83% of the OTC traded derivatives market, the asset class *IR derivatives* clearly dominates *foreign exchange (FX), CDS, equity and commodity derivatives.*

Table 2.1: OTC derivatives by asset class June 30th 2010

	Notional - Total	
Asset class	Trillion USD Eqv.	in %
Interest Rates	451.8	83.0
Foreign Exchanges	53.1	9.8
CDS	30.3	5.6
Equities	6.3	1.1
Commodities	2.9	0.5
Total	544.4	100.0

Source: Bank for International Settlements, semiannual OTC derivatives statistics

An interesting analysis would be to examine which products in the asset class *IR derivatives* contribute mostly to the significant increase from 1998 to 2010. From table 2.2 it is striking that *IR-Swaps* account for a significant portion of the overall traded *IR derivatives*, i.e. 74%. Here, *IR-Swaps* together with *IR-Basis Swaps* and *CC-Swaps* amount to a total of 78.4% of the market which clearly underlines the importance of revisiting the pricing framework. Furthermore table 2.2 shows that the G14 banks Barclays, BNP Paribas, BoA Merrill Lynch, Citi, Credit Suisse, Deutsche Bank, Goldman Sachs, HSBC, J.P. Morgan, Morgan Stanley, RBS, Societe Generale, UBS and Wells Fargo trade a significant portion of the overall traded amount, i.e. 20.1%.

Table 2.2: Interest rate derivatives by product June 30th 2010

	Notional - Total		Notional - G14	
Products	Trillion USD Eqv.	in %	Trillion USD Eqv.	share of G14 in %
CC - Swap	8.9	2.0	3.5	38.7
CC - Swap Exotic	0.8	0.2	0.2	24.1
IR - Cap/Floor	12.1	2.7	3.6	29.4
IR - FRA	53.9	12.0	29.8	55.3
IR - Inflation Swap	1.4	0.3	0.6	43.7
IR - Option	1.3	0.3	0.4	31.7
IR - Option Exotic	0.8	0.2	0.3	34.8
IR - Swap	332.2	74.0	35.0	10.5
IR - Swap Basis	10.7	2.4	4.8	44.9
IR - Swap Exotic	3.9	0.9	1.1	27.8
IR - Swaption	22.9	5.1	11.0	48.2
IR - Unspecified	0.3	0.1	0.1	43.2
Total	449.2	100.0	90.3	20.1

Source: TriOptima, Interest Rate Trade Repository Report
Note: Total of 451.8 in table 2.1 defers from 449.2 due to different sources

Table 2.3 solely focuses on the composition of counterparties trading plain interest rate swaps. Naturally, a great portion of these swaps are traded through financial

institutions, nevertheless the financial products are also relevant for non-financial institutions as they account for 9.2% of the overall traded interest rate swaps. That is why both financial and non-financial institutions must pay further attention to the pricing framework in order to manage their interest rate risks adequately.

Table 2.3: Interest rate swaps by counterparty June 30th 2010

	Notional - Total	
Counterparty	Trillion USD Eqv.	%
Reporting dealers	79.7	22.9
Other financial institutions	235.7	67.8
Non-financial institutions	32.1	9.2
Total	346.5	100.0

Source: Bank for International Settlements, semiannual OTC derivatives statistics

In relation to analysing the interest rate derivatives market, table 2.4 illustrates to what extend the interest rate derivatives are distributed among the most important currencies. Clearly the US Dollar and the Euro are the currencies in which interest rate derivatives are most heavily traded, with significant shares of 38.6% and 33.8%, respectively. The remaining currencies are primarily Japanese Yen and British Pounds. Not surprisingly, these four currencies account for a stunning 93.4% of the total traded amount of interest rate derivatives. Consequently, the data applied in the thesis primarily relies on the two most liquid currencies.

Table 2.4: Interest rate derivatives by currency June 30th 2010

	Notional - Total	
Currency	Trillion USD Eqv.	%
USD	173.3	38.6
EUR	151.8	33.8
JPY	55.9	12.4
GBP	38.5	8.6
AUD	5.7	1.3
CHF	5.0	1.1
CAD	3.2	0.7
SEK	3.2	0.7
Other	12.6	2.8
Total	449.2	100.0

Source: TriOptima, Interest Rate Trade Repository Report

Investigating the market for OTC traded derivatives, it is clear that interest rate derivatives account for 83% of all OTC traded derivatives with *interest rate swaps*

being the most important product. Moreover, the primary market participants are financial institutions, nevertheless non-financial institutions are still engaged to a certain extent. With regards to currencies in which derivatives are denominated, the most liquid currencies are US Dollar, Euro, Japanese Yen and British Pounds. Hence, a correct pricing framework in the swap market is of great importance due to its size and the possibility for financial and non-financial institutions to manage interest rate risks through this market.

3

Pre-crisis pricing framework

The main aspect in pricing interest rate swaps prior to the financial crisis is to determine *one* forward curve. The procedure for constructing the swap curve is generally agreed on upon practitioners, though there exists no single correct methodology. Furthermore, practitioners face the challenge in regards to which interpolation technique to apply as well as incorporating turn of year effects into the forward curve. Essentially, different approaches w.r.t these challenges might result in varying forward and discounting curves. Thus, the construction of the swap curve must be conducted thoroughly while taking these factors into consideration.

3.1 The theoretical framework

In this section the theoretical framework is examined for the single-curve framework. There exist several official benchmarks for interbank term deposits such as Libor, Euribor, Cibor or Tibor. The spot Libor rate is defined as the rate of return from buying 1 unit of a default free zero-coupon bond at time t and selling it at maturity T_n. Hence, the spot Libor rate is in fact the discounting rate:

$$L_{t,T_n} = \frac{1}{\delta_n}\left(\frac{1}{P_{t,T_n}} - 1\right) \tag{3.1}$$

Here, P_{t,T_n} refers to the default free discounting factor where δ_n is the daycount fraction for the interval $[t, T_n]$. Then, the forward Libor rate from T_{n-1} to T_n standing at time t can be estimated by the following equation:

$$F_{t;T_{n-1},T_n} = \frac{1}{\delta_n}\left(\frac{P_{t,T_{n-1}}}{P_{t,T_n}} - 1\right) \tag{3.2}$$

Here, δ_n is now the daycount fraction for the interval $[T_{n-1}, T_n]$. Due to the relation between forward Libor rates and discounting factors in equation 3.2 the single-curve framework avoids arbitrage opportunities.

An instrument that is based on the forward Libor rate is the Forward Rate Agreement (FRA). Taking a long position in a FRA, the payoff at maturity T_n can

be determined by the difference between the spot Libor rate and the fixed rate K:

$$V_{T_n} = \delta_n \left(L_{T_{n-1},T_n} - K \right) \tag{3.3}$$

Determining the value of the FRA at time t the following equation can be applied:

$$V_t = \delta_n \left(E_t[L_{T_{n-1},T_n}] - K \right) P_{t,T_n} \tag{3.4}$$

This imposes the challenge to determine the forward Libor rate. Here, $E_t[\,]$ denotes the expectations operator under the T_n-forward measure Q^{T_n}. At initiation, the FRA rate K is determined such that it sets both legs to par:

$$\delta_n \, K \, P_{t,T_n} = \delta_n \, E_t[L_{T_{n-1},T_n}] \, P_{t,T_n} \tag{3.5}$$

The choice of a zero-coupon bond maturing at time T_n as numeraire is particularly useful when dealing with interest rate derivatives. It follows that any simply-compounded forward rate spanning a time interval, ending in T_n, is a martingale under the T_n-forward measure, i.e.

$$F_{t;T_{n-1},T_n} = E_t[L_{T_{n-1},T_n}] \, . \tag{3.6}$$

The result from the above equation will be examined in more detail in section 3.2. By applying this relation it follows that equation 3.2 can be written as:

$$F_{t;T_{n-1},T_n} = E_t[L_{T_{n-1},T_n}] = \frac{1}{\delta_n} \left(\frac{P_{t,T_{n-1}}}{P_{t,T_n}} - 1 \right) \tag{3.7}$$

Thus, derivatives dependent on future interest rates can be priced by applying forward rates. This feature essentially simplifies the pricing procedure of interest rate derivatives. Now, the pricing of a FRA becomes a straight forward procedure. Simultaneously, an interest rate swap (IRS) can be priced as a portfolio of several FRAs where both legs of the swap must also equal each other at initiation:

$$\textbf{IRS:} \;\; C_N \sum_{n=1}^{N} \Delta_n \, P_{t,T_n} = \sum_{n=1}^{N} \delta_n \, E_t[L_{T_{n-1},T_n}] \, P_{t,T_n} \tag{3.8}$$

Here, C_N is the par swap rate of the N-length IRS at time t, Δ_n and δ_n are the daycount fractions of the fixed and floating leg, respectively. For simplicity it is assumed that the payments of the fixed and floating leg occur simultaneously. In-

serting equation 3.7 into equation 3.8 yields:

$$
\begin{aligned}
C_N \sum_{n=1}^{N} \Delta_n \, P_{t,T_n} &= \sum_{n=1}^{N} \delta_n \, \left(\frac{1}{\delta_n} \left(\frac{P_{t,T_{n-1}} - P_{t,T_n}}{P_{t,T_n}} \right) \right) \, P_{t,T_n} \\
C_N \sum_{n=1}^{N} \Delta_n \, P_{t,T_n} &= \sum_{n=1}^{N} \left(P_{t,T_{n-1}} - P_{t,T_n} \right) \\
C_N \sum_{n=1}^{N} \Delta_n \, P_{t,T_n} &= P_{t,T_0} - P_{t,T_N} \qquad (3.9)
\end{aligned}
$$

The right hand side of the above equation can be considered as a long position in one zero-coupon bond with maturity T_0 and a short position in another zero-coupon bond with maturity T_N. Finally, the swap rate can be determined as:

$$
C_N = \frac{P_{t,T_0} - P_{t,T_N}}{\sum_{n=1}^{N} \Delta_n \, P_{t,T_n}} \qquad (3.10)
$$

Naturally, the swap rate C_N must be equal to the rate on the swap curve with maturity N.

When considering a tenor swap (TS) the present values of both floating legs must likewise equal each other at initiation. The TS can be seen as a portfolio of two IRS of the same maturity with matching fixed legs and two floating legs plus a tenor spread added to the floating leg that is indexed to the shorter tenor. Hence, the required relation among the two floating legs is

$$
\textbf{TS:} \quad \sum_{n=1}^{N} \delta_n \left(E_t[L_{T_{n-1},T_n}] + \tau_N \right) P_{t,T_n} = \sum_{m=1}^{M} \delta_m \, E_t[L_{T_{m-1},T_m}] \, P_{t,T_m} \qquad (3.11)
$$

where τ_N denotes the time t market spread of the length N between the two underlying Libor rates with tenors $n < m$. The lefthand side could resemble the 3M underlying Libor rate whereas the righthand side could reflect the 6M underlying Libor rate. In this example, the 3M Libor payer compensates the higher credit risk inherent in the 6M Libor rate by adding the 3M vs 6M tenor basis spread. Solving for τ_N such that both legs equal each other at initiation yields:

$$
\tau_N = \frac{\sum_{m=1}^{M} \delta_m \, E_t[L_{T_{m-1},T_m}] \, P_{t,T_m} - \sum_{n=1}^{N} \delta_n \, E_t[L_{T_{n-1},T_n}] \, P_{t,T_n}}{\sum_{n=1}^{N} \delta_n P_{t,T_n}} \qquad (3.12)
$$

Ametrano and Bianchetti (2009) consider the above as being the difference in the two swap rates from two plain interest rate swaps with different underlying tenors

but the same maturity $T_N = T_M$:

$$\tau_N = C_M - C_N \tag{3.13}$$

The two approaches expressed in equation 3.12 and 3.13 to determine the tenor basis spread might differ slightly from each other. This is due to different daycount conventions, as the fixed leg usually is payed on 30/360 basis whereas the spread is added to the floating leg that is based on act/360. Here, dealers have different reasons for trading contracts applying the two different approaches (Linderstrøm and Rasmussen, 2011, p.15).

In the case of a cross currency swap (CCS) the interest rate payments of both legs occur in different currencies. From the possible types of CSSs: fixed vs fixed, fixed vs floating and floating vs floating, the latter type is particular important and is used for generating the other types synthetically. Assuming that both legs have the same tenor but depend on different underlying rates a CCS must satisfy the following relation:

$$\begin{aligned} \textbf{CCS:} \quad & \left(-P^f_{t,T_0} + \sum_{n=1}^{N} \delta^f_n \left(E^f_t[L^f_{T_{n-1},T_n}] + b_N \right) P^f_{t,T_n} + P^f_{t,T_N} \right) fx_t \\ & = -P_{t,T_0} + \sum_{n=1}^{N} \delta_n \, E_t[L_{T_{n-1},T_n}] \, P_{t,T_n} + P_{t,T_N} \end{aligned} \tag{3.14}$$

Here, the index f denotes that the variable is relevant for a foreign currency where b_N is the basis spread for length N such that the US Dollar as base currency trades flat against the foreign currency. $E^f_t[\,]$ denotes the expectations operator under the T_n-forward measure $Q^{T_n}_f$ in the foreign currency applying P^f_{t,T_n} as numeraire. The spot exchange rate of US Dollar per foreign currency at time t is represented by fx_t. The US Dollar acts as a base currency but could easily be replaced by another base currency. Similarly to equation 3.12 determining the cross currency basis spread can be done by isolating b_N in equation 3.14 which yields:

$$\begin{aligned} b_N = & \frac{\left(P^f_{t,T_0} - \sum_{n=1}^{N} \delta^f_n \left(E^f_t[L^f_{T_{n-1},T_n}] \right) P^f_{t,T_n} - P^f_{t,T_N} \right) fx_t}{\sum_{n=1}^{N} \delta_n \, P_{t,T_n}} \\ & + \frac{-P_{t,T_0} + P_{t,T_N} + \sum_{n=1}^{N} \delta_n \left(E_t[L_{T_{n-1},T_n}] \right) P_{t,T_n}}{\sum_{n=1}^{N} \delta_n \, P_{t,T_n}} \end{aligned} \tag{3.15}$$

The no-arbitrage condition from equation 3.2 creates the foundation for pricing IRS, TS and CCS as in equations 3.8, 3.11 and 3.14, respectively. Consequently, determining the forward curve enables practitioners to estimate swap rates, tenor basis spreads and cross-currency basis spreads.

3.2 Choice of numeraire

The following section takes its form from the work of Geman, El Karoui, and Rochet (1995), Brigo and Mercurio (2006), Hull (2009) and Björk (2009). In general, a numeraire is a reference asset that is chosen in a way to normalize all other asset prices with respect to it. The bank account is often implicitly used as a risk neutral numeraire. However, this is just one of many possible choices since any positive, non-dividend paying asset can be applied as numeraire. Dealing with interest rate derivatives, the choice of a zero-coupon bond as numeraire is particularly useful.

3.2.1 Martingales

Defining a sequence of random variables $X_0, X_1, \ldots, X_t$, the variable X_t is a martingale if, for all $t > 0$, the following is true:

$$E[X_t | X_{t-1}, X_{t-2}, \ldots, X_0] = X_{t-1} \tag{3.16}$$

Similarly, a variable θ is a martingale if it follows a zero-drift stochastic process

$$d\theta = \sigma \, dz \tag{3.17}$$

where dz is a Wiener process. The volatility parameter σ can be considered a stochastic variable itself or it can depend on θ as well as on other stochastic variables. The convenience of the martingale property is shown in its tremendous applicability in financial literature, where the expected value at any future time is equal to its value today:

$$E[\theta_T] = \theta_t \tag{3.18}$$

The change in θ between time t and time T is the sum of the changes over many small time intervals. Consequently, the expected change must be zero.

3.2.2 The equivalent martingale measure result

The equivalent martingale measure result shows that under the no-arbitrage condition the relationship between two price processes is a martingale. Hull (2009) introduces the market price of risk λ as:

$$\lambda = \frac{\mu - r}{\sigma} \tag{3.19}$$

Here, μ and σ are the return and volatility on θ, respectively, and r is the risk free rate. The market price of risk measures the risk adjusted excess return with respect to securities that depend on θ. Furthermore, the market price of risk must at any given time be the same for all derivatives that are dependent only on θ and t to ensure no arbitrage. From Hull (2009) equation 3.19 only holds for investment assets that provide no income.

Assuming the two prices of traded assets X and N depend on a single source of uncertainty and provide no income during the time of matter, the relationship between the prices of the two assets is a martingale for some choice of the market price of risk. The relationship is defined by:

$$\phi = \frac{X}{N} \tag{3.20}$$

This can be thought of as measuring the price of X in units of N, where the security price of N is referred to the numeraire. Choosing the same market price of risk for instruments X as for a given numeraire N makes the relative price ϕ a martingale. To prove this result, the price processes of X and N are defined as

$$dX = (r + \lambda\sigma_X)Xdt + \sigma_X X dz \tag{3.21}$$

$$dN = (r + \lambda\sigma_N)Ndt + \sigma_N N dz \tag{3.22}$$

where μ is replaced by rewriting equation 3.19. Choosing the same market price of risk for X as for N, i.e. $\lambda = \sigma_X = \sigma_N$, results in a zero-drift relative price process ϕ as written below:

$$d\phi = (\sigma_X - \sigma_N)\phi \, dz \tag{3.23}$$

This is similar to equation 3.17 where now the process of ϕ is a martingale. The derivation from equations 3.21 and 3.22 to equation 3.23 is presented in appendix A.1.

3.2.3 Zero-coupon bond as numeraire

Assuming there exists a numeraire N and a probability measure Q^N, equivalent to the initial risk neutral measure Q^0, the price of any traded asset X relative to N is a martingale under Q^N, i.e.:

$$\frac{X_t}{N_t} = E^N \left[\frac{X_T}{N_T} | \mathcal{F}_t \right] \tag{3.24}$$

This proposition introduced by Geman, El Karoui, and Rochet (1995) provides a fundamental tool for pricing derivatives as it is generally applicable for any positive non-dividend paying numeraire.

When applying the zero-coupon bond as numeraire any simple compounded forward rate is a martingale under the T-forward measure. Hence, the price of an interest rate derivative π_t at time t can be estimated as the discounted expected payoff on claim H at maturity T, conditional on the T-forward measure:

$$\pi_t = P_{t,T} E^T \left[H_T | \mathcal{F}_t \right] \tag{3.25}$$

The reason why the measure Q^T is called forward measure is justified in the following. Recalling equation 3.2 and rewriting it gives:

$$F_{t;T_{n-1},T_n} P_{t,T_n} = \frac{1}{\delta_n} \left(P_{t,T_{n-1}} - P_{t,T_n} \right) \tag{3.26}$$

Now, considering $F_{t;T_{n-1},T_n} P_{t,T_n}$ as a traded asset X_t and by applying P_{t,T_n} as the numeraire N_t the lefthand side (LHS) of equation 3.24 yields:

$$\textbf{LHS: } \frac{F_{t;T_{n-1},T_n} P_{t,T_n}}{P_{t,T_n}} = \frac{1}{\delta_n} \left(\frac{P_{t,T_{n-1}}}{P_{t,T_n}} - 1 \right) = F_{t;T_{n-1},T_n} \tag{3.27}$$

Similarly, the righthand side (RHS) of equation 3.24 can be rewritten as:

$$\textbf{RHS: } E_t \left[\frac{F_{t;T_{n-1},T_n} P_{T_n,T_n}}{P_{T_n,T_n}} | \mathcal{F}_t \right] = E_t[F_{T_{n-1};T_{n-1},T_n}] \tag{3.28}$$

In the following the filtration $\mathcal{F}_t$ is omitted for simplification purposes. Finally, replacing $F_{T_{n-1};T_{n-1},T_n}$ with its equivalent L_{T_{n-1},T_n} on the right hand side of the above equation and setting equations 3.27 and 3.28 equal to each other yields:

$$F_{t;T_{n-1},T_n} = E_t[L_{T_{n-1},T_n}]$$

This corresponds exactly to equation 3.6 applied in the previous section to price interest rate swaps and hence indirectly FRAs.

3.3 Bootstrapping the spot curve

Constructing the spot curve, equivalently denoted as the zero curve in some parts of the literature, is done by bootstrapping the most liquid and dominant instruments for their respective time horizons (Ron, 2000, p.4). The input instruments should cover all areas of the term structure. The bootstrapped spot curve will be

constructed for the US Dollar, applying market data from June 30th 2010.

3.3.1 The short end of the spot curve

The short end of the spot curve is based on short term deposit rates with maturities up until three months. Deposits are OTC traded zero-coupon contracts that start at their reference date and pay the fixed rate of the contract, i.e. deposit rate, up until the corresponding maturity. Here, Libor is the primary global benchmark for short term interest rates as it is widely used as a reference rate for many interest rate contracts. Each day the British Bankers' Association (BBA) calculates Libor rates based on panels of major banks who submit their cost of borrowing unsecured funds for various periods of time and in various currencies. Consequently, using the Libor rates as input instruments for the short end of the spot curve well reflects the liquidity in the money market. From table 3.1 the different Libor rates with maturities up until three months can be seen.

Table 3.1: US Dollar Deposit rates June 30th 2010

Instrument	Start date	End date	Quote (%)
Libor ON	30 Jun 2010	1 Jul 2010	0.30563
Libor 1W	2 Jul 2010	9 Jul 2010	0.32875
Libor 2W	2 Jul 2010	16 Jul 2010	0.33875
Libor 1M	2 Jul 2010	2 Aug 2010	0.34844
Libor 2M	2 Jul 2010	2 Sep 2010	0.43188
Libor 3M	2 Jul 2010	4 Oct 2010	0.53394

Source: British Bankers' Association

In continuation hereof, denoting the spot rate R_{t,T_n} at time t with maturity T_n as the rate to be bootstrapped from market instruments, it can be directly inferred from the Libor deposit rate that:

$$R_{t,T_n} = L_{t,T_n} \tag{3.29}$$

3.3.2 The middle area of the spot curve

The middle area of the spot curve up to three years is constructed by using FRAs or interest rate futures. FRA contracts are forward starting deposits that carry a fixed time horizon to settlement and settle at maturity. This feature makes them preferable to futures that have fixed settlement days and are marked-to-market daily (Ron, 2000, p.6). On the other hand futures are usually more liquid compared to FRAs, why for currencies with highly liquid interest rate futures markets, futures

could even be used out to five years (Ron, 2000, p.10). Consequently, the best mix of both instruments will be chosen to bootstrap the middle area of the spot curve. Table 3.2 shows respective market quotes for US Dollar FRAs.

Table 3.2: US Dollar FRAs June 30th 2010

Instrument	Start date	End date	Quote (%)
FRA 1x4	2 Aug 2010	2 Nov 2010	0.5610
FRA 2x5	2 Sep 2010	2 Dec 2010	0.6170
FRA 3x6	4 Oct 2010	3 Jan 2011	0.7075
FRA 4x7	2 Nov 2010	2 Feb 2011	0.7300
FRA 5x8	2 Dec 2010	2 Mar 2011	0.7450
FRA 6x9	3 Jan 2010	4 Apr 2011	0.7650
FRA 7x10	2 Feb 2011	3 Mai 2011	0.7870
FRA 8x11	2 Mar 2011	2 Jun 2011	0.8170
FRA 9x12	4 Apr 2011	4 Jul 2011	0.8500
FRA 12x15	4 Jul 2011	3 Oct 2011	0.9570
FRA 12x18	4 Jul 2011	3 Jan 2012	1.2100
FRA 12x24	4 Jul 2011	2 Jul 2012	1.4950

Source: Nordea Analytics

Following this, equation 3.30 can be applied to transform forward rates such as quoted in table 3.2 into spot rates R_{t,T_n}. Here, $F_{t;T_{n-1},T_n}$ denotes the forward rate based on a FRA starting at time T_{n-1} and maturing at T_n.

$$R_{t,T_n} = \left[\left(1 + F_{t;T_{n-1},T_n}\right)^{T_n - T_{n-1}} \left(1 + R_{t,T_{n-1}}\right)^{T_{n-1}}\right]^{\frac{1}{T_n}} - 1 \tag{3.30}$$

Whereas FRAs are traded OTC and therefore have the advantage of being more customizable, interest rate futures are traded on exchanges as highly standardized contracts, reducing the credit risk and transaction costs. One popular example is the Eurodollar futures contract traded at the Chicago Mercantile Exchange. It refers to a one million US Dollar deposit with the 3M US Dollar Libor rate as underlying. Consequently, the price of the futures contract $J^{Fut}_{t;T_{n-1},T_n}$ at time t can be estimated as

$$J^{Fut}_{t;T_{n-1},T_n} = 100 - F^{Fut}_{t;T_{n-1},T_n} \tag{3.31}$$

where $F^{Fut}_{t;T_{n-1},T_n}$ is the implied forward rate of the corresponding futures contract.

Extracting forward rates from interest rate futures requires a convexity adjustment. Most interest rate futures have zero convexity, meaning they pay a fixed payoff per basis point change, regardless of the level of the underlying interest rates.

Since FRAs are convex instruments, forward rates backed out of Eurodollar futures contracts are biased. Replicating a short position in a Eurodollar futures with a long position in a FRA results in a portfolio that has net positive convexity. When interest rates rise, being short a Eurodollar future will generate profits that can be reinvested at higher rates. Contrary to this, decreasing interest rate lead to a loss that can be financed at lower rates. This mark-to-market effect is incorporated in the Eurodollar futures price as it is settled daily and must be removed to obtain an unbiased predictor of forward rates and hence eliminate the advantage of being short the Eurodollar future (Kirikos and Novak, 1997, p.1). Thus, the futures contract's price needs to be adjusted by:

$$J^{FutAdj}_{t;T_{n-1},T_n} = J^{Fut}_{t;T_{n-1},T_n} + CA_{t;T_{n-1},T_n} \tag{3.32}$$

Estimating the convexity adjustments $CA_{t;T_{n-1},T_n}$ requires an estimation of the future path of the underlying interest rate until maturity of the futures contract. This is due to the fact that the volatility of the forward rates and their correlation to the spot rates have to be accounted for (Ametrano and Bianchetti, 2009, p.15). A term structure model such as the one proposed by Hull and White (1990) allows to estimate the convexity adjustment in a consistent and rigorous framework:

$$dr = (\theta_t - ar)dt + \sigma dz \tag{3.33}$$

Here, r is the short term interest rate, θ is the long term mean reversion level, a the rate of mean reversion, dz is a Wiener process and σ is the annual volatility of the short rate. In this constant parameter version, a and σ are constants whereas θ_t is a time varying function and chosen such that the model exactly fits the current market term structure of interest rates. Kirikos and Novak (1997) acknowledge that the normal-distributed rate assumption admits the possibility of producing negative interest rates. However, this probability is considered almost negligible (Brigo and Mercurio, 2006, p.74).

Based on Hull & White's one-factor short rate model, Kirikos and Novak (1997) introduce the following formula to estimate the convexity adjustment accordingly as

$$CA_{t;T_{n-1},T_n} = (1 - e^{-Z}) \left[100 - J^{Fut}_{t;T_{n-1},T_n} + 100 \frac{360}{T_n - T_{n-1}} \right] \tag{3.34}$$

where

$$Z = \Lambda + \Phi$$

and

$$\Lambda = \sigma^2 \left(\frac{1 - e^{-2aT_{n-1}}}{2a} \right) \left[\frac{1 - e^{-a(T_n - T_{n-1})}}{a} \right]^2$$
$$\Phi = \frac{\sigma^2}{2} \left(\frac{1 - e^{-a(T_n - T_{n-1})}}{a} \right) \left[\frac{1 - e^{-aT_{n-1}}}{a} \right]^2 .$$

Obviously, the challenge here is to determine the Hull-White parameters a and σ. Practitioners such as Ametrano and Bianchetti (2009), Kirikos and Novak (1997), Ron (2000) as well as Bloomberg agree on applying $a = 0.03$ as rate of mean reversion. No unanimous result for the volatility parameter σ based on the 3M US Dollar Libor rate prevails in the literature. For instance, Kirikos and Novak (1997) apply a volatility parameter of $\sigma = 1.5\%$.

It shows that the Hull & White model is a very convenient short-rate model for determining the convexity adjustment. As no closed-form solution exists for pricing futures one can estimate the volatility parameters a and σ by applying the very convenient closed-form solution from the Hull & White model for pricing interest rate caps that have the same underlying rate as the futures. Hence, one can determine the volatility parameters when calibrating the model to market data. Consequently, determining the volatility parameters and the adjusted spot curve must be done simultaneously using an iterative process (Kirikos and Novak, 1997, p.2). According to Brigo and Mercurio (2006, p.76) a cap can be priced as a portfolio of n caplets, i.e.

$$Cap(t, \tau, N, X) = K \sum_{n=1}^{N} \left[P_{t,T_{n-1}} \Phi(-h_n + \sigma_p^n) - (1 + X_{\tau_n}) P_{t,T_n} \Phi(-h_n) \right] \qquad (3.35)$$

where

$$B_{T_{n-1},T_n} = \frac{1}{a} \left[1 - e^{-a(T_n - T_{n-1})} \right]$$
$$\sigma_p^n = \sigma \sqrt{\frac{1 - e^{-2a(T_{n-1} - t)}}{2a}} B_{T_{n-1},T_n}$$
$$h_n = \frac{1}{\sigma_p^n} \log \frac{P_{t,T_n}(1 + X_{\tau_n})}{P_{t,T_{n-1}}} + \frac{\sigma_p^n}{2}.$$

To this end, K is the nominal value, X the strike or cap rate, Φ the standard normal cumulative distribution and τ denotes the set of times $\{t_0, t_1, \ldots, t_N\}$, meaning that t_n is the difference in years between the payment date d_n of the n-th caplet and the settlement date t, where t_0 is the first reset time.

The iterative process can be described as the following:

1. Choose a, σ and a futures curve.
2. Estimate the corresponding forward by $F_{t;T_{n-1},T_n} = 100 - J^{FutAdj}_{t;T_{n-1},T_n}$.
3. Estimate the corresponding discount curve by relation 3.2.
4. Estimate cap prices for various strikes and maturities by equation 3.35.
5. Calibrate a and σ by minimizing the sum of the squared differences between model and market prices, i.e.

$$SSD = \min \sum_{n=1}^{N} \left(Cap_n^{HW} - Cap_n^{market} \right)^2 .$$

Before this methodology can be applied to observed cap prices, its set-up needs to be tested in order to ensure a controlled process. This is done by choosing a, σ and a fictional futures curve and generating cap prices based on that fictional curve. Afterwards, a and σ are calibrated such that the sum of the squared differences between the model and the previously generated prices is minimized. If the resulting estimates for a and σ equal those initially chosen, then the methodology is validated. The methodology was proven valid.

When determining the rate of mean reversion a and the volatility σ of the short term rate, i.e. the 3M US Dollar Libor rate, there exists a mismatch between the fixing dates of futures and caps. For example, standing at June 30th 2010, the next Eurodollar futures is fixed on Sep 15th 2009 delivering the discounting factor corresponding to Dec 15th 2010. Whereas a cap that will be fixed the first time on Sep 30th 2010 and settled at Jan 3rd 2011, yields the discounting factor corresponding to Jan 3rd 2011. Hence, in order to price caps on a discounting curve that is derived from futures it is necessary to interpolate between discounting factors. Ametrano and Bianchetti (2009) comment that interpolation is already used during the bootstrapping procedure, before actually interpolating the spot curve. Here, the interpolation of discounting factors is done by applying a forward monotone convex spline proposed by Hagan and West (2008). The choice of this interpolation method is discussed in section 3.4.

Table 3.3: Hull-White parameters for US Dollar Eurdollar Futures contracts

Hull-White parameter	Value
Rate of mean reversion a	-0.3346
Volatility σ	0.0067

Source: Inspired by Ametrano and Bianchetti (2009)

Table 3.3 summarizes the estimated Hull-White parameters following the approach of Kirikos and Novak (1997). According to Brigo and Mercurio (2006, p.134) it is common to observe a negative parameter value for the rate of mean reversion a when calibrating the Hull & White model. This means that the short rate is diverging from the long term mean reversion level θ.

Finally, market quotes for different series of Eurodollar futures and their respective convexity adjustments calculated from equation 3.35 using the determined Hull-White parameters as shown in table 3.4. Forward rates are transformed into spot rates by again applying equation 3.30.

Table 3.4: US Dollar Eurdollar Futures contracts June 30th 2010

Instrument	Start date	End date	Quote	Conv. Adj.	Forward
Eurodollar Fut 09/2010	15 Sep 2010	15 Dec 2010	99.350	0.000	0.650
Eurodollar Fut 12/2010	15 Dec 2010	16 Mar 2011	99.230	0.001	0.769
Eurodollar Fut 03/2011	16 Mar 2011	15 Jun 2011	99.160	0.003	0.837
Eurodollar Fut 06/2011	15 Jun 2011	21 Sep 2011	99.065	0.005	0.930
Eurodollar Fut 09/2011	21 Sep 2011	21 Dec 2011	98.920	0.008	1.072
Eurodollar Fut 12/2011	21 Dec 2011	21 Mar 2012	98.710	0.012	1.278
Eurodollar Fut 03/2012	21 Mar 2012	20 Jun 2012	98.510	0.017	1.473
Eurodollar Fut 06/2012	20 Jun 2012	19 Sep 2012	98.280	0.024	1.696
Eurodollar Fut 09/2012	19 Sep 2012	19 Dec 2012	98.050	0.033	1.917
Eurodollar Fut 12/2012	19 Dec 2012	20 Mar 2013	97.810	0.044	2.146
Eurodollar Fut 03/2013	20 Mar 2013	19 Jun 2013	97.605	0.057	2.338

Source: Datastream

3.3.3 The long end of the spot curve

The long end of the spot curve, i.e. from three years onwards, is determined from the observed coupon swap rates. The swaps applied in the bootstrapping procedure have the 3M US Dollar Libor rate as underlying. According to Hagan and West (2006, p.92) equation 3.10 can be used iteratively to solve for P_{t,T_n} assuming P_{t,T_i} is known for $i = 1, 2, \ldots, n-1$, i.e.:

$$P_{t,T_N} = \frac{1 - C_N \sum_{n=1}^{N-1} \delta_n P_{t,T_n}}{1 + C_N \delta_N} \tag{3.36}$$

However, swap rates are only available for certain maturities as can be seen from table 3.5.

Table 3.5: US Dollar Swap rates June 30th 2010

Instrument	Start date	End date	Quote (%)
Swap 1Y	2 Jul 2010	4 Jul 2011	0.710
Swap 2Y	2 Jul 2010	2 Jul 2012	0.951
Swap 3Y	2 Jul 2010	2 Jul 2013	1.305
Swap 4Y	2 Jul 2010	2 Jul 2014	1.686
Swap 5Y	2 Jul 2010	2 Jul 2015	2.036
Swap 6Y	2 Jul 2010	4 Jul 2016	2.330
Swap 7Y	2 Jul 2010	3 Jul 2017	2.553
Swap 8Y	2 Jul 2010	2 Jul 2018	2.732
Swap 9Y	2 Jul 2010	2 Jul 2019	2.880
Swap 10Y	2 Jul 2010	2 Jul 2020	3.007
Swap 12Y	2 Jul 2010	4 Jul 2022	3.215
Swap 15Y	2 Jul 2010	2 Jul 2025	3.423
Swap 20Y	2 Jul 2010	2 Jul 2030	3.588
Swap 25Y	2 Jul 2010	2 Jul 2035	3.661
Swap 30Y	2 Jul 2010	2 Jul 2040	3.701

Source: Nordea Analytics

This lack of liquidity reduces the information set which may lead to inconsistent discounting factors. There exist two alternatives to mitigate this issue. The first one is to interpolate the input swap rates for all expiries that are not quoted and then bootstrap the discount factors directly from this complete information set through equation 3.36. Here, the interpolation technique of choice is similar to the one applied for interpolating the spot curve and will be introduced in the next section 3.4. Having estimated the discount factors P_{t,T_n} the spot rates R_{t,T_n} with $n > 1$ year can be inferred by the following relation:

$$R_{t,T_n} = \left(\frac{1}{P_{t,T_n}}\right)^{\frac{1}{\delta_n}} - 1 \tag{3.37}$$

The second alternative presented by Hagan and West (2006) is an iterative process to bootstrap spot rates from swap rates. Therefore equation 3.36 needs to be rewritten as:

$$r_N = -\frac{1}{\delta_N} \ln \left[\frac{1 - C_N \sum_{n=1}^{N-1} \delta_n P_{t,T_n}}{1 + C_N \delta_N}\right] \tag{3.38}$$

The following procedure describes how to apply this formula in the second alternative:

i) guessing initial rates r_N for each of the quoted expiries, e.g. continuous equivalent of the input swap rates, and replacing C_N with r_N

ii) interpolating r_N for all missing maturities by applying the method of the spot

curve itself, i.e. monotone convex spline

iii) estimating all discount factors P_{t,T_n} and in continuation hereof new estimates for all rates r_N by formula 3.38

iv) iterating the steps ii) and iii) until convergence

Applying the second bootstrap procedure in practice leads to bumpy spot rates with an unsatisfying convergence whereas the first alternative delivered a better fit which is why the latter is the alternative of choice although practitioners such as Hagan and West (2006) argue that this way decouples the interpolation procedure from the bootstrap procedure. Further information on the bootstrap algorithm is provided in West (2011).

3.3.4 The bootstrapped spot curve

After examining each part of the spot curve, the entire curve can be constructed as displayed in table 3.6. Here, the choice between FRAs and futures contracts to determine the middle part of the curve highly depends on each instrument's liquidity. Consequently, the choice of futures contracts over FRAs in this case also reflects the concern for liquidity. Under different circumstances the construction of the spot curve could have a different composition as no general receipt prevails.

A key issue is to decide which instruments to include. Excluding too many key rates runs the risk of excluding market information whereas including too many key rates will lead to overfitting the spot curve. Consequently, this could result in an implausible curve that is subject to arbitrage or failure of the convergence of the bootstrap algorithm (Hagan and West, 2006, p.94).

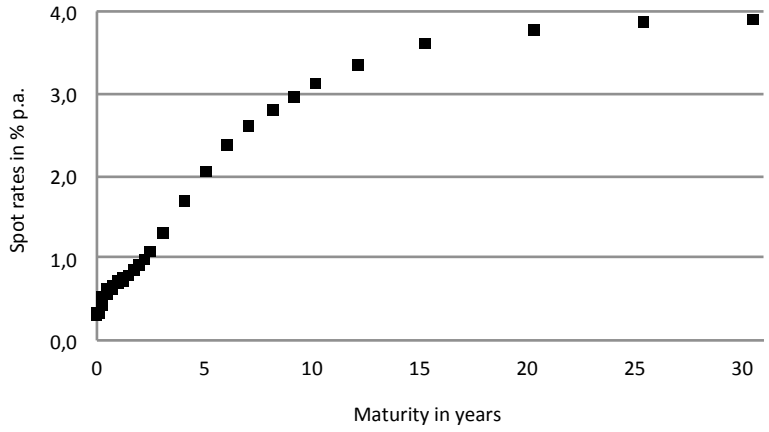


Figure 3.1: The bootstrapped spot curve

Figure 3.1 shows the foundation for determining the complete spot curve which is done by interpolating between the key rates. Which interpolation technique to apply will be investigated in the next section.

Table 3.6: US Dollar Spot curve June 30th 2010

Maturity	Instrument	Spot rate (%)
0,00	Libor ON	0.3056
0,02	Libor 1W	0.3288
0,04	Libor 2W	0.3388
0,09	Libor 1M	0.3484
0,18	Libor 2M	0.4319
0,26	Libor 3M	0.5339
0,46	Eurodollar Fut 09/2010	0.5649
0,51	FRA 3x6	0.6170
0,71	Eurodollar Fut 12/2010	0.6343
0,76	FRA 6x9	0.6644
0,96	Eurodollar Fut 03/2011	0.6851
1,01	FRA 9x12	0.7086
1,23	Eurodollar Fut 06/2011	0.7361
1,26	FRA 12x15	0.7553
1,48	Eurodollar Fut 09/2011	0.7888
1,73	Eurodollar Fut 12/2011	0.8525
1,98	Eurodollar Fut 03/2012	0.9213
2,22	Eurodollar Fut 06/2012	0.9962
2,47	Eurodollar Fut 09/2012	1.0742
3,01	Swap 3Y	1.3107
4,01	Swap 4Y	1.7015
5,01	Swap 5Y	2.0661
6,02	Swap 6Y	2.3771
7,01	Swap 7Y	2.6162
8,01	Swap 8Y	2.8106
9,01	Swap 9Y	2.9734
10,01	Swap 10Y	3.1150
12,02	Swap 12Y	3.3519
15,02	Swap 15Y	3.5946
20,02	Swap 20Y	3.7834
25,02	Swap 25Y	3.8602
30,03	Swap 30Y	3.8986

3.4 Interpolation techniques

This section will at first give an overview of the literature dealing with various interpolation techniques suggested by theorists and practitioners. In continuation hereof, an analysis of the more relevant techniques will be conducted. The aim is then to determine the best applicable technique for interpolating the bootstrapped key rates where the main concern is to secure a continuous, non-negative forward

curve.

3.4.1 Literature review

The academic literature contains many works that examine interpolation methods for the purpose of curve construction. Most of them apply some kind of *spline* technique. In general a *spline* is a function defined piecewise by polynomials of degree k that is continuously differentiable $k-1$ times. The main advantage of piecewise polynomial interpolation is that a large number of data points can be fit with low-degree polynomials (Heath, 1997, p.232).

Linear interpolation on yields, discount factors or the logarithm of these is the simplest example of polynomial splines. This method is stable and trivial to implement, but generates discontinuous forward rates as linear functions are clearly not differentiable, see section 3.4.3. Similarly, *quadratic splines* often produce 'zig-zag' forward curves and therefore are unsuitable to price interest rate derivatives (Hagan and West, 2008, p.7).

To overcome this issue a number of approaches based on *cubic splines* have been introduced in the literature. A cubic spline is a piecewise cubic polynomial that is twice continuously differentiable. Here, McCulloch (1975) started out with a *cubic regression spline* on zero-coupon bond prices though this leads to instabilities in the yields and forward rates. Consequently, it is recommended to apply cubic splines either on yields, the logarithm of zero-coupon bond prices or a similar transformation. Fisher, Nychka, and Zervos (1995) respond in another way to mitigate the oscillating forward curve. They propose to use a cubic spline with a *roughness penalty* to extract the forward curve. A generalized cross-validation method is chosen to stiffen the spline though simultaneously reducing the goodness-of-fit. Waggoner (1997) extents this method by introducing a *variable roughness penalty*.

Other, more relevant variations of cubic splines are the *quadratic-natural cubic spline* proposed by McCulloch and Kochin (2000) and the *Bessel* or *Hermite method* discussed in De Boor (2001). The former method achieves a more stable curve in the long end as opposed to the *natural cubic spline* method, see Burden and Faires (1997, chapter 3.4), that tends to have a 'roller coaster' output curve. This is done by setting the endpoint constraints natural at the long end but quadratic at the short end (Hagan and West, 2006, p.100). The latter method, i.e. the *Hermite method*, requires not just the values of the interpolating functions but also their first derivatives at the node points. Hence, a cubic Hermite interpolant is a piecewise cubic polynomial interpolant with a continuous first derivative.

Despite considerable popularity in financial institutions and in software packages, cubic splines suffer from some well-known problems. There is no guarantee that

cubic splines preserve any convexity or monotonicity that may characterize the original data. Occasionally, cubic splines introduce excess convexity or spurious inflection points. Moreover, cubic interpolations may suffer from a lack of locality, meaning a local pertubation of curve input data modifies sections of the discount curve far away from the pertubed data point (Andersen, 2007, p.229).

Hyman (1983) used a cubic Hermite method to develop a practical algorithm that ensures that in regions of monotonicity in the input data, the interpolating function preserves this property. He introduced a filter that removes most of the unpleasant waviness. The monotone preserving cubic spline is a local method. Hagan and West (2008) state that this approach does not explicitly ensure strictly positive forward rates. Nevertheless, this technique will be examined in more detail in section 3.4.4.

Adams (2001) argues in favour of a *quartic spline* as the smoothest forward rate interpolation scheme. However, his method lacks in two points: firstly, it requires a set of instantaneous forward rates as input and secondly, it demands such high smoothness criteria that any desired stiffness is completely lost from the system (Hagan and West, 2006, p.104).

Recently, Andersen (2007) introduced an approach based on the works of Tanggaard (1997) and Kvasov (2000). He uses a *hyperbolic tension spline* that allows the smooth manipulation of locality and shape preservation, and thereby to overcome the problems of cubic spline interpolation. Here, a tension is added to each end point of a cubic spline as a pulling force. By increasing the tension, excess convexity and spurious inflection points are gradually reduced until the curve eventually approaches a linear spline (Andersen, 2007, p.229).

Most recently, Hagan and West (2008) developed a new interpolation scheme where the spline is constructed based on forward rates such that the interpolated curve is locally monotone and convex if the inputs show the analogous discrete properties. This so called *monotone convex spline* will be investigated more thoroughly in section 3.4.5.

3.4.2 Desirable features

Before some of the above introduced interpolation schemes can be surveyed, it is necessary to determine the criteria for evaluating each scheme. Naturally, it is a prerequisite that each interpolation function is able to reconstruct the inputs at each node for the bootstrapped curve to be seriously considered. Hagan and West (2008) propose to take the following features into consideration:

1. Are the forward rates positive? In order to avoid arbitrage it is necessary to ensure non-negative forward rates.

2. Are the forward rates continuous? Continuity is required to price interest sensitive instruments such as derivatives.

3. How local is the interpolation method? Locality prevails if a change in the input data changes the interpolation function only nearby.

4. How stable is the interpolation scheme? The degree of stability is estimated as a maximum basis point change in the interpolation curve given some basis point change in one of the inputs.

5. How local are hedges? By setting up a portfolio that shall provide an adequate hedge against more general moves in the underlying, it is crucial that the hedge still works when a change in one of the inputs occurs.

The primary focus is to ensure continuous and positive forward rates for pricing interest rate derivatives. Smoothness of the forwards is desired, but should not be achieved at the expense of the other criteria mentioned above. Naturally, hedging and pricing of derivatives goes hand in hand, nevertheless the last criteria will not be discussed as hedging is beyond the scope of the thesis.

3.4.3 Linear interpolation spline

Interpolating spot rates piecewise linearly for $t_{n-1} \leq t \leq t_n$ is done by the following equation:

$$R_t = \frac{t - t_{n-1}}{t_n - t_{n-1}} R_{t_n} + \frac{t_n - t}{t_n - t_{n-1}} R_{t_{n-1}} \tag{3.39}$$

Now, revising equation 3.2 the forward rate can be rewritten as:

$$F_{t;T_{n-1},T_n} = \frac{R_{t_n} t_n - R_{t_{n-1}} t_{n-1}}{t_n - t_{n-1}} \tag{3.40}$$

To this end, denoting f_t as the instantaneous forward rate at time t, i.e. $f_t = \lim_{\epsilon \to 0} f_{0;t,t+\epsilon}$, it must hold that

$$f_t = \frac{d}{dt} R_t \; t \; . \tag{3.41}$$

Consequently, inserting equation 3.39 in 3.41 yields:

$$f_t = \frac{2t - t_{n-1}}{t_n - t_{n-1}} R_{t_n} + \frac{t_n - 2t}{t_n - t_{n-1}} R_{t_{n-1}} \tag{3.42}$$

Comparing both interpolation formulas for the spot and forward rate, i.e equations 3.39 and 3.42, reveals that by the time t reaches t_n, R_{n-1} has been reduced to zero in

the formula for the spot rate whereas this is clearly not the case for the forward rate. This results in the forward curve jumping at each node t_n which is an undesirable feature.

Applying linear interpolation on log spot rates is remarkably popular, being provided as one of the default methods by many software vendors (Hagan and West, 2006, p.96). Equation 3.39 can be modified in the following way:

$$ln(R_t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} ln(R_{t_n}) + \frac{t_n - t}{t_n - t_{n-1}} ln(R_{t_{n-1}}) \tag{3.43}$$

Taking the exponential, it can be simplified as:

$$R_t = R_{t_n}^{\frac{t-t_{n-1}}{t_n - t_{n-1}}} R_{t_{n-1}}^{\frac{t_n - t}{t_n - t_{n-1}}} \tag{3.44}$$

Finally, interpolating on the logarithm of discount factors can be conducted similarly through:

$$P_t = P_{t_n}^{\frac{t-t_{n-1}}{t_n - t_{n-1}}} P_{t_{n-1}}^{\frac{t_n - t}{t_n - t_{n-1}}} \tag{3.45}$$

This method corresponds to piecewise constant forward curves and is occasionally called *raw* or *exponential interpolation*. As it is very stable and trivial to implement it is often used to identify mistakes in fancier models (Hagan and West, 2008, p.5). By construction, raw interpolation has a constant instantaneous forward rate on each interval $t_{n-1} \leq t \leq t_n$ that must be equal to the discrete forward rate. Thus, the raw method guarantees that all instantaneous forward rates are positive, this is not the case for the interpolation on log spot rates.

Linear interpolation on log rates or log discount factors are popular choices that lead to stable and fast bootstrapping procedures (Ametrano and Bianchetti, 2009, p.21). Piecewise linear splines have an excellent degree of locality. Unfortunately, they produce insufficient forward rates with a 'zig-zag' or piecewise-constant shape. McCulloch and Kochin (2000) point out that a discontinuous forward curve implies either implausible expectations about future short-term interest rates or implausible expectations about holding period returns.

3.4.4 Cubic hermite spline with a Hyman filter

The choice of the cubic Hermite interpolant is superior compared to the usual cubic spline as this interpolant may have a more pleasing visual appearance and allows to preserve monotonicity if the original data is monotonic (Heath, 1997, p.234).

Firstly, a description of the piecewise polynomial Hermite interpolant will be

given. Here, all x_{t_n} that belong to the interval $[x_{t_1}, x_{t_N}]$ have corresponding data points defined by $f_{t_n} = f(x_{t_n})$. The local mesh spacing can be defined as

$$\Delta f_{t_{n+1/2}} = f_{t_{n+1}} - f_{t_n} \quad , \quad \Delta x_{t_{n+1/2}} = x_{t_{n+1}} - x_{t_n} \tag{3.46}$$

where the slope of the piecewise linear interpolant between the data points is

$$\Delta S_{t_{n+1/2}} = \frac{\Delta f_{t_{n+1/2}}}{\Delta x_{t_{n+1/2}}} \, . \tag{3.47}$$

The data is *locally monotone* at x_{t_n} if $S_{t_{n+1/2}} S_{t_{n-1/2}} > 0$ whereas the interpolant is *piecewise monotone* if $P(x_t)$ is monotone between f_{t_n} and $f_{t_{n+1}}$ for x_t between x_{t_n} and $x_{t_{n+1}}$ (Hyman, 1983, p.646). The cubic Hermite interpolant is defined for $t_1 \leq t_n < t_N$ as:

$$P(x_t) = c_1 + (x_t - x_{t_n})c_2 + (x_t - x_{t_n})^2 c_3 + (x_t - x_{t_n})^3 c_4 \tag{3.48}$$

Given the data points f_{t_n} a numerical approximation of the slope f'_{t_n} at x_{t_n} is needed for $t_1 \leq t_n \leq t_N$ in order to estimate the coefficients where $x_{t_n} \leq x_t \leq x_{t_{n+1}}$:

$$\begin{aligned} c_1 &= f_{t_n} \\ c_2 &= f'_{t_n} \\ c_3 &= \frac{3S_{t_{n+1/2}} - f'_{t_{n+1}} - 2f'_{t_n}}{\Delta x_{t_{n+1/2}}} \\ c_4 &= \frac{2S_{t_{n+1/2}} - f'_{t_{n+1}} - f'_{t_n}}{\Delta x^2_{t_{n+1/2}}} \end{aligned} \tag{3.49}$$

Interestingly, 3.48 becomes a local interpolation formula once f'_{t_n} is given. If changes should be made to either f_{t_n} or f'_{t_n}, the interpolant changes only in the region $[x_{t_{n-1}}, x_{t_{n+1}}]$. Here, localness is a desirable feature if just a few data points need to be readjusted as it avoids recalculating the interpolation function at all data points. Hence, to gain total localness for 3.48 global continuity in the second derivative must be sacrificed (Hyman, 1983, p.646).

Approximations of f'_{t_n} can be done either locally or non-locally. The former uses only f_{t_n} values near x_t to calculate the derivative, whereas the latter uses all f_{t_n} to obtain the derivative by solving a linear equation system. Hyman (1983) proposes different local methods to approximate the first derivative such as Akima, Fritsch-Butland, Parabolic and Fourth-order finite difference. Following his advice to apply the parabolic method for an unequally spaced mesh as it provides the

highest accuracy, the following equation will be used:

$$f'_{t_n} = \frac{\Delta x_{t_{n-1/2}} S_{t_{n+1/2}} + \Delta x_{t_{n+1/2}} S_{t_{n-1/2}}}{x_{t_{n+1}} - x_{t_{n-1}}} \quad , \quad t_1 < t_n < t_N \tag{3.50}$$

At the boundaries the parabolic methods uses an uncentered difference approximation to determine the derivative at t_1 and t_N:

$$f'_{t_n} = \frac{(2\Delta x_{t_{n+1/2}} \Delta x_{t_{n+3/2}}) S_{t_{n+1/2}} - \Delta x_{t_{n+1/2}} S_{t_{n+3/2}}}{\Delta x_{t_{n+1/2}} + \Delta x_{t_{n+3/2}}} \quad , \quad t_n = t_1 \tag{3.51}$$

$$f'_{t_n} = \frac{(2\Delta x_{t_{n-1/2}} \Delta x_{t_{n-3/2}}) S_{t_{n-1/2}} - \Delta x_{t_{n-1/2}} S_{t_{n-3/2}}}{\Delta x_{t_{n-1/2}} + \Delta x_{t_{n-3/2}}} \quad , \quad t_n = t_N \tag{3.52}$$

Filtering f'_{t_n} according to equation 3.53 before interpolating with equation 3.48 will retain the important local monotonic properties of the data.

$$f'_{t_n} = \begin{cases} min[\, max(0, f'_{t_n}), 3\, min(|\, S_{t_{n-1/2}} \,|, |\, S_{t_{n+1/2}} \,|)] & f'_{t_n} \geq 0 \\ max[\, min(0, f'_{t_n}), -3\, min(|\, S_{t_{n-1/2}} \,|, |\, S_{t_{n+1/2}} \,|)] & f'_{t_n} < 0 \end{cases} \tag{3.53}$$

This simple constraint can convert an unacceptable geometric interpolant into an excellent one. If the data is convex, a good geometric interpolant should preserve this convexity (Hyman, 1983, p.648-654). Ametrano and Bianchetti (2009) found the classic Hyman monotonic cubic filter applied to spline interpolation on log discount factors to be the easiest and best approach. In theory, there is no mechanism which ensures that the generated forward rates are positive (Hagan and West, 2008, p.8).

3.4.5 Monotone convex spline

Introducing the monotone convex method, Hagan and West (2008) base their work on ideas from Hyman (1983) but now explicitly guarantee continuous forward rates that are positive. The main difference is that their interpolation algorithm is based on the interpolation between forward rates and not spot rates or discount factors. This section follows the structure and content of the paper by Hagan and West (2008, p.8-13).

Given spot rates as inputs, discrete forward rates will be calculated as

$$f^d_{t_n} = \frac{R_{t_n} t_n - R_{t_{n-1}} t_{n-1}}{t_n - t_{n-1}} \tag{3.54}$$

such that $f^d_{t_n}$ belongs not only to time t_n but to the entire interval $[t_{n-1}, t_n]$. The

instantaneous forward rate for time t_n is determined as:

$$f_{t_n} = \frac{t_n - t_{n-1}}{t_{n+1} - t_{n-1}} f^d_{t_{n+1}} + \frac{t_{n+1} - t_n}{t_{n+1} - t_{n-1}} f^d_{t_n} \quad , \quad i = 1, 2, \ldots, n-1 \tag{3.55}$$

Similarly, the boundaries are selected so that $f'(t_0) = 0 = f'(t_N)$:

$$f_{t_0} = f^d_{t_1} - \frac{1}{2}(f_{t_1} - f^d_{t_1}) \tag{3.56}$$

$$f_{t_N} = f^d_{t_N} - \frac{1}{2}(f_{t_{N-1}} - f^d_{t_N}) \tag{3.57}$$

Hence, if the discrete forward rates are positive so is f_{t_n} for $n = 1, 2, \ldots, N-1$. The next step is then to define an interpolation function f on the interval $[t_0, t_N]$ for $f_0, f_1, \ldots, f_N$ that satisfies the following conditions (arranged in a decreasing order of necessity):

i) $\frac{1}{t_n - t_{n-1}} \int_{t_{n-1}}^{t_n} f_t \, dt = f^d_{t_n}$ so the discrete forward is recovered by the curve.

ii) f is positive.

iii) f is continuous.

iv) If $f^d_{t_{n-1}} < f^d_{t_n} < f^d_{t_{n+1}}$ then $f(t)$ is increasing $[t_{n-1}; t_n]$ and if $f^d_{t_{n-1}} > f^d_{t_n} > f^d_{t_{n+1}}$ then $f(t)$ is decreasing on $[t_{n-1}, t_n]$.

Hagan and West (2008) propose a normalized function g defined on $[0, 1]$:

$$g(x) = f(t_{n-1} + (t_n - t_{n-1})x) - f^d_{t_n} \tag{3.58}$$

Setting $x = \frac{t - t_{n-1}}{t_n - t_{n-1}}$ and rearranging yields:

$$f(t) = f^d_{t_n} + g\left(\frac{t - t_{n-1}}{t_n - t_{n-1}}\right) \tag{3.59}$$

Here, the function f represents the interpolation function and is determined by the discrete forward rate plus an adjustment factor estimated from the function g. Hagan and West (2008) choose the function of g to be piecewise quadratic such that it by construction satisfies conditions i) and iii), and where g is adjusted to satisfy iv). A posteriori, ii) is satisfied if the other constraints are satisfied.

Determining the piecewise quadratic function for g of the functional form $g(x) =$

$K + Lx + Mx^2$ is done by setting up three equations with three unknowns, i.e.

$$g(0) = f_{t_{n-1}} - f^d_{t_n} \tag{3.60}$$
$$g(1) = f_{t_n} - f^d_{t_n} \tag{3.61}$$
$$0 = \int_0^1 g(x)dx \tag{3.62}$$

which can be solved to define the function for g as

$$g(x) = g(0)[1 - 4x + 3x^2] + g(1)[-2x + 3x^2] \, . \tag{3.63}$$

The subscript on g is disregarded as g is adjusted piecewise for each interval. The function g is differentiated such that the slope can be determined for a given x in the interval $[t_{n-1}, t_n]$.

$$g'(x) = g(0)(-4 + 6x) + g(1)(-2 + 6x) \tag{3.64}$$
$$g'(0) = -4g(0) - 2g(1) \tag{3.65}$$
$$g'(1) = 2g(0) + 4g(1) \tag{3.66}$$

Here, determining the slope at the start and end point of each interval is crucial in order to ensure monotonicity, i.e. condition iv). By applying equations 3.65 and 3.66 one can estimate the solution of $g'(0) = 0$ and $g'(1) = 0$ which gives the following:

$$g'(0) = 0 \quad \rightarrow \quad g(1) = -2g(0) \tag{3.67}$$
$$g'(1) = 0 \quad \rightarrow \quad g(0) = -2g(1) \tag{3.68}$$

The resulting two lines divide the $g(0)/g(1)$ plane into eight sectors. Here, modifying g in each sector is essential such that it ensures monotonicity as well as continuity. The latter is guaranteed by ensuring that the boundaries of any two sectors coincide. The treatment for every diametrically opposite pair of sectors is the same which reduces the adjustment on g to only four cases. This is illustrated in figure 3.2.

(i) In these sectors, $g(0)$ and $g(1)$ are of opposite signs and $g'(0)$ and $g'(1)$ are of the same sign, so g is monotone and does not need to be modified.

(ii) In these sectors, all values $g(0)$ and $g(1)$, as well as $g'(0)$ and $g'(1)$ are of opposite sign, meaning g is currently not monotone and therefore needs to be adjusted. Furthermore, the formulas for (i) and (ii) need to agree on the boundary $\mathcal{A}$ to ensure continuity.

(iii) The situation here is the same as in the previous case. Now the formulas for

(i) and (iii) need to agree on the boundary $\mathcal{B}$ to ensure continuity.

(iv) In these sectors, $g(0)$ and $g(1)$ are of the same sign so at first it appears that g does not need to be modified. Unfortunately, this is not the case. Modification will be needed to ensure that the formulas for (ii) and (iv) agree on $\mathcal{C}$ and likewise, the ones for (iii) and (iv) agree on $\mathcal{D}$.

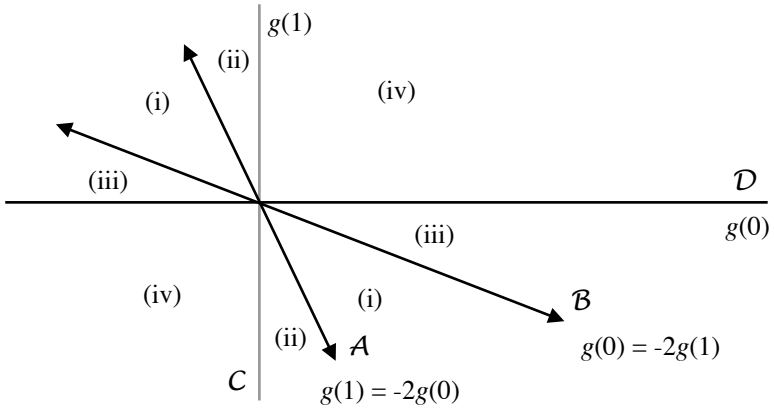


Figure 3.2: The reformulated possibilities for g

Source: Hagan and West (2008, figure 4)

The origin is a special case. If $g'(0) = 0 = g'(1)$ then $g(x) = 0$ for all x, and $f^d_{t_{n-1}} = f^d_{t_n} = f^d_{t_{n+1}}$ such that $f(t) = f^d_{t_n}$ for t ϵ $[t_{n-1}, t_n]$.

Adjusting the function g for each sector is done accordingly:

(i) The function g is not modified in this sector. On $\mathcal{A}$, g is equal to $g(x) = g(0)(1 - 3x^2)$ and on $\mathcal{B}$ we have $g(x) = g(0)(1 - 3x + \frac{3}{2}x^2)$.

(ii) Adjusting g is done by inserting a flat segment which changes to a quadratic at exactly the right moment to ensure that $\int_0^1 g(x) = 0$.

$$g(x) = \begin{cases} g(0) & 0 \leq x \leq \eta \\ g(0) + (g(1) - g(0)) \left(\frac{x-\eta}{1-\eta}\right)^2 & \eta \leq x \leq 1 \end{cases} \tag{3.69}$$

$$\eta = \frac{g(1) + 2g(0)}{g(1) - g(0)} \tag{3.70}$$

As required, g reduces to $g(x) = g(0)(1-3x^2)$ on $\mathcal{A}$ as $\eta \to 0$ if $g(1) \to -2g(0)$ which ensures that the boundaries between sector (i) and (ii) coincide.

(iii) Again, a flat segment is inserted.

$$g(x) = \begin{cases} g(1) + (g(0) - g(1)) \left(\frac{\eta - x}{\eta}\right)^2 & 0 \leq x \leq \eta \\ g(1) & \eta \leq x \leq 1 \end{cases} \tag{3.71}$$

$$\eta = 3\frac{g(1)}{g(1) - g(0)} \tag{3.72}$$

As required, g reduces to $g(x) = g(0)(1 - 3x + \frac{3}{2}x^2)$ on $\mathcal{B}$ as $\eta \to 1$ if $g(1) \to -\frac{1}{2}g(0)$ which ensures that the boundaries between sector (i) and (iii) coincide.

(iv) Here, the function g is determined such that it reduces to the one defined in (ii) as it approaches $\mathcal{C}$, and to the one defined in (iii) as it approaches $\mathcal{D}$.

$$g(x) = \begin{cases} A + (g(0) - A) \left(\frac{\eta - x}{\eta}\right)^2 & 0 \leq x \leq \eta \\ A + (g(1) - A) \left(\frac{x - \eta}{1 - \eta}\right)^2 & \eta \leq x \leq 1 \end{cases} \tag{3.73}$$

$$\eta = \frac{g(1)}{g(1) - g(0)} \tag{3.74}$$

$$A = -\frac{g(0)g(1)}{g(0) + g(1)} \tag{3.75}$$

Here, the first line satisfies (iii) as $\mathcal{A} = 0$ if $g(1) = 0$ and likewise, the second line satisfies (ii) $A = 0$ if $g(0) = 0$.

Similarly, the functional form for the spot rate $r(t)$ can be determine based on the given functional form for $f(t)$. However, this involves an adjustment of the function g in each of the four sectors such that monotonicity and continuity keep guaranteed. Taking this into consideration, the functional form for the spot rate is then given by:

$$\begin{aligned} r(t)t &= \int_0^t f(s)ds \\ r(t)t &= \int_0^{t_{n-1}} f(s)ds + \int_{t_{n-1}}^t f(s)ds \\ r(t)t &= r_{t_{n-1}} t_{n-1} + \int_{t_{n-1}}^t f(s)ds \end{aligned}$$

$$r(t)t = r_{t_{n-1}}t_{n-1} + (t - t_{n-1})f^d_{t_n} + \int_{t_{n-1}}^{t} g(s)ds$$
$$r(t) = \frac{1}{t}\left(r_{t_{n-1}}t_{n-1} + (t - t_{n-1})f^d_{t_n} + \int_{t_{n-1}}^{t} g(s)ds\right) \tag{3.76}$$

where n is found such that $t_{n-1} \leq t < t_n$ and $f(t) = f^d_{t_n} + g\left(\frac{t-t_{n-1}}{t_n - t_{n-1}}\right)$, as defined in equation 3.59.

3.4.6 Evaluation of Interpolation techniques

Before all interpolation schemes can be evaluated with respect to the desirable features introduced in section 3.4.2, it is necessary to specify how stable each interpolation scheme is. This can be done by measuring how much the interpolated curve changes, if one of the inputs changes. Here, the norm $M(r)$ is measured on spot rates - both as inputs and outputs, i.e.

$$\parallel M(r) \parallel = \sup_t \max_{t_n} \left|\frac{\delta r(t)}{\delta R_{t_n}}\right| \tag{3.77}$$

where the input is defined as R_{t_n} and the bootstrapped output is $r(t)$. The maximum difference in the suprenum norm is estimated between the two bootstrapped curves (one with unchanged inputs and one with a change in one of the inputs) by testing at discrete points along the curve in steps of 10/360. Table 3.7 shows the norms for each interpolation method in basis points if e.g. the input spot rate of the 3-year swap rate is increased by one basis point.

Table 3.7: Stability of interpolation methods, Norm in bps

Instrument	Linear on log discount factors	Linear on log spot rates	Cubic Hermite with a Hyman filter	Monotone convex
Eurodollar Fut 09/2010	0.9	0.9	1.4	1.1
Eurodollar Fut 03/2011	1.0	1.0	1.3	1.2
Swap 3Y	1.0	1.0	1.0	1.0
Swap 5Y	1.0	1.0	1.0	1.0

Note: The input spot rate of each instrument is changed by +1 bp.

Ideally, a desirable maximum change in the output curve would similarly be one basis point. The *linear interpolation methods* are therefore superior to both more advanced methods, *cubic Hermite spline with a Hyman filter* and the *monotone convex spline*, in terms of stable outputs. However, they produce non-continuous

forward rates which makes them inappropriate to price interest rate derivatives. Table 3.8 summarizes the different properties of each interpolation scheme.

Table 3.8: Comparison of interpolation methods

Interpolation spline	Forwards positive?	Forward smoothness	Method local?	Method stable?
Linear on log discount factors	yes	not continuous	excellent	excellent
Linear on log spot rates	no	not continuous	excellent	excellent
Cubic Hermite with a Hyman filter	no	continuous	very good	good
Monotone convex	yes	continuous	very good	good

Source: Hagan and West (2008, table 1)

It can be inferred that both the *cubic Hermite spline with a Hyman filter* and the *monotone convex spline* produce continuous forward rates but are still characterized by a very good locality and a good degree of stability compared to *linear splines* on log spot rates and log discount factors.

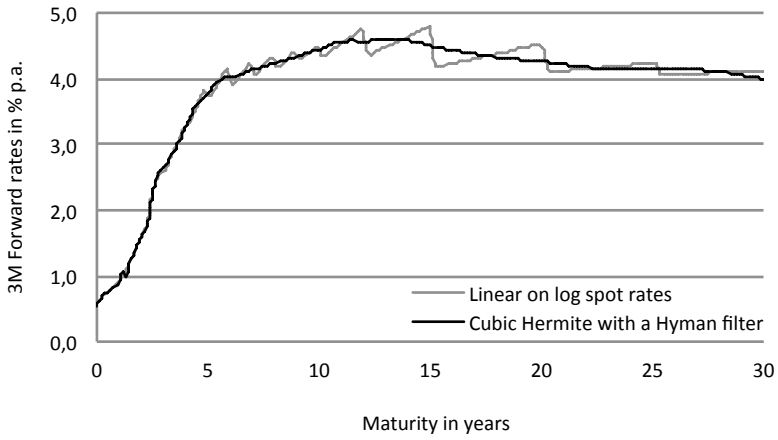


Figure 3.3: Linear on log spot rates vs Cubic Hermite with a Hyman filter

The resulting 3M US Dollar forward curves as they are displayed in figures 3.3 and 3.4 underscore the superiority of the more advanced methods. They show a smooth behaviour whereas the linear methods produce jagged forward curves as theoretically predicted.

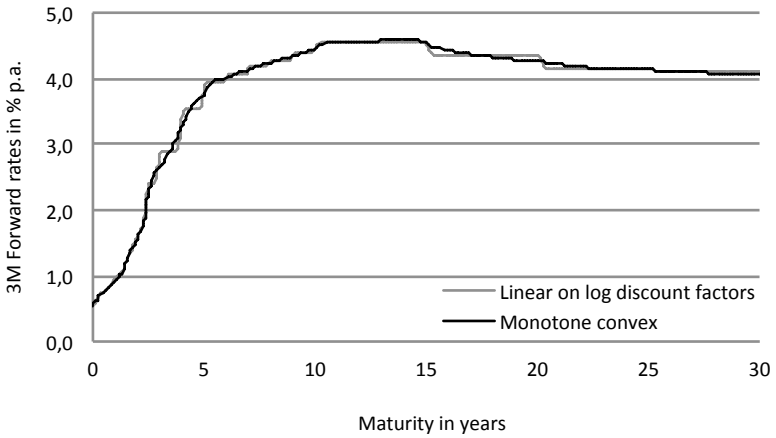


Figure 3.4: Linear on log discount factors vs Monotone convex

Although the *cubic Hermite spline with a Hyman filter* has a similar smooth behaviour as the *monotone convex spline*, the latter is the method of choice as it is the only method, according to Hagan and West (2008), where simultaneously

(1) all input instruments are exactly reproduced as outputs of the bootstrapping procedure,

(2) the forward curve is guaranteed to be positive if the inputs are positive and arbitrage free, and

(3) the forward curve is continuous.

Furthermore,

(4) the method is local, i.e. changes in inputs at a certain location do not affect in any way the output at other locations, and

(5) the method is stable, i.e. as inputs change, the outputs change more or less proportionately.

Finally, interpolating between the key rates of the spot curve bootstrapped in section 3.3 by applying the *monotone convex spline* yields the complete spot curve shown in figure 3.5.

Figure 3.5: The complete spot curve

3.5 Turn of year effect

The turn is the period between the last business day of the current calendar year and the first business day of the new year. Normally, the turn of year effect is a jump observed in market quotations of rates spanning across the end of the year. Ametrano and Bianchetti (2009) acknowledge that the turn effect is due to the increased search for liquidity by financial institutions due to regulatory requirements just before the periodic balance sheet is announced.

The jump that is caused by the turn effect varies with maturity of the underlying deposit rates. Typically, deposit rates with short maturities cause larger jumps. Here, the Overnight deposit rate is exposed to the largest jumps as such a deposit runs from the last business day of the current year until the first business day of the following year. Similarly, jumps in Tomorrow Next and Spot Next deposit rates occur one and two business days before. Longer maturing deposit rates experience similar although smaller jumps when crossing the year border. The reason for decreasing jumps with increasing tenors is the weight given to the jumping rate. Consider the 1M deposit rate as a weighted average of 22 Overnight rates. If the 1M deposit spans across the end of year, there will be a single Overnight rate with the weight of 1/22 that crosses the year end and displays the jump. Hence, the turn of year effect is smaller for longer tenors. Table 3.9 presents historical turn of year effects for different tenors. Noticeably, the turn of year effect for the Overnight (O/N) rate for the years 09/10 and 08/09 are negative. This results seems counterintuitive but might be a result of other dominating factors.

The turn effects for the 1M US Dollar Libor rate, illustrated in figure 3.6, are marked with two rectangular shapes. Here, the jumps occur two business days before the first business day of December, i.e. at the 29th November 2007 and

Table 3.9: Turn of year effects in bps (US Dollar)

Year	O/N	1M	3M
09/10	-1.1	0.1	0.7
08/09	-0.4	46.9	12.1
07/08	32.3	40.3	3.2
06/07	3.6	2.9	0.5
Average	8.6	22.6	4.1

Source: Datastream

27th November 2008 respectively. It is obvious that these jumps are of significant amplitude and thus need to be taken into consideration.

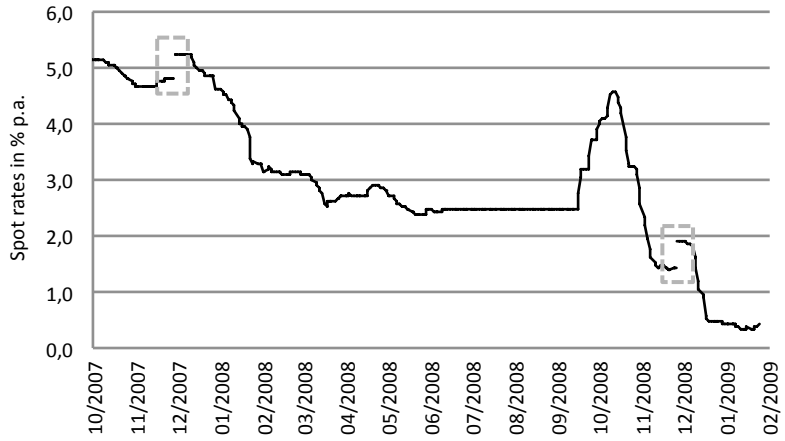


Figure 3.6: The turn of year effect on 1M US Dollar Libor rate

Ametrano and Bianchetti (2009) state that the turn of year effect is generally only observable at the end of the first two years and becomes negligible thereafter. Implementing these two turn effects is done by adding the average historical jump of the 3M US Dollar Libor rate from the last four years, i.e. 4.1 basis points, to the spot rates from January 2011 throughout the whole term structure and additionally 50 percent of that as a roughly chosen jump for the second year to the spot rates from January 2012 onwards. This choice can be supported by two facts: firstly, the instruments in the middle area of the spot curve are based on a 3M underlying tenor meaning it is straightforward to apply a historical value from the same time series, and secondly, modeling jumps through more advanced models does not necessarily yield better estimates as the evolvement of interest rates is highly complex and beyond the scope of the thesis. Consequently, keeping it simple serves the purpose of introducing turn of year effects but is by no means a standard practitioners

approach.

The 2011 and 2012 turn of year jumps are clearly observable in the resulting forward curve displayed in figure 3.7. Accordingly, these jumps can be found in the spot curve as well where each turn effect induces just one discontinuity opposed to two discontinuities in the forward curve.

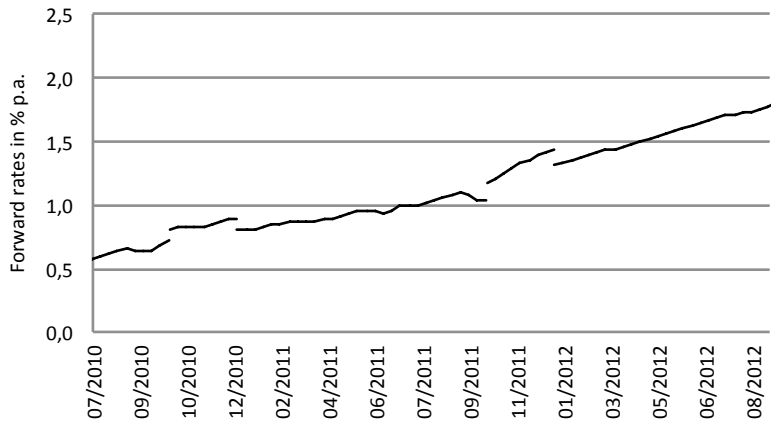


Figure 3.7: The forward curve including turn of year effects

The discontinuities that are introduced by the turn of year effect may seem counterintuitive in relation to the previous section that stressed the importance of a smooth continuous forward curve. Despite this fact, the turn of year effect simply corresponds to the true and detectable market effects that should be in any yield curve used to mark-to-market interest rate derivatives (Ametrano and Bianchetti, 2009, p.27).

3.6 Deriving the swap curve

Finally, having implemented the turn effects in the previous section the swap curve can now be derived from equation 3.10 based on the adjusted discount factors. The swap curve for the US market as of June 30th 2010 is illustrated in figure 3.8.
Here, the derived and quoted swap curve coincide only with few basis points deviation. It is at the three to four year maturities that the maximum difference of minus five basis points occur where the derived swap curve exceeds the market swap curve. Interestingly, the derived swap curve becomes smaller than the market swap curve at the maturity of 15 years and onwards, though only with one basis point at its maximum. From this it is clear that the market swap curve can be reproduced when applying an appropriate methodology.

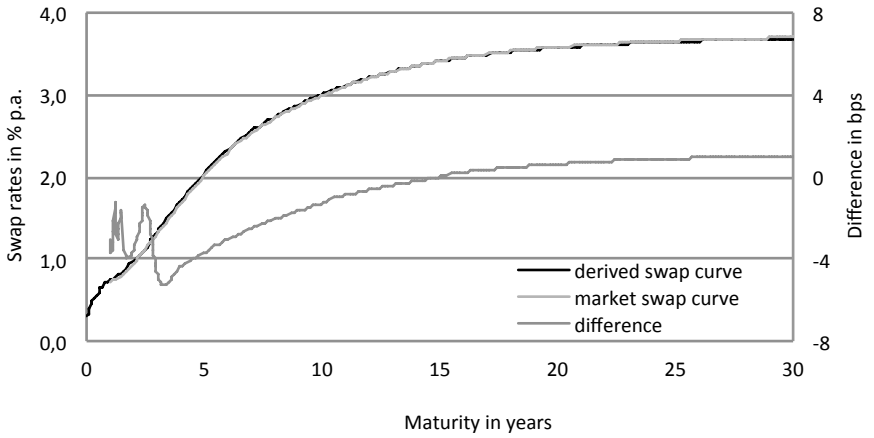


Figure 3.8: Reconstruction of the US Dollar swap curve

Concluding this chapter, the pre-crisis pricing framework of interest rate swaps does in fact incorporate different basis spreads though only when settling a *single* swap contract. This means there is no consistent pricing framework as neither the tenor nor the cross-currency basis spreads are incorporated in the forward curve. At this point, the swap rate even for a plain interest rate swap does not reflect the market uncertainties appropriately. Hence, the applicability of the derived swap curve is questionable in the pre-crisis pricing framework.

4

Distress in the financial markets

Prior to August 2007, interest rates quoted in the market were consistent with what is known from standard textbooks. The market quotes of Forward Rate Agreements had a precise relationship with the spot Libor rates which they are indexed to. Similarly, Libor and Overnight Index Swap (OIS) rates were chasing each other closely. Lastly, flows of interest rate payments with the same maturity that differ only in their payment frequency were considered equivalent, apart from a very little basis spread (Morini, 2009, p.2).

However, a series of events should trigger distress in the financial markets: Bear Stearns announced in late July 2007 that two of its hedge funds lost nearly their entire values and just a few days later BNP Paribas closed two of its funds because it could not value the assets in them. The European Central Bank responded by pumping billions of Euro in the money market and likewise other Central Banks began to intervene (Guillén, 2009, p.1). The liquidity crisis widened the basis, so that market rates that were formerly consistent with each other suddenly revealed a degree of incompatibility that worsened as time passed by (Mercurio, 2009, p.2).

Forward rates implied by two consecutive deposits started to differ significantly from the quoted FRA rates. Simultaneously, the gap between Libor and OIS rates rose to an extent that is no longer negligible. Similarly, the spread between two swap rates with the same maturity but based on different underlying tenors experienced a huge increase. Also cross-currency basis spreads were not unaffected by the increased tension in the money market. All these rates which were so closely interconnected, suddenly diverged, with each one now incorporating its own liquidity and credit premium according to Mercurio (2009).

The purpose of the following sections is to examine the above mentioned indicators for distress in the financial market during the financial crisis in more detail and thus to underscore the necessity of a multi-curve pricing framework for interest rate derivatives.

4.1 FRA and implied forward rates

As introduced in the theoretical framework, section 3.1, the forward rate $F_{t;T_{n-1},T_n}$ can be estimated from two spot rates $R_{t,T_{n-1}}$ and R_{t,T_n} in the following way:

$$F_{t;T_{n-1},T_n} = \left[\frac{(1 + R_{t,T_n})^{T_n}}{(1 + R_{t,T_{n-1}})^{T_{n-1}}} \right]^{\frac{1}{\delta_n}} - 1$$

Here, δ is the daycount fraction for the interval $[T_{n-1}, T_n]$.

Although the real-world interbank market is not populated by completely risk less banks, the risk in the interbank lending market was considered negligible as the large majority of the banks had a very low risk profile. Designated banks that contribute to the Libor panels are selected to be the upper part of the banking world in terms of credit standing, reputation and activity in the cash markets. A population that was considered virtually risk less before the crisis (Morini, 2009, p.5). Thus, the Libor rate L_{t,T_n} was regarded as a good approximation to the risk free spot rate R_{t,T_n}. The equation for the forward rate is then:

$$F_{t;T_{n-1},T_n} = \left[\frac{(1 + L_{t,T_n})^{T_n}}{(1 + L_{t,T_{n-1}})^{T_{n-1}}} \right]^{\frac{1}{\delta_n}} - 1 = \frac{1}{\delta_n} \left(\frac{P_{t,T_{n-1}}}{P_{t,T_n}} - 1 \right)$$

Furthermore, the payoff of a long position in a FRA contract at maturity T_n is known to be $V_{T_n} = \delta_n \left(L_{T_{n-1},T_n} - K \right)$. The FRA is quoted through its equilibrium rate $F_{t;T_{n-1},T_n} = E_t[L_{T_{n-1},T_n}]$, which in theory should correspond to the FRA rate K making such a deal fair at initiation. Figure 4.1 depicts the spread between quoted FRA rates and their equivalent implied forward rates estimated from Libor rates.

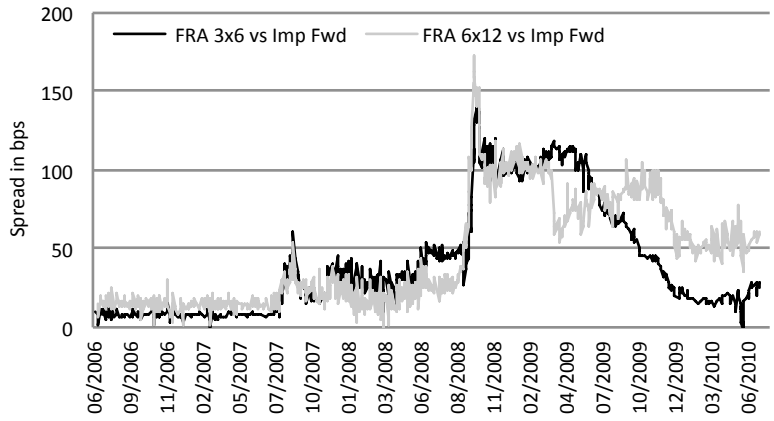


Figure 4.1: US Dollar FRA vs implied forward rate

Source: Datastream

The spread between the 3x6 FRA rate and the implied 3M forward rate stayed in the range between 0 and 10 basis points up until August 2007. From this point on the standard replication forward rate has exceeded the market FRA rate remarkably. The spread moved erratically around different levels, averaging to 53.2 basis points between August 2007 and June 2010. A similar behaviour can be noticed for the spread between the 6x12 FRA rate and the implied 6M forward rate, though not recovering as much since July 2009 as in the previous case.

Surprisingly, these spreads do not necessarily lead to arbitrage opportunities. The following sketch is inspired by Mercurio (2009). Denoting the FRA rate and the forward rate implied by two deposits with maturities T_1 and T_2 as F_{FRA} and F_{Libor} respectively, and assuming that $F_{Libor} > F_{FRA}$ also seen from figure 4.1, the following strategy could be applied to take advantage from such an arbitrage opportunity:

a) Buy $(1+\delta_n F_{Libor})$ deposits with maturity T_2, paying $(1+\delta_n F_{Libor})P_{t,T_2} = P_{t,T_1}$ US Dollar

b) Sell 1 deposit with maturity T_1, receiving P_{t,T_1} US Dollar

c) Enter a long position in a FRA, paying out at time T_1

$$\frac{\delta_n(L_{T_1,T_2} - F_{FRA})}{1+\delta_n L_{T_1,T_2}}$$

Naturally, the value of this strategy at initiation is zero. The value at time T_1 can be estimated by b) plus c), i.e.

$$\frac{\delta_n(L_{T_1,T_2} - F_{FRA})}{1+\delta_n L_{T_1,T_2}} - 1 = -\frac{1+\delta_n F_{FRA}}{1+\delta_n L_{T_1,T_2}}$$

which is negative if rates are assumed to be positive. This residual debt can be payed by selling $1+\delta_n F_{Libor}$ deposits with maturity T_2. Hence, the amount of

$$\frac{1+\delta_n F_{Libor}}{1+\delta_n L_{T_1,T_2}} - \frac{1+\delta_n F_{FRA}}{1+\delta_n L_{T_1,T_2}} = \frac{\delta_n(F_{Libor} - F_{FRA})}{1+\delta_n L_{T_1,T_2}} > 0$$

will remain in cash at T_1, which is equivalent to $\delta_n(F_{Libor} - F_{FRA})$ received at T_2. Whenever a zero investment today generates a positive, risk free gain in the future, this is defined as arbitrage. Here, the gain at T_1 is stochastic whereas the gain at T_2 is deterministic. However, this FRA replication strategy lacks some issues that under current market conditions cannot be neglected anymore:

i) The counterparties in a) and c) are exposed to default risk.

ii) There exists the possibility of a liquidity crunch at time 0 or at T_1.

iii) Regulatory requirements with regards to capital injections may influence the credit worthiness of market participants.

Either of these events could mean a loss at T_2 that may exceed the positive gain from the replication strategy. Therefore, this strategy does not necessarily constitute an arbitrage opportunity as it is no longer risk free. The divergence in both forward rates F_{Libor} and F_{FRA} can be seen as representative of the market estimate of future credit and liquidity issues (Mercurio, 2009, p.7).

A second example that formerly equivalent rates differ in the aftermath of the financial crisis, can be inferred solely from the market for FRAs. Based on given market quotes for both a 1x4 and a 4x7 FRA, it is possible to estimate the rate for a 1x7 FRA by the following formula:

$$F_{Imp}^{1x7} = \frac{(1 + \delta_n^{1x4} F_{FRA}^{1x4})(1 + \delta_n^{4x7} F_{FRA}^{4x7}) - 1}{\delta_n^{1x7}}$$

Now, comparing this estimate with the quoted FRA rate in the market, it can be seen from figure 4.2 that the latter exceeds the implied rate substantially since September 2007, averaging to 20.6 basis points until June 2010. This divergence demonstrates that each underlying tenor contains its own credit and liquidity premia. In fact, the premia for the 6M tenor is larger than for the 3M tenor as will be explained in section 4.3.

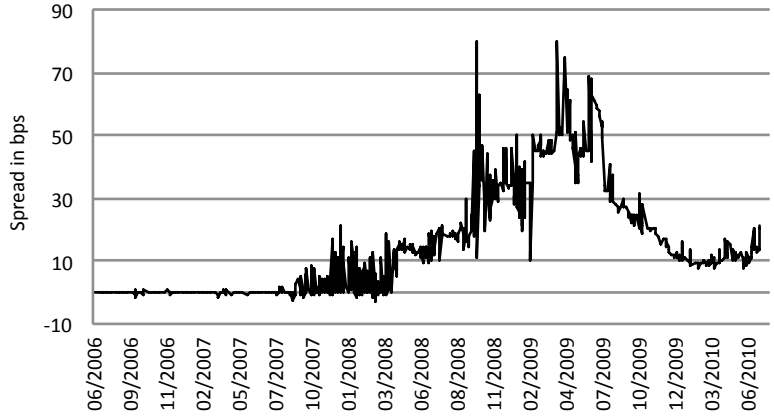


Figure 4.2: US Dollar 1x7 FRA vs implied forward rate from 1x4 and 4x7 FRAs

Source: Datastream

4.2 Libor-OIS spread

An OIS is a fixed vs. floating interest rate swap with the floating leg tied to a published index of a daily overnight reference rate, i.e. the Federal Funds rate in the US market and EONIA in the Euro market. As the OIS rate can be considered as lending for a very short period of time, it is typically associated with almost negligible liquidity or credit risk (Morini, 2009, p.10). Furthermore, the OIS rate typically gives an indication of the market's expectations towards future lending transactions over the swap term. Oppositely, the Libor fixing is meant to capture the rates paid on unsecured interbank deposit at large, internationally active banks (Michaud and Upper, 2008, p.48). The Libor quotes should give an indication for the typical default or liquidity risk contained by players in the Libor-world. Consequently, the spread between the OIS and the US Libor rate can be considered as an indicator for the credit and liquidity risks that may affect counterparties when lending for periods longer than one day.

Morini (2009) stresses that liquidity problems, besides credit issues, are among the main reasons for explaining the Libor-OIS spread. Building a common understanding, Acerbi and Scandolo (2008) specify three kinds of liquidity risk: funding liquidity risk, market liquidity risk and systemic liquidity risk. Similarly, Michaud and Upper (2008) decompose the risk premium into several factors reflecting the characteristics of the borrowing bank as well as market-wide conditions:

$$riskpremium = tprem + credit + bliq + mliq + micro$$

Here, bank specific variables are used to distinguish between the default risk ($credit$) and a premium related to the funding ability of the borrowing bank ($bliq$). Market-wide conditions include the uncertainty in the overnight rates ($tprem$), the liquidity in the market ($mliq$) and factors related to the fixing process and the microstructure of the market ($micro$).

Generally, distinguishing between the important factors that define the risk premium can be tricky considering the fact that there are no financial instruments that are directly related to any of the individual factors proposed. Moreover, funding liquidity risk for a bank is usually strongly correlated to its risk of default as funding problems lead to higher funding costs, making it more likely that the bank will default. In fact, one has to be careful to draw too precise a line between credit and liquidity risk, though it may lead to an unnecessary multiplication of the actual risk factors (Morini, 2009, p.11). Challenges in distinguishing between liquidity and credit risks are addressed in Duffie and Singleton (1997) and Collin-Dufresne and Solnik (2001).

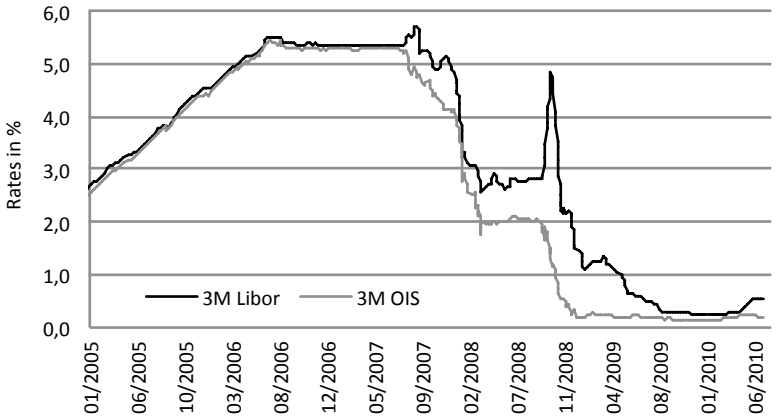


Figure 4.3: US Dollar 3M Libor vs 3M OIS rate

Source: Bloomberg

The relationship between the 3M US Dollar Libor and the 3M OIS rate is shown in figure 4.3. Obviously, both rates tracked each other very closely before the financial crisis emerged.

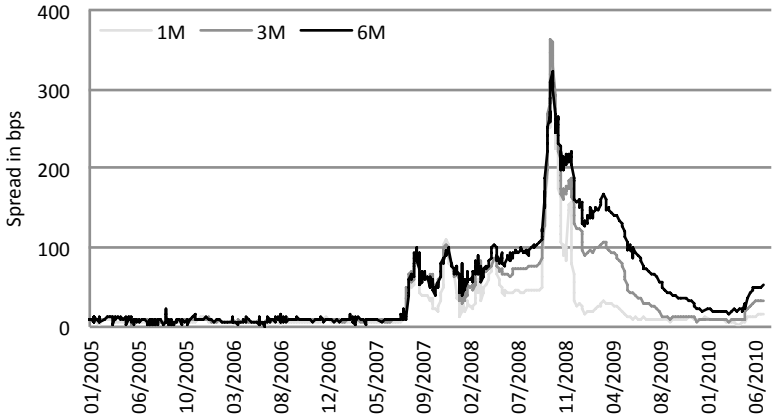


Figure 4.4: US Dollar Libor - OIS spreads

Source: Bloomberg

Figure 4.4 illustrates the US Dollar Libor-OIS spread not just for 3M rates but also for the 1M and 6M rates. Here, the spreads for the different tenors have been at constant levels at roughly 10 bps up until August 2007. From that point on losses linked to sub-prime mortgages in the US started to emerge. Noticeably, the spreads reach their maximum in the early stages of September 2008, just after the US government seizes Fannie Mae and Freddie Mac and when Lehman Brothers announced a USD 4 billion loss eventually leading to their collapse. Obviously,

the significance in the spreads increases for longer tenors. Not until the beginning of 2010 did the spreads return to more normal levels. In April 2010 the financial problems in Greece reached a turning point forcing the IMF to give a Euro 110 billion loan. As a consequence of the financial problems in Greece, the spreads experienced a steep increase at that point, introducing new unforeseen risks to the market. From the US Libor-OIS spreads in figure 4.5, it is clear that the market on June 30th 2010 still is filled with credit and liquidity problems.

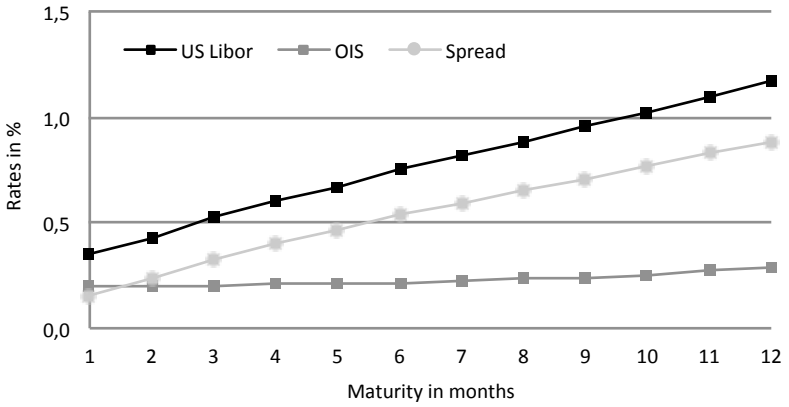


Figure 4.5: US Dollar Libor - OIS spreads June 30th 2010

Source: Datastream

Given the Libor-OIS spread was almost negligible before the financial crisis it was considered reasonable to apply both rates as risk free rates. The significant increases in the Libor-OIS spread starting around August 2007 has led to the OIS being a more appropriate measure for a risk free rate. It has therefore become popular among practitioners to build a risk free term structure based on the OIS as opposed to the Libor rate (Morini, 2009, p.10).

4.3 Tenor basis spread

The exchange of floating rate payments based on two different indices is known as money market basis swap, tenor swap or basis swap. If both indices are default free rates then such a swap should trade flat. However, this is not the case anymore as shown in the previous sections 4.1 and 4.2. The gap between FRA market quotes and the corresponding replicated forward rate can be bridged using tenor basis spreads, recalling that a swap is merely a portfolio of FRAs. These spreads are estimated as the difference between the fixed legs of two interest rate swaps with

matching maturities, the same reference rate but with different fixing frequencies of the floating legs.

Although no-one indeed borrows or lends cash in a basis swap and the counterparty default risk normally is mitigated by requiring collateral, the payments are indexed to a risky, unsecured Libor that has a built-in credit premium. Large spreads need to be paid by the payer of the leg with shorter tenor to the payer of the leg with longer tenor, and became especially important during the financial crisis. Market participants justify this by an axiom already introduced in Tuckman and Porfirio (2003):

Lending at 12M Libor involves more counterparty/liquidity risk than rolling lending at 6M Libor.

In a risky world, the counterparty that receives payments with the longer tenor (12M) suffers a higher counterparty/liquidity risk that will be compensated by a higher market level of the 12M Libor compared to the one implied by the 6M Libor. Assuming the counterparty/liquidity risk can be eliminated by collateralization and indexing to Libor rather than lending, this higher level of the 12M Libor is not justified anymore by a higher risk. Consequently, the receiver of the leg with the longer tenor in a basis swap will have to compensate this advantage by adding a spread to the leg with the shorter tenor (Morini, 2009, p.19). This also explains why the FRA rate always needs to be lower than or equal to the corresponding replicated forward rate. Naturally, this explanation can be used analogically for other tenors.

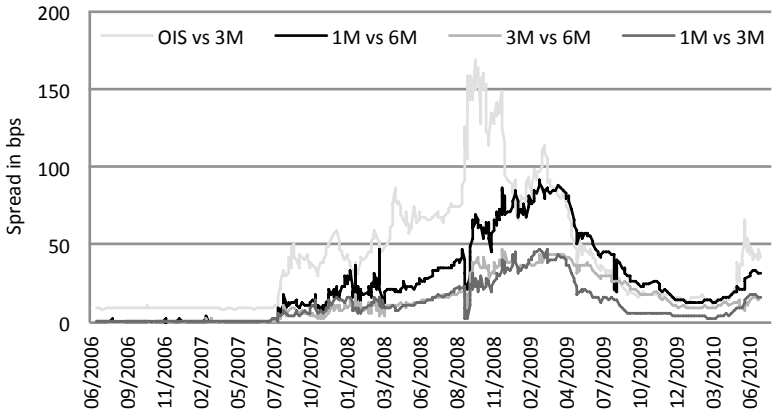


Figure 4.6: US Dollar 1-year tenor basis spreads

Source: Bloomberg

Figures 4.6 and 4.7 present the evolvement of the tenor basis spreads OIS vs 3M,

1M vs 3M, 3M vs 6M and 1M vs 6M for a maturity of 1 and 10 years, respectively. It can easily be seen that the tenor spreads were never higher than approximately 10 basis points until August 2007. Hereafter, all tenor spreads increased substantially. Interestingly, the OIS vs 3M spread peaked exactly in September 2008 when Lehman Brothers went into bankruptcy. This is in line with the patterns found in the Libor-OIS spread and the gap between FRA market rates and their replicated forward rates. However, the remaining tenor spreads (1M vs 3M, 3M vs 6M and 1M vs 6M) reacted oppositely and went down. Here, especially the spreads for the 10-year maturity decreased significantly.

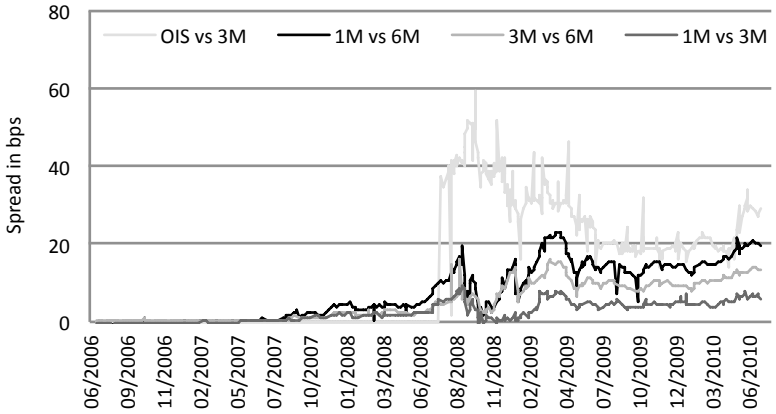


Figure 4.7: US Dollar 10-year tenor basis spreads

Source: Bloomberg

Other observations from figures 4.6 and 4.7 are that the OIS vs 3M spread is of highest magnitude and amplitude compared to other spreads. More generally, spreads decrease for longer maturities. For example, the spread for a OIS vs 3M basis swap with a maturity of 1 year at June 30th 2010 is 42.6 basis points and thus larger than the spread for the same swap with a maturity of 10 years, i.e. 29.1 basis points.

Morini (2009) invokes various fundamental explanations to justify the axiom that lending at a longer tenor involves more counterparty/liquidity risk than rolling lending at a shorter tenor. The following points elaborate on these explanations and try to analyse the foundations of the axiom from a qualitative point of view.

a) **Lower loss due to default**
Considering a 12M lender and a 6M roller, and a default in the period from 6-12 months from today, then the 12M lender loses all the interest whereas the 6M roller loses only the interest for the 6-12 month period as the interest for 0-6 month period was already cashed in. Although both lenders lose their notional, the 6M roller is clearly better off.

b) **Exiting at par when credit conditions worsen**
Again considering the same two lenders, the 6M roller is in a superior position if the credit quality of the borrower worsens as he can stop lending with no cost after 6 months. Instead the 12M lender would have to unwind its deposit at a cost that incorporates the increased risk of default. However, this also implies that the 12M lender has a monetary advantage if the credit standing of the borrower improves. Thus, the expected gain of the 6M roller compared to the 12M lender when the counterparty's credit worthiness worsens is compensated by its expected loss when the counterparty's credit worthiness improves.
This symmetry can be broken down by taking bid-ask spreads and commercial reasons into consideration. Firstly, bid-ask spreads restrain a counterparty from permanently exiting and entering a position at a theoretical fair value. Secondly, it is not common in the financial industry to unwind a financing contract since the counterparty's reputation has a strong impact on its funding costs. Also a downgrade of the borrower's rating might have negative consequences with respect to the lender's regulatory capital. Admittedly, both facts are difficult to quantify.

c) **Liquidity advantage in 6M lending**
It can be advantageous to exit at par after 6 months since the lender may be in need of funding liquidity. This argues in favor of the 6M roller over the 12M lender. Here, the same considerations as in b) apply, meaning there exists a symmetry in whether it is favorable to be either the 6M roller or the 12M lender. However, there are elements in reality that break down the symmetry. One factor is the bid-ask spread, the other one is a bias towards unwinding when it already involves a fair value loss but this only is true if the unwinding is performed for credit reasons. In case of funding liquidity reasons, the symmetry is broken only if a correlation between credit and liquidity risk is assumed. Disentangling credit and liquidity factors is not a clear cut case as mentioned before (Michaud and Upper, 2008, p.38).

d) **Libor anomalies**
Madigan (2008) states that rumors have been circulating about banks purposely misreporting their lending rates to the BBA in an attempt to drive benchmark rates down. Another and maybe more severe incentive for Libor panel banks to quote a lower interest rate publicly than they would be prepared to pay in a private transaction, is to forestall questions that might be asked in relation to their own credit and liquidity situation. Michaud and Upper (2008) enforce this speculation by pointing out that banks with increasing credit risk, measured in the CDS market, do not appear to have quoted significantly higher Libor rates than banks with lower credit risk. Either credit risk had no impact on the funding of Libor banks or the Libor did not reflect the actual funding cost in particular for distressed banks. This hints that the Libor was not a reliable indicator for interbank borrowing during the financial crisis.

e) **Banks in the Libor panel are revised**
If the credit standing of a bank in the Libor panel worsens, its borrowing rate cannot be representative anymore and the bank will be excluded from the panel. The BBA is committed to review the panels at least twice a year to ensure that just the upper part of the banking world in terms of credit standing, reputation and activity in the cash-markets is represented in the panel. Similarly, when replicating the flows of the shorter tenor leg of a basis swap, the lender has the option to check after the 6 months if the credit standing of the borrower is still sufficient enough to belong to the uppermost group of banks in terms of credit standing. Thus, the 6M leg implicitly embeds this bias towards the uppermost group of banks.

f) **Turn-of-the-year effect**
When an announcement date of the financial reports is between inception and maturity of a deal, it is more likely that one of the counterparties will have its credit standing revised during the life of the deal. This automatically puts more pressure on longer tenor rates versus shorter tenor rates.

All reasons listed in the points from a) to f) evidently explain why it is not reasonable to assume a zero tenor basis spread when entering a basis swap. Summarizing this section, it is important to outline that the counterparty risk in the shorter leg (e.g. 6M) is lower than in the longer leg (e.g. 12M) because the replication strategy of the 6M leg includes the possibility of moving to a better counterparty in terms of credit standing after 6 months. Consequently, the survival probability of the borrower of the 6M leg in the 6-12M period is higher than the survival probability of the

borrower of the 12M leg in the same period. This is why, on average, the 6M leg embeds less counterparty risk than the 12M leg, justifying a positive basis spread to be added on the shorter leg when the flows are replicated in a collateralized basis swap free of counterparty risk (Morini, 2009, p.23).

4.4 Cross-currency basis spread

A cross-currency basis swap is essentially an exchange of a floating rate in one currency for a floating rate in another currency. Cross-currency swaps serve several purposes where the most dominant one is transforming a liability, i.e. typically a loan, or an asset from one currency into another. In a *constant notional* CCS the two counterparties exchange notional values at the actual FX spot rate both at initiation and at termination in the two currencies. To reduce the FX risk that is inherent in the *constant notional* CCS, participants in the interbank market often trade a *mark-to-market* CCS where the notional is adjusted according to the current FX spot rate at each fixing point of the underlying. The change in the notional is payed out in the currency that is marked-to-market (Linderstrøm and Rasmussen, 2011, p.16). As the currency risk is reduced in the *mark-to-market* CCS, the market risk is isolated to only depend on the relative change in the CCS basis spread. Typically, the cross-currency basis spread is added to the weaker currency in a CCS, e.g. on the leg of the Euro in relation to the US Dollar. Why this is the case follows from the following.

Having a fictional *constant notional* cross-currency basis swap that exchanges a risk free overnight rate in one currency for a risk free overnight rate in another currency, the swap must therefore trade flat according to Tuckman and Porfirio (2003) because both legs can be replicated by floating rate notes paying a risk free rate. Intuitively, when paying for 1 unit of currency A at initiation, receiving the risk free rate associated to that currency A and receiving 1 unit of currency A at maturity, this must be worth 1 unit of currency A today. Hence, the fixing and discounting rate are the same resulting in a payoff today equal to the initial notional value. Similarly, this is also the case for the other leg of the swap, where the notional is determined by the FX spot rate. Consequently, both floating rate notes trade at par and that is why the exchange of the two risk free overnight rates in the fictional CCS is fair standing today.

Opposed to the fictional swap, actual cross-currency swaps exchange Libor or other benchmark rates. Here, Tuckman and Porfirio (2003) consider a cross-currency swap of Libor rates as a portfolio of three imaginary swaps: a cross-currency basis swap of overnight, risk free rates; a money market basis swap of domestic Libor

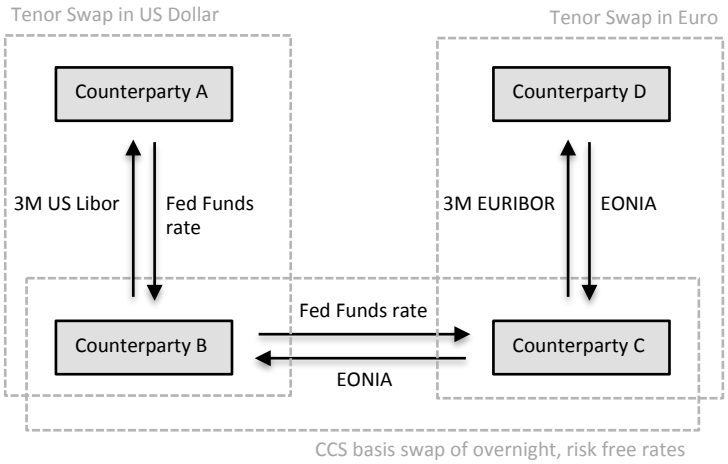


Figure 4.8: Decomposing a CCS

versus domestic overnight, risk free rates; and a money market basis swap of foreign Libor versus foreign overnight, risk free rates - see figure 4.8 exemplarily for the cross USD/EUR. This indicates that the observed cross-currency basis swap spreads stem from the difference between the domestic and foreign term structures of credit spreads. In determining to which leg the spread is added, one may consider an example. Given that the domestic 3M Libor contains more credit risk than the foreign 3M Libor, then in a risk neutral world, a stream of domestic Libor payments would be worth more than the stream of foreign Libor payments. Consequently, the foreign Libor plus a spread would trade fair agains the domestic Libor (Tuckman and Porfirio, 2003, p.4).

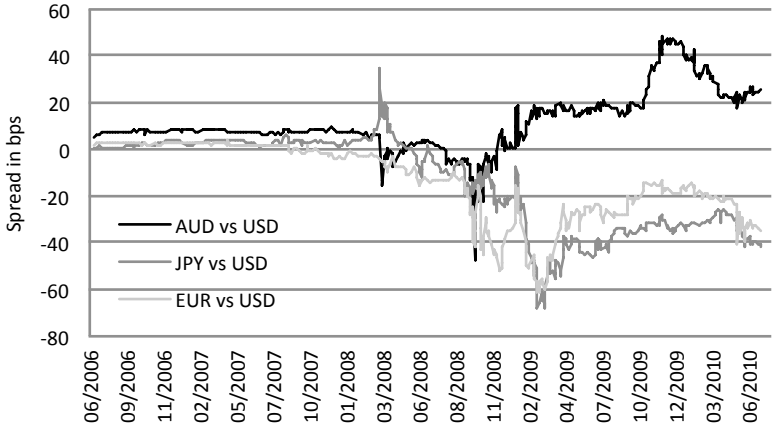


Figure 4.9: CCS 5-year basis spreads

Source: Bloomberg

Figure 4.9 illustrates the 5-year cross-currency basis swap spreads for the Aus-

tralian Dollar, Japanese Yen and the Euro, all against the US Dollar. Here, all three spreads have been small, positive and steady at their respective levels up until the beginning of 2008. The Euro vs US Dollar spread becomes and stays negative from November 2007 until the end of the period, experiencing the largest difference on March 3rd 2009 with -61.3 basis points. Interestingly, the Yen vs US Dollar spread experiences a significant increase to 34.3 basis points the 18th of March 2008, i.e. two days after the US government orchestrated and backed up the acquisition of Bear Stearns by JP Morgan Chase in order to calm the markets down, afterwards becoming negative and only occasionally positive with a few basis points. This spread stays negative from November 14th 2008 and shows a similar behavior to the Euro vs US Dollar spread. Likewise, other cross-currency basis spreads against the US Dollar such as the British Pound and Swiss Franc experienced a behavior similar to the Euro spread. Oppositely, the spread for the Australian Dollar is mostly positive, only subject to significant decreases in 2008, and returns to a higher positive value throughout the observation period. Moreover, the spreads increase for longer maturities. At the climax of the financial crisis, the negativity in the cross-currency basis spreads are a sign of increased demand for US Dollar liquidity (Linderstrøm and Rasmussen, 2011, p.16). Furthermore, Fujii, Shimada, and Takahashi (2009c) state that it is most probable that there is a dominant contribution from the liquidity factor in CCS basis spreads. Consequently, forcing the inclusion of unreasonably high credit risk among Libor rates and currencies of developed nations to explain the wide CCS basis in the market, even in the very short maturities. Concluding this section, the various spreads against the US Dollar have moved from constant levels around a few basis points before the crisis to levels significant different from those levels after the crisis.

Recapping this chapter, the financial markets experienced distress never seen before the financial crisis emerged. Indicators such as the gap between FRA market rates and their replicated forward rates, Libor-OIS spreads, tenor and cross-currency basis spreads moved to levels far away from what was known to the markets beforehand. These developments challenge traditional pricing and hedging methodologies that were used throughout the financial industry.

5

Post-crisis pricing framework

The explosion of basis spreads quoted in the market for single-currency interest rate swaps reflects a preference for higher frequency payments from the market participants, e.g. quarterly instead of semiannually. Lenders require a higher compensation for taking longer dated risk and borrowers are willing to pay this compensation in order to reduce the liquidity risk when re-financing. These frictions have induced a segmentation of the interest rate market into sub areas, mainly corresponding to instruments with OIS, 1M, 3M, 6M and 12M underlying tenor rates. Each of them is in principle determined by different internal dynamics, liquidity and credit risk premia, reflecting the different expectations and interests of market participants. Furthermore, Bianchetti (2010) stresses that the market segmentation was already present and well understood before the credit crunch, but not effective due to negligible basis spreads.

Generally, this evolution in the financial markets has triggered a revision of the methodology applied to price and hedge interest rate derivatives that depend on the present value of interest rate-linked cash flows. The pre-crisis approach outlined in chapter 3 is no longer consistent as it does not take the market information carried by basis swap spreads into account, which are now much larger than in the past and thus no longer negligible. Similarly, the segmentation of the interest rate market is not taken into consideration. Consequently, pricing and hedging interest rate derivatives on a single yield curve that consists of several different underlying tenors can lead to dirty results, as the yield curve incorporates the different dynamics and eventually the inconsistencies of distinct market areas, making prices and hedge ratios less stable and more difficult to interpret (Bianchetti, 2010, p.7).

Another paradox the financial industry has been struggling with is that discounting must be unique, i.e. two identical future cash flows of whatever origin must display the same present value. Other than that, valuing a cross-currency basis swaps and a plain vanilla swap creates arbitrage between both instruments. Therefore a framework is needed such that both instruments can be valued to par without inconsistencies (Chibane and Sheldon, 2009, p.1). Unfortunately, such a framework is not easy to construct as a consistent credit and liquidity theory to

account for the interest rate market segmentation does not yet prevail. Such a theory would also explain why the asymmetries examined e.g. in section 4.1 do not necessarily lead to arbitrage opportunities once counterparty and liquidity risks are taken into account (Bianchetti, 2010, p.7).

Instead of explicitly modeling credit and liquidity effects which would expose the pricing of derivatives even more to model risk, Mercurio (2009) states that practitioners seem to agree on an empirical approach. They deal with the above discrepancies by segmenting market rates, constructing as many forward curves as possible tenors and generating future cash flows through the curves associated to the underlying tenor. These are then discounted by applying a unique discounting curve that represents the funding costs of the respective financial institution. Assuming different curves for different rate lengths immediately invalidates the classic pricing approach, which was built on the cornerstone of a unique and fully consistent zero-coupon curve, used both in the generation of future cash flows and in the calculation of their present values (Mercurio, 2009, p.2).

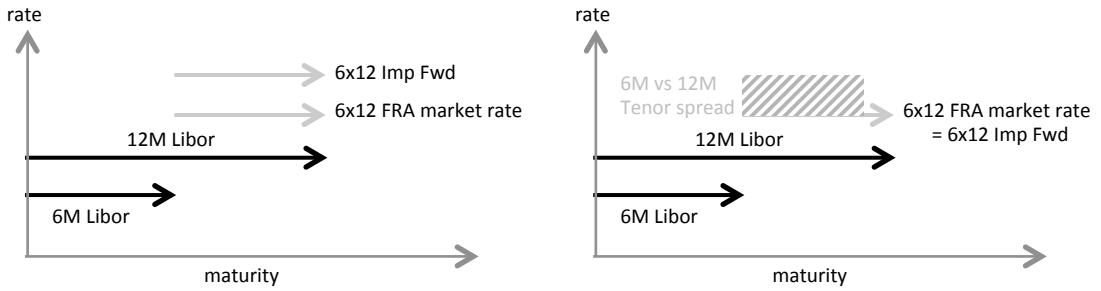
The lost analogies and inconsistencies are reconciled into a common pricing framework by properly taking cross-currency and tenor basis spreads, that originate from quotes in the swap markets, into account. This chapter constitutes the main part of the thesis and starts with revisiting the no-arbitrage condition and subsequently introducing a new theoretical pricing framework. Lastly, the theory from the post-crisis pricing framework will be applied to derive forward surfaces and discounting curves, enabling the pricing of various kinds of interest rate swaps to be consistent with each other.

5.1 The no-arbitrage condition

The large discrepancies between the market FRA rates and their implied forward rates during the financial crisis have a significant impact on the theoretical framework for pricing interest rate derivatives when applying the standard replication method. Formerly, it was commonly agreed on that placing money over a 12M period should yield the same as investing in two 6M deposits as plotted in figure 5.1.

From section 4.1 it is clear that the standard textbook relation shown in equation 5.1 no longer holds:

$$F_{t;T_{n-1},T_n} = \frac{1}{\delta_n}\left(\frac{P_{t,T_{n-1}}}{P_{t,T_n}} - 1\right) \tag{5.1}$$



(a) Standard replication (b) Basis-consistent replication

Figure 5.1: No-arbitrage condition

Consequently, the purpose of this section is to construct a basis-consistent replication of FRA rates, incorporating the increased credit and liquidity premia in the financial markets. This is done by replicating a FRA contract with a tenor swap.

Recalling from section 3.1 that tenor swaps are quoted as two standard fixed-for-floating swaps with the same fixed leg and different floating legs, where the spread τ_N denotes the time t market spread of the length N between the two fixed legs with the same maturity. This is similar to considering two floating legs where one leg pays at a higher frequency than the other leg. Here, the tenor spread is subtracted from the lower frequency leg. As swaps are collateralized contracts that suffer no risk of default but are indexed to Libor rates that are now perceived as risky, it is important to take this into consideration (Morini, 2009, p.15).

Inspired by Morini (2009), defining the frequencies of the two floating legs as α and 2α, e.g. 6M (α) and 12M (2α), the price of a tenor swap with maturity $T_N = 2\alpha$ can be computed as the expectation of the Libor-dependent payoff discounted with the risk free rate:

$$
\begin{aligned}
&TS_{t,2\alpha,\tau_{2\alpha}} \\
&= E_t[P_{t,\alpha}\alpha L_{t,\alpha} + P_{t,2\alpha}\alpha L_{\alpha,2\alpha} - P_{t,2\alpha}2\alpha(L_{t,2\alpha} - \tau_{2\alpha})] \\
&= E_t[P_{t,2\alpha}\alpha L_{\alpha,2\alpha}] + P_{t,\alpha}\alpha L_{t,\alpha} - P_{t,2\alpha}2\alpha(L_{t,2\alpha} - \tau_{2\alpha}) \\
&= E_t[P_{t,2\alpha}\alpha L_{\alpha,2\alpha}] + P^{OIS}_{t,\alpha}\alpha\frac{1}{\alpha}\left(\frac{1}{P^L_{t,\alpha}} - 1\right) - P^{OIS}_{t,2\alpha}2\alpha\left(\frac{1}{2\alpha}\left(\frac{1}{P^L_{t,2\alpha}} - 1\right) - \tau_{2\alpha}\right) \\
&= E_t[P_{t,2\alpha}\alpha L_{\alpha,2\alpha}] + P^{OIS}_{t,\alpha}\left(\frac{1}{P^L_{t,\alpha}} - 1\right) - P^{OIS}_{t,2\alpha}\left(\left(\frac{1}{P^L_{t,2\alpha}} - 1\right) - 2\alpha\tau_{2\alpha}\right) \\
&= E_t[P_{t,2\alpha}\alpha L_{\alpha,2\alpha}] - P^{OIS}_{t,2\alpha}\left(\frac{1}{P^L_{t,2\alpha}} - 1 - 2\alpha\tau_{2\alpha} - \frac{P^{OIS}_{t,\alpha}}{P^{OIS}_{t,2\alpha}}\left(\frac{1}{P^L_{t,\alpha}} - 1\right)\right) \qquad (5.2)
\end{aligned}
$$

Here, P^{OIS} denotes the risk free discounting factor which from section 4.2 in most cases can be estimated from the OIS rate and where P^L is the corresponding Libor

discounting factor. If $\tilde{K}(\tau_{2\alpha})$ is defined as

$$\tilde{K}(\tau_{2\alpha}) = \left(\frac{1}{P^L_{t,2\alpha}} - 1 - 2\alpha\tau_{2\alpha} - \frac{P^{OIS}_{t,\alpha}}{P^{OIS}_{t,2\alpha}} \left(\frac{1}{P^L_{t,\alpha}} - 1 \right) \right) / \alpha \tag{5.3}$$

then equation 5.2 reduces to

$$TS_{t,2\alpha,\tau_{2\alpha}} = E_t[P_{t,2\alpha}\alpha L_{\alpha,2\alpha}] - P^{OIS}_{t,2\alpha}\tilde{K}(\tau_{2\alpha})\alpha \ . \tag{5.4}$$

Now, consider a collateralized FRA contract given in the following equation:

$$FRA^{Col}_{t;\alpha,2\alpha,K} = E_t[P_{t,2\alpha}\alpha(L_{\alpha,2\alpha} - K)] \tag{5.5}$$

Interestingly, the tenor swap in equation 5.4 is equal to a collateralized FRA contract when $K = \tilde{K}(\tau_{2\alpha})$ and where the spread is given by $\tau_{2\alpha}$. Both the FRA contract and the tenor swap involve the exchange of two legs. One leg is deterministic and fixed today. For the FRA that is the payment at 2α and for the basis swap it is the payment of the 2α leg at 2α less the first payment of the α leg, only leaving one stochastic leg that for both contracts corresponds to the payment of $L_{t,\alpha,2\alpha}$ at α.

As the tenor swap in equation 5.4 must equal zero at initiation, the defined $\tilde{K}(\tau_{2\alpha})$ from equation 5.3 must replicate the FRA rate using only deterministic factors. Consequently, constructing a basis-adjusted FRA rate is done by analyzing the defined $\tilde{K}(\tau_{2\alpha})$:

$$\begin{aligned}
\tilde{K}(\tau_{2\alpha}) &= \left(\frac{1}{P^L_{t,2\alpha}} - \frac{P^{OIS}_{t,\alpha}}{P^{OIS}_{t,2\alpha}} \frac{1}{P^L_{t,\alpha}} + \frac{P^{OIS}_{t,\alpha}}{P^{OIS}_{t,2\alpha}} - 1 - 2\alpha\tau_{2\alpha} \right) / \alpha \\
&= \frac{1}{P^L_{t,\alpha}} \left(\frac{P^L_{t,\alpha}}{P^L_{t,2\alpha}} - \frac{P^{OIS}_{t,\alpha}}{P^{OIS}_{t,2\alpha}} \right) + \left(\frac{P^{OIS}_{t,\alpha}}{P^{OIS}_{t,2\alpha}} - 1 - 2\alpha\tau_{2\alpha} \right) / \alpha \\
&= F^{OIS}_{t;\alpha,2\alpha} + \frac{1}{P^L_{t,\alpha}}(F^L_{t;\alpha,2\alpha} - F^{OIS}_{t;\alpha,2\alpha}) - 2\tau_{2\alpha}
\end{aligned} \tag{5.6}$$

Alternatively this yields:

$$\tilde{K}(\tau_{2\alpha}) = F^L_{t;\alpha,2\alpha} + \left(\frac{1}{P^L_{t,\alpha}} - 1 \right) (F^L_{t;\alpha,2\alpha} - F^{OIS}_{t;\alpha,2\alpha}) - 2\tau_{2\alpha} \tag{5.7}$$

From the above result, a FRA contract is fairly priced when K is set equal to $\tilde{K}(Z_{2\alpha})$ where $Z_{2\alpha}$ is the equilibrium value for the tenor spread $\tau_{2\alpha}$. Hence, a basis-consistent

replication of a FRA rate can be determined by:

$$F^{Z}_{t;\alpha,2\alpha} = F^{L}_{t;\alpha,2\alpha} + \left(\frac{1}{P^{L}_{t,\alpha}} - 1\right)(F^{L}_{t;\alpha,2\alpha} - F^{OIS}_{t;\alpha,2\alpha}) - 2Z_{2\alpha} \tag{5.8}$$

During the crisis, the substantial difference between the market FRA rate and the standard replicated one originates from the significant increase in the tenor spreads $Z_{2\alpha}$ (Morini, 2009, p.16). Despite the presence of considerable tenor spreads, the FRA rates can be correctly replicated using equation 5.8, exemplarily illustrated for the 6x12 FRA in figure 5.2.

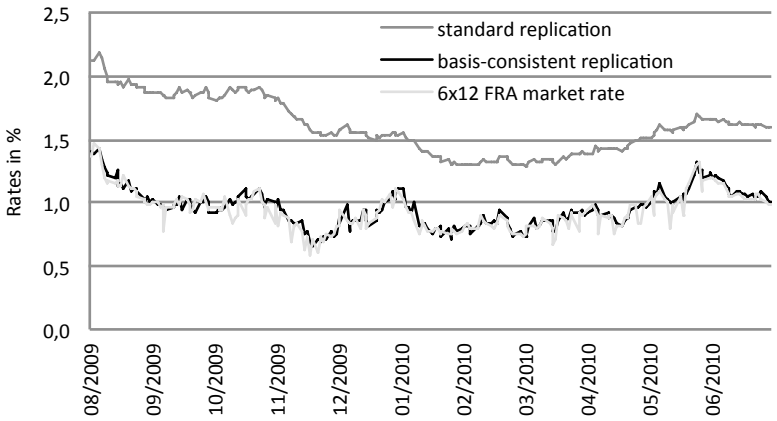


Figure 5.2: Basis-consistent replication of 6x12 FRA rates

Source: Bloomberg, Nordea Analytics

5.2 The theoretical framework

The basis-consistent replication of FRA rates can be extended to construct a framework where interest rate swaps, tenor swaps and cross-currency swaps can be priced consistently with each other. Firstly, the case of uncollateralized swaps will be explored to make the reader familiar with the general procedure. Thereafter, the focus shifts towards the more advanced collateralized swaps as they make up for the biggest part in the OTC markets. Here, there will be distinguished between constant notional swaps and mark-to-market swaps to derive the forward surface. The theoretical framework is primarily based on the findings in Fujii, Shimada, and Takahashi (2009a), Fujii, Shimada, and Takahashi (2009b), Fujii, Shimada, and Takahashi (2010) and Linderstrøm and Rasmussen (2011).

5.2.1 Swap curve construction without collateral

Following the notation introduced in the pre-crisis framework, m and n correspond to semiannual and quarterly payments for the underlying Libor rates. Likewise, the swap rate, the tenor spread of a 3M vs 6M tenor swap and the cross-currency basis spread where the US Dollar as base currency trades flat against the foreign currency f, all with maturity N, are denoted as C_N, τ_N and b_N, respectively. Consequently, the required conditions for a bank located in the US that identifies the 3M US Dollar Libor rate as its appropriate discounting rate are given as follows:

$$\textbf{IRS:}\ C_M \sum_{m=1}^{M} \Delta_m P_{t,T_m} = \sum_{m=1}^{M} \delta_m \, E_t[L_{T_{m-1},T_m}] \, P_{t,T_m} \tag{5.9}$$

$$\textbf{TS:}\ \sum_{n=1}^{N} \delta_n \left(E_t[L_{T_{n-1},T_n}] + \tau_N\right) P_{t,T_n} = \sum_{m=1}^{M} \delta_m \, E_t[L_{T_{m-1},T_m}] \, P_{t,T_m} \tag{5.10}$$

$$\begin{aligned}\textbf{CCS:}\ & - P_{t,T_0} + \sum_{n=1}^{N} \delta_n \, E_t[L_{T_{n-1},T_n}] \, P_{t,T_n} + P_{t,T_N} \\ & = \left(-P^f_{t,T_0} + \sum_{n=1}^{N} \delta^f_n \left(E^f_t[L^f_{T_{n-1},T_n}] + b_N\right) P^f_{t,T_n} + P^f_{t,T_N} \right) fx_t \end{aligned} \tag{5.11}$$

Here, it is assumed that $N = 2M$ and fx_t symbolizes the exchange rate of US Dollar per foreign currency at time t. In fact, equation 5.1 holds between the 3M forward rate and the discounting rate allowing to rewrite equation 5.10 as:

$$P_{t,T_0} - P_{t,T_N} + \tau_N \sum_{n=1}^{N} \delta_n P_{t,T_n} = \sum_{m=1}^{M} \delta_m \, E_t[L_{T_{m-1},T_m}] \, P_{t,T_m} \tag{5.12}$$

Eliminating the floating legs from the above equation and equation 5.9 yields:

$$C_M \sum_{m=1}^{M} \Delta_m \, P_{t,T_m} - \tau_N \sum_{n=1}^{N} \delta_n P_{t,T_n} = P_{t,T_0} - P_{t,T_N} \tag{5.13}$$

Once the set of $\{P_{t,T}\}$ is sequentially estimated and properly interpolated, the set of 3M US Dollar forward Libor rates $\{E_t[L_{T_{n-1},T_n}]\}$ can be recovered from equation 5.1. Similarly, the set of 6M US Dollar forward Libor rates $\{E_t[L_{T_{m-1},T_m}]\}$ can be backed out from equation 5.10. It is possible to extend this procedure by adding more tenor swaps to derive additional forward curves for tenors such as 1M and 12M. By adding the CCS condition 5.11, exemplarily for the cross EUR/USD, it is then possible to extract the Euro discounting curve and the 3M Euro forward curve. As the 3M US Dollar Libor rate is assumed to be the discounting rate, the LHS of

equation 5.11 has to be zero and thus can be rewritten to:

$$P^f_{t,T_0} - P^f_{t,T_N} - b_N \sum_{n=1}^{N} \delta^f_n \, P^f_{t,T_n} = \sum_{n=1}^{N} \delta^f_n \, E^f_t[L^f_{T_{n-1},T_n}] \, P^f_{t,T_n} \tag{5.14}$$

Moreover, the constraint from a EUR IRS is:

$$C^f_N \sum_{n=1}^{N} \delta^f_n \, P^f_{t,T_n} = \sum_{n=1}^{N} \delta^f_n \, E^f_t[L^f_{T_{n-1},T_n}] \, P^f_{t,T_n} \tag{5.15}$$

As before, by eliminating the floating parts from equations 5.14 and 5.15, the following equation can be used to extract the set of foreign discounting factors $\{P^f_{t,T_n}\}$ sequentially as all other inputs are quoted in the market:

$$(C^f_N + b_N) \sum_{n=1}^{N} \delta^f_n \, P^f_{t,T_n} = P^f_{t,T_0} - P^f_{t,T_N} \tag{5.16}$$

Applying appropriate spline methods, such as presented in section 3.4, gives a continuous discounting curve. Now, the set of 3M Euro forward Libor rates $\{E^f_t[L^f_{T_{n-1},T_n}]\}$ can be obtained from equation 5.15 by using the determined discounting factors. Again, it is possible to extend this procedure by adding more Euro tenor swaps to derive additional forward curves for tenors such as 1M, 6M and 12M. Having a domestic Libor as discounting rate, the cross-currency basis spread does not affect the US Dollar discounting factors as can be seen from equation 5.13, whereas the Euro discounting factors depend not only on the EUR IRS quotes but also on the basis spreads in the EUR/USD CCS.

However, the case looks different for foreign banks which have their funding bases in the US market and thus still apply a US Dollar rate as discounting rate. In the following, the same procedure will be carried out under the assumption that the 3M US Dollar Libor is the discounting rate from the perspective of a foreign bank.

Now, the initial conditions for IRS, TS, CCS can be reformulated as:

$$\textbf{IRS: } C_M^f \sum_{m=1}^{M} \Delta_m^f P_{t,T_m}^f = \sum_{m=1}^{M} \delta_m^f \, E_t^f[L_{T_{m-1},T_m}^f] \, P_{t,T_m}^f \tag{5.17}$$

$$\textbf{TS: } \sum_{n=1}^{N} \delta_n^f \, (E_t^f[L_{T_{n-1},T_n}^f] + \tau_N^f) \, P_{t,T_n}^f = \sum_{m=1}^{M} \delta_m^f \, E_t^f[L_{T_{m-1},T_m}^f] \, P_{t,T_m}^f \tag{5.18}$$

$$\begin{aligned} \textbf{CCS: } & - P_{t,T_0} + \sum_{n=1}^{N} \delta_n \, E_t[L_{T_{n-1},T_n}] \, P_{t,T_n} + P_{t,T_N} \\ & = \left(-P_{t,T_0}^f + \sum_{n=1}^{N} \delta_n^f \, (E_t^f[L_{T_{n-1},T_n}^f] + b_N) \, P_{t,T_n}^f + P_{t,T_N}^f \right) f x_t \end{aligned} \tag{5.19}$$

It can be seen that the RHS of equations 5.17 and 5.18 are equivalent, hence:

$$\begin{aligned} C_M^f \sum_{m=1}^{M} \Delta_m^f \, P_{t,T_m}^f &= \sum_{n=1}^{N} \delta_n^f \, (E_t^f[L_{T_{n-1},T_n}^f] + \tau_N^f) \, P_{t,T_n}^f \\ C_M^f \sum_{m=1}^{M} \Delta_m^f \, P_{t,T_m}^f - \sum_{n=1}^{N} \delta_n^f \, \tau_N^f \, P_{t,T_n}^f &= \sum_{n=1}^{N} \delta_n^f \, E_t^f[L_{T_{n-1},T_n}^f] \, P_{t,T_n}^f \end{aligned} \tag{5.20}$$

As the 3M US Dollar Libor rate is still assumed to be the discounting rate, the LHS of equation 5.19 has to be zero and thus can be rewritten as:

$$\begin{aligned} \sum_{n=1}^{N} \delta_n^f \, (E_t^f[L_{T_{n-1},T_n}^f] + b_N) \, P_{t,T_n}^f &= P_{t,T_0}^f - P_{t,T_N}^f \\ \sum_{n=1}^{N} \delta_n^f \, E_t^f[L_{T_{n-1},T_n}^f] \, P_{t,T_n}^f + \sum_{n=1}^{N} \delta_n^f \, b_N \, P_{t,T_n}^f &= P_{t,T_0}^f - P_{t,T_N}^f \end{aligned} \tag{5.21}$$

Inserting equation 5.20 into 5.21 and further simplifying yields:

$$C_M^f \sum_{m=1}^{M} \Delta_m^f \, P_{t,T_m}^f + \sum_{n=1}^{N} \delta_n^f \, (b_N - \tau_N^f) \, P_{t,T_n}^f = P_{t,T_0}^f - P_{t,T_N}^f \tag{5.22}$$

Now, it is straightforward to derive the continuous set of Euro discounting factors $\{P_{t,T_n}^f\}$ from this formula by appropriate splining as before. Consecutively, the set of 6- and 3M Euro forward Libor rates , i.e. $\{E_t^f[L_{T_{m-1},T_m}^f]\}$ and $\{E_t^f[L_{T_{n-1},T_n}^f]\}$, can be calculated from equations 5.17 and 5.18, respectively. Following this approach under the assumption of the 3M US Dollar Libor rate being the discounting rate, the Euro discounting and two Euro forward curves can be extracted, enabling the foreign bank to consistently mark-to-market EUR IRS with EUR TS and EUR/USD CCS at the same time. Naturally, further TS conditions could be added to this framework in order to extract forward curves for other tenors.

Fujii, Shimada, and Takahashi (2009b) highlight the relation among discounting and tenor-dependent yield curves. Following their work, the term $\Delta_m^f \, P_{t,T_m}^f$ can be approximated by:

$$\Delta_m^f \, P_{t,T_m}^f \simeq \frac{\Delta_m^f}{2}(P_{t,T_m-3m}^f + P_{t,T_m}^f) \tag{5.23}$$

Now, simplifying equation 5.22 by replacing $\Delta_m^f = 2\Delta_n^f$ yields:

$$\sum_{n=1}^{N} \left[\Delta_n^f \, C_M^f + \delta_n^f \, (b_N - \tau_N^f)\right] P_{t,T_n}^f \simeq P_{t,T_0}^f - P_{t,T_N}^f \tag{5.24}$$

This approximation indicates that the effective swap rate, meaning the swap rate that correctly accounts for tenor and cross-currency basis spreads, differs from the normal swap rate in the following relation:

$$C_{M^{eff}}^f \simeq C_M^f + \frac{\delta_n^f}{\Delta_n^f} \, (b_N - \tau_N^f) \tag{5.25}$$

Consequently, having a position in a plain vanilla interest rate swap exposes the holder inevitably to tenor and cross-currency basis spreads. This might seem counterintuitive in the first place but is nevertheless one of the lessons to be learned from the financial crisis.

From both cases, the bank located in the US and the foreign bank, it is obvious that a different choice of Libor as discounting rate leads to different discounting curves, which lead to different present values even for the same cash flow. The following example serves the purpose to show that this ambivalence just reflects the asymmetry of funding costs of financial firms while simultaneously allowing for arbitrage.

Considering a high-rated financial firm located in the US that can borrow at 3M US Dollar Libor flat and therefore it is appropriate to also apply this rate as the discounting rate. The question of interest is how much does it cost the same firm to borrow in Euro. Firstly, the firm could borrow US Dollar in the US market and by entering a EUR/USD CCS swap them into Euro. Here, the firm receives US Dollar payments from the swap which will be passed through to repay for the US Dollar loan, implying total funding cost of 3M Euro Libor plus a basis spread. Hence, the firm can borrow cash in Euro at a cheaper cost than in the European domestic market due to the usually negative EUR/USD basis spread as seen previously in figure 4.9. This brings the US firm in the position to provide loans in the European domestic market at 3M Euro Libor flat and cashing in a profit.

On the other hand, a high-rated financial firm located in Europe cannot raise any profit by lending at 3M Euro Libor flat in the domestic market as its funding costs is as well 3M Euro Libor flat. In case this firm wants to provide US Dollar loans at 3M US Dollar Libor flat to its clients, the firm needs to borrow Euro in the domestic market and again by entering a EUR/USD CCS swap them into US Dollar. Here, the firm pays the 3M US Dollar Libor to the CCS counterparty by passing through the repayments from their clients in return for receiving 3M Euro Libor plus basis spread. In line with the previous case, this spread is usually negative and thus forcing the firm to take a loss as it provides loans at a yield below its funding cost.

Both financial firms need to apply the reference rate that best represents their respective funding costs. Otherwise the pricing of financial products is biased and does not reflect their true values. If the high-rated European financial firm would have chosen the 3M US Dollar Libor as its appropriate discounting rate, it would have misvalued its provided US Dollar loans by seeing a profit although it is indeed a loss. Fujii, Shimada, and Takahashi (2009b) mention that banks experience difficulties in the price agreement when closing a swap as each bank discounts the cash flows using its own funding cost. This might force banks to move away from mark-to-market towards a mark-to-model approach (Whittall, 2010a, p.1). Moreover, Fujii, Shimada, and Takahashi (2009b) emphasize that the coexistence of different funding currencies within a single firm needs to be avoided to diminish arbitrage within the system. By incorporating cross-currency basis spreads correctly in the post-crisis pricing framework it is ensured that the asymmetry of funding costs is still reflected when pricing swaps. In fact, foreign financial firms take advantage of the asymmetry of funding cost by entering CCS in emerging markets where basis spreads are astonishingly large and negative (Fujii, Shimada, and Takahashi, 2009b, p.8).

5.2.2 Swap curve construction with collateral

Swap markets experienced tremendous growth in the late 1980's and early 1990's, when an increasingly diverse group of counterparties entered the markets. Consequently, market practitioners developed a number of credit enhancements to improve the quality of the swap contracts, seeking to mitigate the exposure to counterparty risk. Johannes and Sundaresan (2007) clearly agree on the most important credit enhancement being the posting of collateral in the amount of the current mark-to-market value of the swap contract. Overall, 70 percent of all OTC derivatives transactions were subject to collateral agreements in the year 2010 compared to only 30 percent in 2003 (ISDA, 2010, p.10). Inevitably, this questions the fundamentals

and especially the justification of Libor discounting for collateralized swap contracts. That is also why the majority of banks recognize that the overnight indexed swap rate should be used to discount future cash flows of collateralized swap contracts (Whittall, 2010a, p.1). This subsection will focus on the effects on the valuation of swap contracts when posting collateral that reduces the credit risk and changes the funding cost significantly.

In the following the generic pricing of collateralized trades will be discussed. As noted in Piterbarg (2010), collateralized contracts are based on the credit support annex (CSA) to the International Swaps and Derivatives Association master agreement. That is why these contracts are often referred to as CSA trades. In a collateralized swap contract, the collateral is posted from the counterparty to the institution that has a positive present value of the contract. To compensate the counterparty putting up collateral, the institution needs to pay the margin called *collateral rate* on the outstanding collateral to the payer. The legal collateral agreement between the two parties can vary substantially in the independent amount, minimum transfer amount and threshold amount. Here, the independent amount must be posted independently of the exposure between the two parties whereas the threshold amount is the lower bound before additional collateral is required. The minimum transfer amount is a further aspect of a collateral agreement, primarily required to reduce operational burdens. Additionally, Linderstrøm and Rasmussen (2011) comment on the difficulties pricing such contracts when asymmetries in the above described amounts occur and refer to credit valuation adjustments (CVA) when pricing such contracts. This section will not investigate this topic. Common practice is thus to use collateral in currencies from developed countries such as US Dollar, Euro and Japanese Yen, where the mark-to-market is done frequently (Fujii, Shimada, and Takahashi, 2009b, p.9). The collateral rate is typically the overnight rate for the corresponding currency, e.g. the Federal Funds rate for the US Dollar. As collateral is used to offset liabilities in case of a default, it can be considered a risk free investment why the collateral rate can be seen as a proxy for the risk free rate (Piterbarg, 2010, p.97).

Pricing collateralized contracts faces the challenge of asymmetry that arises from the credit risk which was briefly discussed above. In order to deal with this problem, the rest of the paper will similar to Fujii, Shimada, and Takahashi (2009b) assume perfect and continuous collateralization in cash with a threshold amount of zero, i.e. collateral is posted continuously and the posted amount of cash is 100 percent of the contract's present value. Actually, 82 percent of the overall posted collateral is indeed cash according to ISDA's margin survey 2010. This approximation seems reasonable when the best practice in the market would be to adjust the collateral on

a daily basis (Fujii, Shimada, and Takahashi, 2009b, p.9). This simplification allows neglecting the counterparty default risk and recovers the symmetry in the collateral payments. Moreover, this allows the decomposition of the cash flow of a collateralized swap into a portfolio of independently collateralized strips of payments.

Considering a stochastic process of the collateral account $V(t)$ with an appropriate self financing strategy under the risk neutral measure, one could invest the posted collateral at the risk free interest rate but would need to pay the collateral rate. The process of the collateral account is then given by

$$dV(s) = y(s)V(s)ds + a(s)dh(s) \tag{5.26}$$

where, $y(s) = r(s) - c(s)$ is the difference of the risk free rate $r(s)$ and the collateral rate $c(s)$ in the domestic currency at time s, $h(s)$ is the time s value of the derivative maturing at T with the cashflow $h(T)$ and $a(s)$ is the number of positions of the derivative. Integrating equation 5.26 yields the following:

$$V(T) = e^{\int_t^T y(u)du}V(t) + \int_t^T e^{\int_s^T y(u)du}a(s)dh(s) \tag{5.27}$$

The trading strategy of perfect and continuous collateralization with a threshold of zero is given such that:

$$\begin{aligned} V(t) &= h(t) \\ a(s) &= e^{\int_t^s y(u)du} \end{aligned} \tag{5.28}$$

Applying the above trading strategy to equation 5.27 determines the value of the collateral account at time T:

$$V(T) = e^{\int_t^T y(s)ds}h(T) \tag{5.29}$$

The present value of the underlying derivative $h(t)$ is then given by the following:

$$h(t) = E_t^Q\left[e^{-\int_t^T (r(s)-y(s))ds}h(T)\right] = E_t^Q\left[e^{-\int_t^T c(s)ds}h(T)\right] \tag{5.30}$$

Here, $E_t^Q[\ ]$ denotes the expectation operator under the money-market account Q at time t. From this result it is clear that the discounting of future cash flows of collateralized trades is done at the collateral rate and not the Libor rate. Especially in distressed markets, the Libor rate for a corresponding currency can differ significantly from the collateral rate. Therefore, it follows that Libor discounting is not appropriate for the pricing of collateralized trades.

For some trades, collateral can alternatively be posted in a foreign currency. The procedure for determining at what factor discounting should be done is very similar to the procedure described previously, though subject to modifications. Considering the process of the collateral account V^f:

$$dV^f(s) = y^f(s)V^f(s)ds + a(s)d\left[\frac{h(s)}{fx(s)}\right] \tag{5.31}$$

The foreign exchange rate at time s is given by $fx(s)$ where $y^f(s) = r^f(s) - c^f(s)$ is the difference of the risk free rate $r^f(s)$ and the collateral rate $c^f(s)$ of the foreign currency f at time s. Again, integrating the process for the collateral account gives:

$$V^f(T) = e^{\int_t^T y^f(s)ds}V^f(t) + \int_t^T e^{\int_s^T y^f(u)du}a(s)d\left[\frac{h(s)}{fx(s)}\right] \tag{5.32}$$

Now, the trading strategy for collateral posted in a foreign currency can be adopted:

$$\begin{aligned} V^f(t) &= \frac{h(t)}{fx(t)} \\ a(s) &= e^{\int_t^s y^f(u)du} \end{aligned} \tag{5.33}$$

This reduces equation 5.32 to:

$$V^f(T) = e^{\int_t^T y^f(s)ds}\frac{h(t)}{fx(t)} \tag{5.34}$$

Then, the present value of the underlying derivative in terms of the domestic currency is given by:

$$\begin{aligned} h(t) = V^f(t)fx(t) &= E_t^Q\left[e^{-\int_t^T r(s)ds}V^f(T)fx(T)\right] \\ &= E_t^Q\left[e^{-\int_t^T r(s)ds}e^{\int_t^T (r^f(s)-c^f(s))ds}h(T)\right] \end{aligned} \tag{5.35}$$

Again, from the above equation it becomes clear that for contracts where collateral is posted in a foreign currency, Libor discounting is inappropriate. The difference from the previous example is now that the collateral earns the foreign risk free interest rate less the foreign collateral rate as opposed to the domestic rates.

Analyzing the matter on hand from a different perspective would be to consider the above results in terms of funding cost on a collateralized trade. If there exists an expected future cash inflow from the underlying contract, i.e. a positive present value, then the counterparty immediately posts collateral on which the receiver pays the collateral rate and is obliged to return the whole amount in the end. This can be considered a loan allocated from the payer of the collateral, where the funding of the

position is done at the collateral rate. Oppositely, if there exists an expected future cash outflow, i.e. a negative present value, then the required posting of collateral can be interpreted as a loan provided to the counterparty at the same rate. That is why, when compared to non-collateralized trades, i.e. Libor funding, the party that has an expected cash inflow benefits from a lower funding rate whereas the party that has an expected cash outflow suffers from the lower rate of return when giving the loan.

In order to price collateralized swaps it is critical to determine the discounting curve based on an overnight rate. Here, it is very convenient to use the OIS rate already quoted in the market. By assuming that the OIS rate is continuously and perfectly collateralized with a zero threshold and approximating the daily compounding with continuous compounding, the condition from the OIS can be derived from equation 5.30 such that:

$$S_N \sum_{n=1}^{N} \Delta_n E_t^Q \left[e^{-\int_t^{T_n} c(s)ds} \right] = \sum_{n=1}^{N} E_t^Q \left[e^{-\int_t^{T_n} c(s)ds} \left(e^{\int_{T_{n-1}}^{T_n} c(s)ds} - 1 \right) \right] \tag{5.36}$$

Here, S_N denotes the time t par rate for the length N OIS, where $c(t)$ is the overnight rate, i.e. the collateral rate, at time t. Now, by defining the discounting factor as

$$D_{t,T} = E_t^Q \left[e^{-\int_t^T c(s)ds} \right] \tag{5.37}$$

equation 5.36 reduces to

$$\textbf{OIS:} \;\; S_N \sum_{n=1}^{N} \Delta_n D_{t,T_n} = D_{t,T_0} - D_{t,T_N} \; . \tag{5.38}$$

As seen previously, obtaining the continuous set of discounting factors $\{D_{t,T}\}$ can be done by appropriate splining. Deriving the discounting factors from the collateral rate proves to be a useful result in order to calculate the different forward Libor rates as will be described in the following section. This of course assumes that the OIS market is available up until the necessary maturity.

5.2.3 Collateralized swaps with Constant Notional CCS

In the single currency case it is common to use collateral in the same currency as the one the swap is denominated in. From the previous sections it is clear that the calculations of forward Libor rates can now be conducted by using the derived discounting factors from the OIS. This is done in a straightforward procedure by replacing the set of $\{P_{t,T}\}$ with $\{D_{t,T}\}$ in the corresponding equations 5.9 and 5.10,

respectively. However, more challenging is the case of the multi currency approach. Here, the constant notional CCS (CNCCS) also includes payments from a foreign currency which complicates the determination of the term structures due to the involvement of both the risk free and collateral rate simultaneously seen in equation 5.35. Therefore, this section seeks to determine the term structures in a multi currency setup using the constant notional swap as calibration instrument. Among market practitioners it is common to post US Dollar as collateral for multi currency trades.

Clarifying, $D_{t,T}$ is the discounting factor referring to the collateral rate c whereas $P_{t,T}$ is the discounting factor referring to the risk free rate r. Facing the problem of involving both the risk free and collateral rate at the same time, Piterbarg (2010) argues that one can assume the collateral rate, i.e. the Fed Fund rate for US Dollar, to also be the risk free rate. This assumption implies that the domestic discounting factor, i.e. for the US Dollar, is given by:

$$D_{t,T} = E_t^Q \left[e^{-\int_t^T c(s)ds} \right] = E_t^Q \left[e^{-\int_t^T r(s)ds} \right] = P_{t,T} \tag{5.39}$$

Firstly, the relations for a US Dollar collateralized US Dollar swaps are given by

$$\textbf{OIS:}\ S_N \sum_{n=1}^{N} \Delta_n D_{t,T_n} = D_{t,T_0} - D_{t,T_N} \tag{5.40}$$

$$\textbf{IRS:}\ C_M \sum_{m=1}^{M} \Delta_m\, D_{t,T_m} = \sum_{m=1}^{M} \delta_m\, E_t^c[L_{T_{m-1},T_m}]\, D_{t,T_m} \tag{5.41}$$

$$\textbf{TS:}\ \sum_{n=1}^{N} \delta_n\, (E_t^c[L_{T_{n-1},T_n}] + \tau_N)\, D_{t,T_n} = \sum_{m=1}^{M} \delta_m\, E_t^c[L_{T_{m-1},T_m}]\, D_{t,T_m} \tag{5.42}$$

Likewise, the conditions for a swap collateralized and denominated in a foreign currency are:

$$\textbf{OIS}^f\textbf{:}\ S_N^f \sum_{n=1}^{N} \Delta_n^f D_{t,T_n}^f = D_{t,T_0}^f - D_{t,T_N}^f \tag{5.43}$$

$$\textbf{IRS}^f\textbf{:}\ C_M^f \sum_{m=1}^{M} \Delta_m^f\, D_{t,T_m}^f = \sum_{m=1}^{M} \delta_m^f\, E_t^{c^f}[L_{T_{m-1},T_m}^f]\, D_{t,T_m}^f \tag{5.44}$$

$$\textbf{TS}^f\textbf{:}\ \sum_{n=1}^{N} \delta_n^f\, (E_t^{c^f}[L_{T_{n-1},T_n}^f] + \tau_N^f)\, D_{t,T_n}^f = \sum_{m=1}^{M} \delta_m^f\, E_t^{c^f}[L_{T_{m-1},T_m}^f]\, D_{t,T_m}^f \tag{5.45}$$

Here, $D_{t,T}$ and $D_{t,T}^f$ are used as numeraire under the expectation operators $E_t^c[\,]$ and $E_t^{c^f}[\,]$, respectively. From the above it is then possible to derive the discounting factors $D_{t,T}$ and $D_{t,T}^f$ based on the OIS conditions and thereafter sequentially derive

the forward Libor rates. Again, it is possible to extend this setup with additional TS conditions.

Considering the popular choice of having US Dollar collateralized swaps denominated in a foreign currency, the following condition for determining the foreign interest rates is given by

$$\textbf{CNCCS:}\ \sum_{n=1}^{N} \delta_n^f \left(E_t^f[L^f_{T_{n-1},T_n}] + b_N\right) P^f_{t,T_n} - P^f_{t,T_0} + P^f_{t,T_N} = V_N \tag{5.46}$$

where

$$V_N = \left\{ \sum_{n=1}^{N} \delta_n \; E_t^c[L_{T_{n-1},T_n}] \; D_{t,T_n} - D_{t,T_0} + D_{t,T_N} \right\} / fx_t \; . \tag{5.47}$$

Even though the US Dollar discounting factors and forward rates can be estimated from equations 5.40, 5.41 and 5.42, it is still not possible to determine the foreign discounting factors $\{P^f_{t,T_n}\}$ and forward rates $\{E_t^f[L^f_{T_{n-1},T_n}]\}$ only from these standard set of swaps. Here, $E_t^f[\ ]$ denotes that the expectation is taken under the forward measure where $P^f_{t,T}$ is used as numeraire. It would be necessary to attain additional information from US Dollar collateralized IRS and TS denominated in the foreign currency if available in the market to overcome this problem. This would yield the following conditions:

$$\textbf{IRS:}\ \tilde{C}^f_M \sum_{m=1}^{M} \Delta^f_m \, P^f_{t,T_m} = \sum_{m=1}^{M} \delta^f_m \, P^f_{t,T_m} \, E_t^f[L^f_{T_{m-1},T_m}] \tag{5.48}$$

$$\textbf{TS:}\ \sum_{n=1}^{N} \delta_n^f \left(E_t^f[L^f_{T_{n-1},T_n}] + \tilde{\tau}^f_N\right) P^f_{t,T_n} = \sum_{m=1}^{M} \delta^f_m \, P^f_{t,T_m} \, E_t^f[L^f_{T_{m-1},T_m}] \tag{5.49}$$

In the above conditions, $\tilde{C}^f_M$ and $\tilde{\tau}^f_N$ denote the par rates of swaps denominated in a foreign currency but collateralized in US Dollar. Thus, it is important to distinguish them from C^f_M and τ^f_N which are par rates of swaps denominated and collateralized in a foreign currency. As done in section 5.2.1, eliminating the floating legs in equations 5.48 and 5.49 and substituting the remaining expression into equation 5.46 yields:

$$\sum_{n=1}^{N} \delta^f_n (b_N - \tilde{\tau}^f_N) P^f_{t,T_n} + \tilde{C}^f_M \sum_{m=1}^{M} \Delta^f_m P^f_{t,T_m} - V_N = P^f_{t,T_0} - P^f_{t,T_N} \tag{5.50}$$

Equation 5.50 allows the determination of the set of discounting factors $\{P^f_{t,T}\}$ and subsequently the estimation of the US Dollar collateralized foreign forward rates

by applying an appropriate spline method. Unfortunately, it may be difficult to obtain market quotes for US Dollar collateralized swaps denominated in low popular currencies. Consequently, information necessary to apply equations 5.48 and 5.49 could be unavailable. Alternatively, a convenient approach to overcome this problem is to assume that

$$E_t^f[L^f_{T_{n-1},T_n}] = E_t^{c^f}[L^f_{T_{n-1},T_n}] \,. \tag{5.51}$$

This approximation needs the dynamic properties of both the foreign risk free and collateral rate to be similar with each other. If that is the case then 5.51 is a valid assumption following the argumentation of Piterbarg (2010). Again, given the information from equations 5.40, 5.41, 5.42 and 5.43, the set of forward rates can be estimated from equation 5.46. This also implies that the correction arising from the change of numeraire is neglected. Lastly, applying the interest rate parity is another option to derive the foreign forward rates.

Furthermore, it is to mention that there also exists the possibility for US Dollar swaps to be collateralized by a foreign currency. In such a case, the approach diverges from US Dollar collateralized swaps denominated in a foreign currency as the foreign risk free and collateral rate are not assumed to be equal.

5.2.4 Collateralized swaps with Mark-to-Market CCS

The mark-to-market CCS has many similarities to the constant notional CCS as the counterparties still exchange the Libor in one currency and the Libor plus a spread in another currency with notional exchanges at initiation and maturity. The difference is now that the notional of the currency that pays Libor flat, usually the stronger currency such as the US Dollar, is adjusted at every fixing date based on the FX spot rate at that time. Hence, the difference between the notional defined in the previous and the following period is exchanged at each fixing date. It is important to stress that the notional of the other currency is kept constant throughout the contract. Due to this ongoing adjustment of the notional, the mark-to-market cross-currency swap (MtMCCS) has become popular among market participants as the contract reduces the risk arising from fluctuating exchange rates and consequently the credit exposure of both counterparties.

When pricing a mark-to-market swap it is very convenient to consider it as a portfolio of one-period constant notional CCS contracts. From this, the notional adjustment that occurs in the MtMCSS at the start of the $(i+1)$-th period is equivalent to the net effect from the final notional exchange of the (i)-th CNCCS and the $(i+1)$-th CNCCS.

Assuming the US Dollar to be the collateral currency of choice the determination of the present value of the foreign leg is relatively straightforward as the notional is kept constant throughout the contract. The present value of the foreign leg in the most general form is then given by

$$\begin{aligned} PV^f = &- \sum_{n=1}^{N} E_t^{Q_f} \left[e^{-\int_t^{T_{n-1}} r^f(s) - y(s)\, ds} \right] \\ &+ \sum_{n=1}^{N} E_t^{Q_f} \left[e^{-\int_t^{T_n} r^f(s) - y(s)\, ds} \left(1 + \delta_n^f (L^f_{T_{n-1},T_n} + b_N) \right) \right] \end{aligned} \tag{5.52}$$

where $E_t^{Q_f}$ denotes the expectation operator under the money-market account Q in a foreign currency f at time t, r^f is the foreign risk free rate, y is the difference between the US Dollar risk free and the collateral rate and b_N is still the cross-currency basis spread of length N. Rewriting the equation above to

$$\begin{aligned} PV^f = &- E_t^{Q_f} \left[e^{-\int_t^{T_0} r^f(s) - y(s)\, ds} \right] + E_t^{Q} \left[e^{-\int_t^{T_N} r^f(s) - y(s)\, ds} \right] \\ &+ \sum_{n=1}^{N} \delta_n^f E_t^{Q_f} \left[e^{-\int_t^{T_n} r^f(s) - y(s)\, ds} (L^f_{T_{n-1},T_n} + b_N) \right] \end{aligned} \tag{5.53}$$

reveals that the first two terms are equal to the fixed leg of a usual IRS whereas the last term resembles the floating leg. Likewise, the US Dollar leg of a MtMCCS in its most general form can be formulated as:

$$\begin{aligned} PV^{USD} &= - \sum_{n=1}^{N} E_t^{Q} \left[\frac{e^{-\int_t^{T_{n-1}} c(s) ds}}{fx_{T_{n-1}}} \right] + \sum_{n=1}^{N} E_t^{Q} \left[\frac{e^{-\int_t^{T_n} c(s) ds}\ (1 + \delta_n L_{T_{n-1},T_n})}{fx_{T_{n-1}}} \right] \\ &= - E_t^{Q} \left[\frac{e^{-\int_t^{T_0} c(s) ds}}{fx_{T_0}} \right] + E_t^{Q} \left[\frac{e^{-\int_t^{T_N} c(s) ds}}{fx_{T_{N-1}}} \right] + \sum_{n=1}^{N} \delta_n E_t^{Q} \left[\frac{e^{-\int_t^{T_n} c(s) ds}\ L_{T_{n-1},T_n}}{fx_{T_{n-1}}} \right] \end{aligned} \tag{5.54}$$

Here, E_t^Q denotes the expectation operator under the US Dollar money-market account Q at time t, c is the collateral rate in US Dollar and $fx_{T_{n-1}}$ stands for the exchange rate of US Dollar per foreign currency at time T_{n-1}. Importantly, to follow a consistent notation the index n denotes quarterly payments, having a 3M tenor as underlying, but could be easily replaced with any other payment frequency.

In the following, pricing a MtMCCS is done with US Dollar collateral, under the assumption that the collateral rate, i.e. the Fed Fund rate, is also the risk free rate:

$$r(s) = c(s) \tag{5.55}$$

This assumption implies that $y(s) = r(s) - c(s) = 0$ and it again follows that the domestic risk-free discounting factor, i.e. for the US Dollar, is equal to discounting factor from the collateral rate, see equation 5.39. Note that one cannot assume risk free rates in several currencies to be equal to their respective collateral rates simultaneously if the pricing framework should be consistent with FX forwards and CCS. Taking the aforementioned assumption into account equation 5.53 can be simplified to

$$PV^f = -P^f_{t,T_0} + P^f_{t,T_N} + \sum_{n=1}^{N} \delta^f_n P^f_{t,T_n}(E^f_t[L^f_{T_{n-1},T_n}] + b_N) \tag{5.56}$$

where the expectation operator is replaced with $E^f_t[\ ]$ as here in particular P^f_{t,T_n} is used as numeraire. Noticeably, this leg is equivalent to the left hand side of equation 5.46. Different is the case of determining the present value of the US Dollar leg where the notional is adjusted at each fixing date. Hence, it follows from 5.51 that equation 5.54 can be modified such that

$$PV^{USD} = \frac{-D_{t,T_0}}{fx_t} + \frac{D_{t,T_N}}{FX_{t,T_{N-1}}} + \sum_{n=1}^{N} \frac{\delta_n D_{t,T_n} E^c_t[L_{T_{n-1},T_n}]}{FX_{t,T_{n-1}}} \tag{5.57}$$

where the expectation operator is replaced with $E^c_t[\ ]$ as here in particular D_{t,T_n} is used as numeraire. Moreover, $FX_{t,T_{n-1}}$ denotes the time t forward exchange rate at T_{n-1} whereas fx_t still denotes the spot exchange rate for US Dollar per foreign currency at time t. Assuming that the Libor rate is also the risk free rate, as in section 5.2.1, this would result in a zero present value of the USD leg where the second term cancels out the first one. However, this assumption can be somewhat difficult to justify considering the inherent risk reflected in the Libor rates. That is why a clear distinction must be made between the two rates which leads to an additional model dependent term that must be taken into consideration. The present value depends on the covariance between the risk free zero-coupon bonds and the FX rate. This is due to the model dependent term that appears from the change of numeraire from the money market account Q into the risk free zero-coupon bond price D_{t,T_n} with maturity T_n applying equation 5.39. Being able to determine the model dependent term, a repetition of the same discussion as following equation 5.46 will be allowed after replacing V_N with PV^{USD}.

The determination of the model dependent term, i.e. the convexity correction, can be simplified by assuming that both the risk free and the FX rate follow a geometric brownian motion. The forward FX rate and the US Dollar forward risk

free bond are estimated as:

$$FX_{t,T_{n-1}} = fx_t \frac{D_{t,T_{n-1}}}{P^f_{t,T_{n-1}}} \qquad FB_{t;T_{n-1},T_n} = \frac{D_{t,T_{n-1}}}{D_{t,T_n}} \tag{5.58}$$

The deterministic log-normal volatilities and the correlation between the forward exchange rate $FX_{t,T_{n-1}}$ and the US Dollar forward risk free bond $FB_{t;T_{n-1},T_n}$ are denoted as $\sigma_{FX_{n-1}}(t)$, $\sigma_{FB_{n-1,n}}(t)$ and $\rho_{n-1}(t)$, respectively. In the simplest case, the present value of the US Dollar leg can be corrected such that:

$$\begin{aligned} PV^{USD} = & \frac{-D_{t,T_0}}{fx_t} + \frac{D_{t,T_N}}{FX_{t,T_{N-1}}} \\ & + \sum_{n=1}^{N} \frac{\delta_n D_{t,T_n} E^c_t[L_{T_{n-1},T_n}]}{FX_{t,T_{n-1}}} \, e^{\int_t^{T_{n-1}} \rho_{n-1}(s)\, \sigma_{FX_{n-1}}(s)\, \sigma_{FB_{n-1,n}}(s) ds} \end{aligned} \tag{5.59}$$

Consequently, setting $PV^f = PV^{USD}$ the formula for the mark-to-market cross-currency swap can be fixed as:

$$\begin{aligned} \textbf{MtMCCS:} \quad & \sum_{n=1}^{N} \delta^f_n \, (E^f_t[L^f_{T_{n-1},T_n}] + b_N) P^f_{t,T_n} - P^f_{t,T_0} + P^f_{t,T_N} = \frac{-D_{t,T_0}}{fx_t} + \frac{D_{t,T_N}}{FX_{t,T_{N-1}}} \\ & + \sum_{n=1}^{N} \frac{\delta_n D_{t,T_n} E^c_t[L_{T_{n-1},T_n}]}{FX_{t,T_{n-1}}} \, e^{\int_t^{T_{n-1}} \rho_{n-1}(s)\, \sigma_{FX_{n-1}}(s)\, \sigma_{FB_{n-1,n}}(s) ds} \end{aligned} \tag{5.60}$$

From the above, the simplest method for curve calibration is described in the following procedure:

i) Firstly, determining the US Dollar forward curves from the OIS, IRS and TS conditions as described in the previous section and using them as inputs.

ii) Secondly, the volatility for the Fed Fund rate can be extracted from the OIS option market whereas the forward FX volatility can be directly read from the vanilla FX option market.

iii) Thirdly, the correlation between the forward risk free bond and forward FX rate can be determined based on either historical data or from the use of quanto products.

iv) Lastly, if the market is liquid enough, the forward FX rates can be directly determined from FX forward contracts. On the other hand, if the market turns out to be illiquid, determining the forward exchange rate can be done from applying swap data only.

As the maturity of the forward FX rate is shorter than the one of the MtMCCS by one period, having the set of foreign discounting factor $\{P^f_{t,T_n}\}$ from the previous section allows for the sequential derivation of $\{E^f_t[L^f_{T_{n-1},T_n}]\}$ from equation 5.60. Concludingly, under these simplified assumptions it is then possible to construct consistent forward curves for both currencies when the MtMCCS contract is collateralized with US Dollars.

According to Fujii, Shimada, and Takahashi (2010) the overnight rate controlled by the central bank is not necessarily equal to the risk free rate and thus imposing $r(s) = c(s)$ leaves no freedom to calibrate FX forwards and MtMCCS. In the following, pricing a MtMCCS is done with the US Dollar being again the collateral currency of choice but under the assumption that the collateral rate, i.e. the Fed Fund rate, is not equal to the risk free rate:

$$r(s) \neq c(s) \tag{5.61}$$

As this implies that $y(s) = r(s) - c(s) \neq 0$ the estimation of the foreign leg is not as straightforward as previously. Therefore it is necessary to repeat some common theory. Having three currencies j,k and l the following triangle relation must hold for spot exchange rates:

$$fx^{j,k}_t = fx^{j,l}_t \times fx^{l,k}_t \tag{5.62}$$

Otherwise, the differences will quickly vanish due to arbitrage if the FX spot markets do not suffer from illiquidity. Likewise, this relation should also hold in the FX forward markets if collateral agreements are not in place as the market is supposed to be default free. However, it is not a trivial issue in the presence of collateral (Fujii, Shimada, and Takahashi, 2009a, p.12). Considering a US Dollar collateralized FX forward contract and sticking to the usual notation the present value needs to be zero at initiation, i.e.:

$$FX_{t,T_N} E_t\left[e^{-\int_t^{T_N} r(s)ds} e^{-\int_t^{T_N} y(s)ds}\right] = fx_t E^f_t\left[e^{-\int_t^{T_N} r^f(s)ds} e^{-\int_t^{T_N} y(s)ds}\right] \tag{5.63}$$

Here, E_t and E^f_t denote the expectation operators under the risk free forward measures for each currency where the risk free zero-coupon bonds P_{t,T_N} and P^f_{t,T_N} are used as numeraire, respectively. Solving equation 5.63 for the FX forward rate

yields:

$$FX_{t,T_N} = fx_t \frac{P_{t,T_N}}{P^f_{t,T_N}} \left(\frac{E_t \left[e^{-\int_t^{T_N} y(s)ds} \right]}{E^f_t \left[e^{-\int_t^{T_N} y(s)ds} \right]} \right) \tag{5.64}$$

If $y(s)$ is stochastic, the currency triangle in 5.62 holds only among FX crosses that share the same collateral currency. However, separate quotes for different collateral currencies are unobservable in the real market. Furthermore, closing the hedges within each collateral currency is unrealistic and would cause unnecessary complications (Fujii, Shimada, and Takahashi, 2009a, p.13). However, assuming $y(s)$ to be a deterministic function of time s, the FX forward rate becomes independent from the choice of collateral as it is apparent when shortening equation 5.64 to:

$$FX_{t,T_N} = fx_t \frac{P_{t,T_N}}{P^f_{t,T_N}} \tag{5.65}$$

Hence, the cross currency triangle in 5.62 holds among FX forwards even when they contain multiple collateral currencies. The correction from the simplifying assumption that $y(s)$ is deterministic arises from the covariance between $e^{-\int_t^{T_N} y(s)ds}$ and other stochastic variables such as the Libor and FX rates. Fujii, Shimada, and Takahashi (2009a) expect the absolute size and volatility of this correction to be within the current bid/offer spreads which they see supported by the fact that FX forward rates for different collateral currencies are not quoted in the market. Now, further rearranging relation 5.65 yields

$$FX_{t,T_N} = fx_t \frac{D_{t,T_N}}{D^f_{t,T_N}} e^{-\int_t^{T_N} y^{USD,f}(s)} \tag{5.66}$$

where $y^{USD,f}(s)$ is similarly defined as a deterministic function of time s:

$$\begin{aligned} y^{USD,f}(s) &= y(s) - y^f(s) \\ &= \left[r(s) - r^f(s) \right] - \left[c(s) - c^f(s) \right] \end{aligned} \tag{5.67}$$

Obviously it is possible to bootstrap the spread $\{y^{USD,f}(s)\}$ from relation 5.66 if FX forward markets are liquid up until the necessary maturity. Alternatively, the spread can be bootstrapped from CNCCSs since it will be the only unknown variable after simplification and as each other swap, a CNCCS needs to have a present value of zero at initiation.

Even though the spread might be deterministic, the correction term on the US Dollar leg as introduced in the case where $r(s) = c(s)$, see equation 5.59, must

be taken into consideration here as well. Revisiting the model depending term it becomes more clear when the Libor rate is decomposed into a risk free and a residual part, i.e.:

$$L_{T_{n-1},T_n} = \frac{1}{\delta_n}\left(\frac{1}{D_{T_{n-1},T_n}} - 1\right) + B_{T_{n-1},T_n} \tag{5.68}$$

Here, the term B_{T_{n-1},T_n} reflects the residual part in the US Dollar Libor L_{T_{n-1},T_n} at time T_{n-1}. If deterministic, the residual part can be considered as the spread between the respective Libor and the risk free rate:

$$B_{T_{n-1},T_n} = L_{T_{n-1},T_n} - r_{T_{n-1},T_n} \tag{5.69}$$

Substituting equation 5.68 into 5.54 gives the present value of the US Dollar leg:

$$PV^{USD} = \sum_{n=1}^{N} \delta_n D_{t,T_n}\ E_t^c\left[\frac{B_{T_{n-1},T_n}}{fx_{T_{n-1}}}\right] \tag{5.70}$$

The correction arising from the change of numeraire needs to be considered similarly as in the case where $r(s)$ is assumed to be equal to $c(s)$. Hence, the formula for the US Dollar leg can be determined as:

$$PV^{USD} = \sum_{n=1}^{N} \frac{\delta_n D_{t,T_n} B_{T_{n-1},T_n}}{FX_{T_{n-1}}}\ e^{\int_t^{T_{n-1}} \rho_{n-1}(s)\ \sigma_{FX_{n-1}}(s)\ \sigma_{FB_{n-1,n}}(s)ds} \tag{5.71}$$

As in 5.70 the leg for the foreign currency can be modified:

$$\begin{aligned} PV^f = &\sum_{n=1}^{N} \delta_n^f D_{t,T_n}^f\ e^{-\int_t^{T_n} y^{USD,f}(s)ds}\left(B_{t,T_n}^f + b_N\right) \\ &+ \sum_{n=1}^{N} D_{t,T_{n-1}}^f\ e^{-\int_t^{T_{n-1}} y^{USD,f}(s)ds}\left(e^{-\int_{T_{n-1}}^{T_n} y^{USD,f}(s)ds} - 1\right) \end{aligned} \tag{5.72}$$

Finally, setting both legs equal enables the determination of the set of Libor-OIS forward spreads $\{B_{t,T_n}^f\}$ which then can be used to determine the forward surface of the foreign currency.

Comparing the framework introduced in the beginning of this section where $r(s) = c(s)$ with the one examined above, i.e. $r(s) \neq c(s)$, it becomes clear that both result in a forward surface of the foreign currency where the former applies Libor forward rates as input whereas the latter uses Libor-OIS spreads. Moreover, it is to emphasize that the latter framework is closer to reality as the surface of interest

can be calibrated not just to the OIS, IRS, TS and CCS markets but also to FX forward markets which completes a consistent pricing framework. For completeness it must be clarified that constructing a forward surface can also be done in the two remaining cases where $r(s) = c(s)$ and spreads are used as inputs as well as in the case where $r(s) \neq c(s)$ and forwards are used as inputs, see table 5.1.

Table 5.1: Matrix for surface construction from MtMCCS

	Assumption	
Input	$r(s) = c(s)$	$r(s) \neq c(s)$
Forwards	✓	
Spreads		✓

5.3 Deriving the discounting curves

Uncollateralized derivatives should be discounted by each financial institution using its own funding cost. Naturally, this entails discussions whenever two counterparties have a different perception of the other's funding cost and therefore the following part of the thesis will focus solely on collateralized swaps neutralizing the ambiguity of discounting. Collateral is commonly posted in US Dollar and it is known from the discussion in section 5.2.2 that if the US Dollar collateral rate is equal to the risk free rate then discounting is done at the risk free rate for any given currency. Here, the tendency among practitioners is to apply OIS discounting rates as opposed to the now riskier Libor rates.

The OIS rates listed in table 5.2 can be used to sequentially estimate the set of discounting factors $\{D_{t,T}\}$ and $\{D^f_{t,T}\}$ where the former denotes US Dollar and the latter the foreign discounting factor, i.e. for Euro. For this purpose, solving equations 5.40 and 5.43 for D_{t,T_N} and D^f_{t,T_N}, respectively, yields:

$$D_{t,T_N} = \frac{1 - S_N \sum_{n=1}^{N-1} \delta_n D_{t,T_n}}{1 + S_N \delta_N} \tag{5.73}$$

$$D^f_{t,T_N} = \frac{1 - S^f_N \sum_{n=1}^{N-1} \delta^f_n D^f_{t,T_n}}{1 + S^f_N \delta^f_N} \tag{5.74}$$

The resulting discounting curves are displayed in figure 5.3. Unfortunately, OIS rates are only available up to a maturity of 10 years in the case of US Dollar whereas OIS Euro swaps, i.e. EONIA swaps, are traded up to 30 years.

Table 5.2: US Dollar and Euro OIS rates June 30th 2010

Maturity	Start date	End date	USD Quote (%)	EUR Quote (%)
1W	2 Jul 2010	9 Jul 2010	0.1904	0.4370
2W	2 Jul 2010	16 Jul 2010	0.1910	0.4560
3W	2 Jul 2010	23 Jul 2010	0.1930	0.4660
1M	2 Jul 2010	2 Aug 2010	0.1940	0.4700
2M	2 Jul 2010	2 Sep 2010	0.1980	0.4800
3M	2 Jul 2010	4 Oct 2010	0.2030	0.4890
4M	2 Jul 2010	2 Nov 2010	0.2070	0.5040
5M	2 Jul 2010	2 Dec 2010	0.2100	0.5200
6M	2 Jul 2010	3 Jan 2011	0.2170	0.5380
7M	2 Jul 2010	2 Feb 2011	0.2230	0.5520
8M	2 Jul 2010	2 Mar 2011	0.2320	0.5670
9M	2 Jul 2010	4 Apr 2011	0.2430	0.5840
10M	2 Jul 2010	2 May 2011	0.2560	0.5980
11M	2 Jul 2010	2 Jun 2011	0.2760	0.6120
1Y	2 Jul 2010	4 Jul 2011	0.2925	0.6245
2Y	2 Jul 2010	2 Jul 2012	0.5535	0.8050
3Y	2 Jul 2010	2 Jul 2013	0.9290	1.0240
4Y	2 Jul 2010	2 Jul 2014	1.3390	1.3210
5Y	2 Jul 2010	2 Jul 2015	1.7100	1.5840
6Y	2 Jul 2010	4 Jul 2016	1.9955*	1.8147
7Y	2 Jul 2010	3 Jul 2017	2.2278*	2.0255
8Y	2 Jul 2010	2 Jul 2018	2.4224*	2.1945
9Y	2 Jul 2010	2 Jul 2019	2.5862*	2.3420
10Y	2 Jul 2010	2 Jul 2020	2.7230	2.4665

Note: * denotes interpolated values applying monotone convex method
Sources: Bloomberg, Nordea Analytics

Figure 5.3: US Dollar and Euro discounting curves

5.4 Deriving the forward surfaces

In order to generate the forward surface, i.e. forward curves for each possible tenor, it is necessary to apply the derived discounting curves from the previous section. The methodologies applied to derive the forward surfaces were theoretically explained in sections 5.2.3 and 5.2.4 and will be exemplified in the following. The coming sections distinguish between the forward surfaces derived from constant notional and mark-to-market swaps.

5.4.1 Forward surface for Constant Notional Swaps

The construction of the forward surface of US Dollar collateralized US Dollar rates when applying constant notional swaps is done in the following order:

a) *3M curve:* US Dollar interest rate swaps with the floating leg paying every three months are used to extract the 3M forward rates. Under the condition that both the fixed and the floating leg have a present value of zero, the forward rates can be optimized by using an iterative process such as Excel's solver. The applied swap rates are displayed in table 5.3.

Table 5.3: US Dollar input data

Maturity in years	IRS (3M) in %	OIS vs 3M in bps	1M vs 3M in bps	3M vs 6M in bps	3M vs 12M in bps
1	0.719	42.6	15.6	15.6	45.1
2	0.968	41.4	14.4	17.0	44.6
3	1.328	39.9	13.3	16.3	42.8
4	1.705	36.6	11.6	15.6	41.0
5	2.047	33.7	10.1	15.1	39.1
6	2.336	34.2*	9.0	14.5	37.2*
7	2.563	33.6*	8.0	14.1	35.3
8	2.743	32.0*	7.3	13.8	33.9*
9	2.889	30.3*	6.6	13.5	32.5*
10	3.014	29.1	6.1	13.4	31.1

Note: * denotes estimated values
Source: Bloomberg

b) *OIS, 1M, 6M, 12M curves:* Backing out the remaining forward curves can be done by applying tenor spreads by matching the 3M floating leg, i.e. the derived forward curve in a) with that of another tenor. In particular, the tenor spreads OIS vs 3M, 1M vs 3M, 3M vs 6M and 3M vs 12M, as listed in table 5.3, are needed to get the OIS, 1M, 6M and 12M forward curves. Here it is

important to outline that the spreads are added to the forward rates of the leg with the shorter tenor. Again Excel's solver is used to calibrate the forward rates one at a time in order to reduce calculation time.

Noticeably, the OIS vs 3M quotes for maturities 6 to 9 years are estimated due to the lack of quotes for these maturities. This problem is circumvented by interpolating the quoted OIS rates from 1 to 5 and 10 years using a monotone convex spline, see section 3.4. Furthermore, the quotes for the 3M vs 12M spread for maturities 6, 8 and 9 years were linearly interpolated.

c) *Interpolation:* The resulting forward curves are characterized by a linear 'zig-zag' behavior, especially in the short end of each curve. This pattern is most extreme for the OIS curve and flattens out for longer tenors. Figure 5.4 shows this exemplarily for the 1M forward curve. Consequently, all surface spanning tenors need to be interpolated by appropriate spline techniques. All interpolation schemes introduced in section 3.4 are piecewise splines that reproduce all inputs as outputs. But this is not sought here in particular as 'zig-zag' forwards shall be interpolated directly. After testing various polynomial interpolations, a 4th order polynomial delivered the best fit in terms of smallest sum of squared errors and smoothness. Then, the coefficients are used to build a complete curve for each tenor.

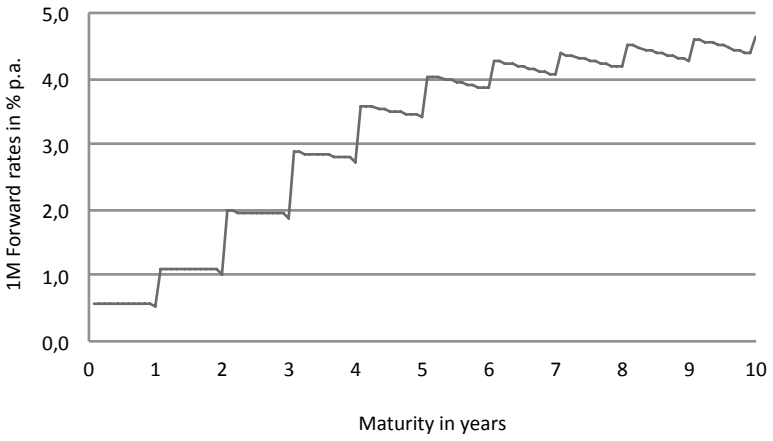


Figure 5.4: US Dollar 1M forward curve

d) *Turns:* Similar to the pre-crisis pricing framework, the historical average turn of year effects for OIS, 1M and 3M noted in table 3.9 are applied for the first turn effect, i.e. when turning from 2010 to 2011, and 50 percent of these effects as roughly chosen jumps for the second turn effect, i.e. from 2011 to

2012. The forward curves in figure 5.5 display the turn effects on each tenor's forward curve.

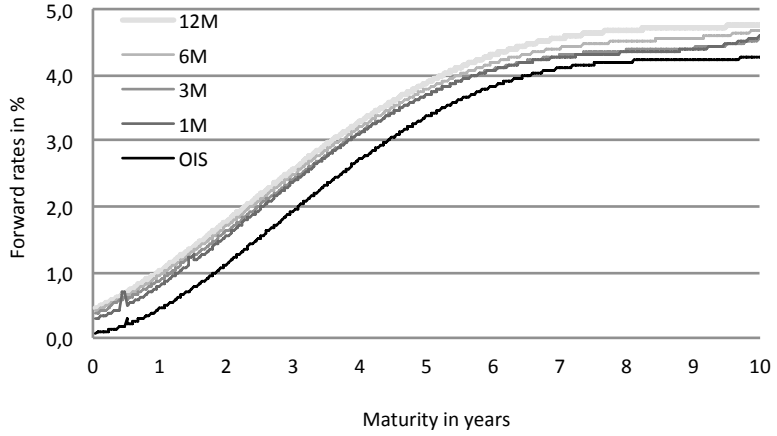


Figure 5.5: US Dollar forward curves including turn effects

e) *Stubs:* Having added turn effects, the last step is to interpolate the stub floating rates in between the surface spanning tenors OIS, 1M, 3M, 6M and 12M. The British Bankers' Association provides fixings also for tenors such as 2M, 4M etc. but nevertheless derivatives based on these tenors are not traded very frequently which is why interpolation is used rather than direct fixings. In fact, the stubs are interpolated piecewise linearly. Other interpolation schemes examined in section 3.4 were based on spot rates and therefore more concerned about the resulting forward rates, now forward rates are directly interpolated. Moreover, practitioners such as Linderstrøm and Scavenius (2010) apply linear interpolation between stubs.

Finally, the US Dollar forward surface is depicted in figure 5.6 in the case where the collateral is posted in US Dollars. It can be seen that the upward slope is biggest between a maturity of 1 to 4 years. Moreover, forward rates usually increase with longer tenors except the 1M forward curve that peaks in the long end of the surface. Interestingly, the OIS curve lies substantially below the remaining tenor curves which underscores the necessity of decoupling forward and discounting curves in comparison to the pre-crisis framework.

Having the US Dollar forward surface it is possible to price any derivative that depends on future interest rates and is collateralized in US Dollar. Here, tenors up to 12M and maturities up to 10 years are available. By reconstructing the input data the methodology is validated and hence applicable for pricing purposes. Figure

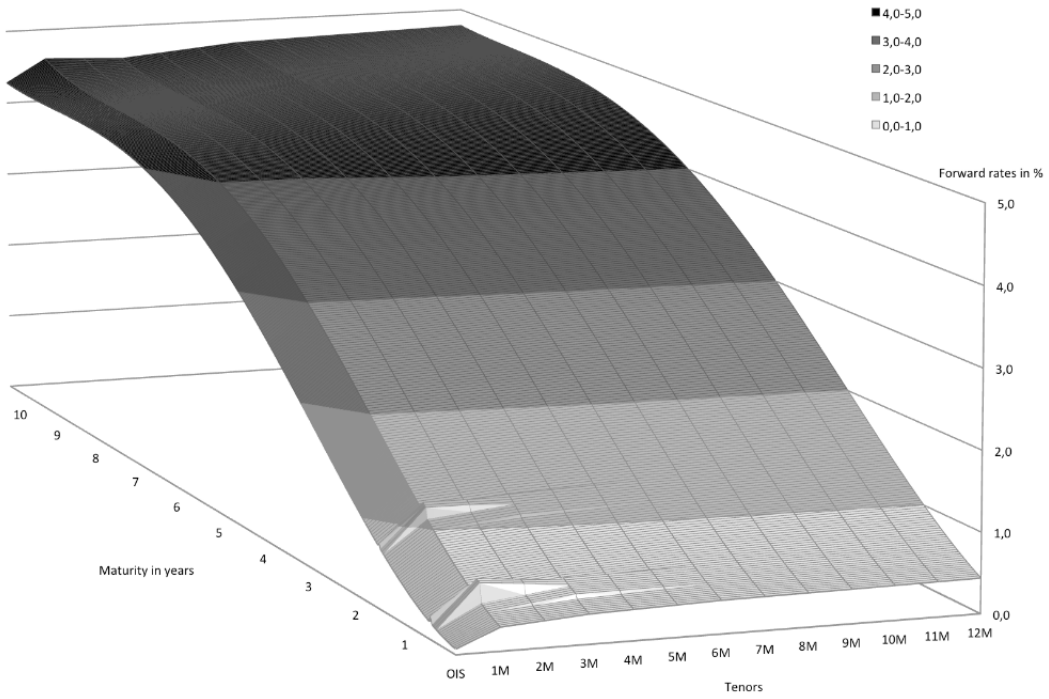


Figure 5.6: US Dollar collateralized US Dollar forward surface

Inspired by Linderstrøm and Scavenius (2010)

5.7 compares the input swap rates for a fixed vs 3M floating swap as they are quoted in the market with the swap rates backed out from the US Dollar surface. It can be inferred that the post-crisis framework is able to reproduce the input swap rates reasonably. The maximum difference of 5 basis points occurs in the very short end of the curve and is due to interpolation and turn effects. Here, it exceeds the usual bid/ask spread for IRS of 1 basis point.

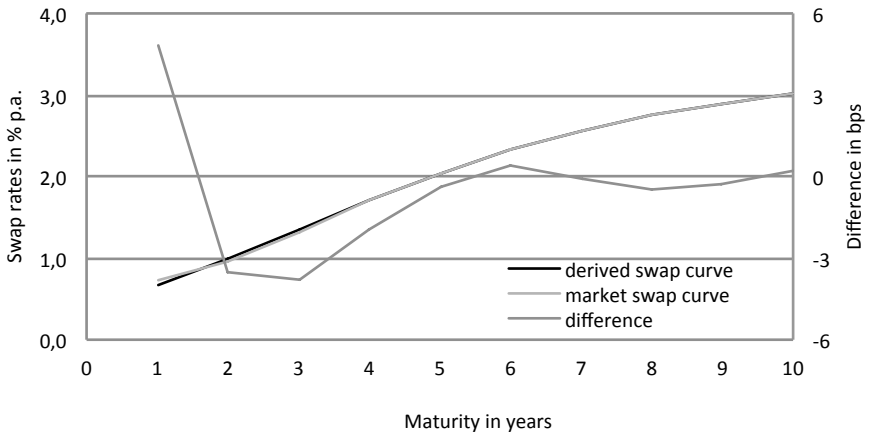


Figure 5.7: Reconstruction of the US Dollar swap curve

The construction of the Euro forward surface is conducted in a similar procedure when collateral is still posted in US Dollar:

a) *3M curve:* The 3M Euro curve can be determined by spreading it to the known 3M US Dollar forward curve. Here, a constant notional cross-currency swap that exchanges 3M Libor fixings in the two currencies Euro and US Dollar is the right instrument to calibrate the 3M forward curve. This is done by applying equation 5.46 where the cross-currency spreads is added to the Euro leg and noticeably the EUR/USD spreads are negative, see table 5.4. Moreover, it is assumed that both the foreign risk free and collateral rate share the same dynamic properties, i.e. equation 5.51 is applied, and thus the correction arising from the change of numeraire is neglected.

Table 5.4: Euro input data

Maturity in years	EUR/USD CCS in bps	OIS vs 3M in bps	1M vs 3M in bps	3M vs 6M in bps	3M vs 12M in bps
1	-46.0	36.1	26.6	19.4	32.1
2	-45.5	36.0	27.0	19.2	32.3
3	-42.8	35.8	25.0	19.8	30.8
4	-38.8	31.9	22.8	18.3	28.8
5	-34.4	31.4	22.1	17.3	26.8
6	-32.4	32.0	21.4	16.6	25.1
7	-30.0	30.7	20.9	15.9	23.8
8	-27.9	30.6	20.5	15.2	22.6
9	-25.7	29.7	20.1	14.5	21.6
10	-23.5	29.5	19.9	13.6	20.4

Source: Bloomberg

b) *OIS, 1M, 6M, 12M curves:* These curves are determined in the exact same manner as in the US Dollar case. Table 5.4 presents the tenor spreads applied in the optimization. Here, the Euro OIS rates are available for all relevant maturities and therefore no interpolation is needed.

c) *Interpolation:* Again, after interpolating the 'zig-zag' forward curves with a polynomial interpolation of 4th order, the resulting curves become smoother.

d) *Turns:* Here, the same intuition is applied as in the US Dollar case. Historical turn effects for the last four years are depicted in table 5.5. From this, the turn effect for the first year is equivalent to 12.4, 23.5 and 5.1 basis points for the OIS, 1M and 3M tenors, respectively. The turn effects for the second

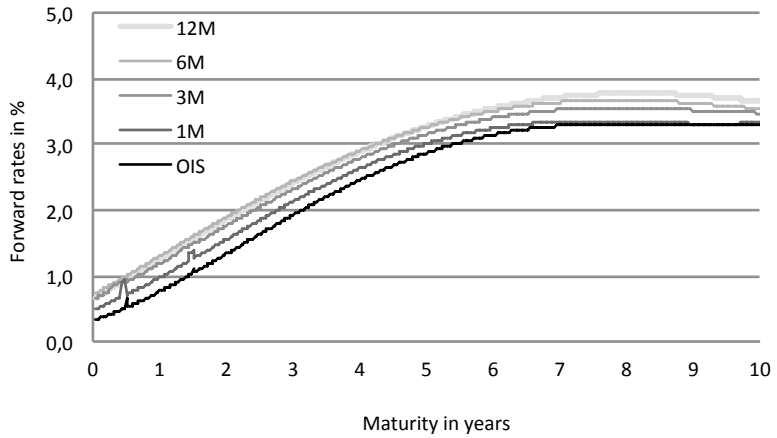


Figure 5.8: Euro forward curves including turn effects

year are roughly estimated to be 50 percent of the turn effects in the first year. Again, the forward curves in figure 5.8 display the turn effects on the respective tenor's forward curve.

Table 5.5: Turn of year effects in bps (Euro)

Year	O/N	1M	3M
09/10	8.9	2.8	1.1
08/09	8.7	21.8	9.5
07/08	29.8	64.0	6.0
06/07	2.0	5.4	3.7
Average	12.4	23.5	5.1

Source: Datastream

e) *Stubs:* Interpolating the stub floating forward rates in between the surface spanning tenors OIS, 1M, 3M, 6M and 12M is again done be using linear interpolation.

Finally, the surface of US Dollar collateralized Euro forward rates is presented in figure 5.9. Compared with the US Dollar surface, the level of the Euro surface in the long end is approximately one percentage-point lower. Furthermore, the slope between a maturity of 1 to 4 years is not as steep as in the US Dollar surface.

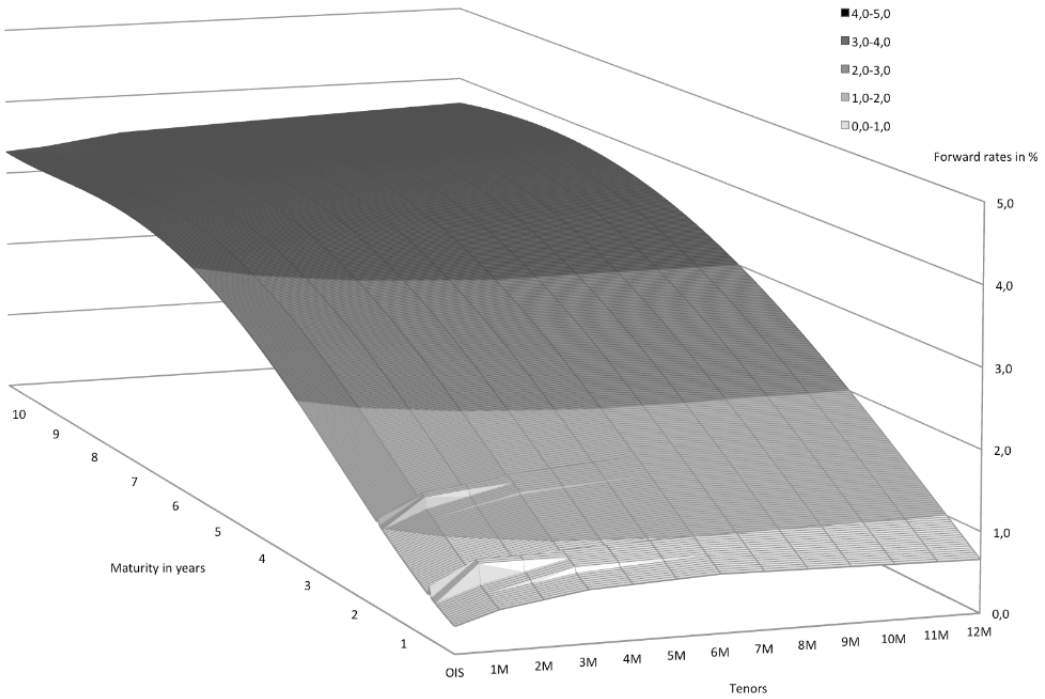


Figure 5.9: US Dollar collateralized Euro forward surface

Inspired by Linderstrøm and Scavenius (2010)

5.4.2 Forward surface for Mark-to-Market Swaps

Extending the surface for pricing mark-to-market swaps the following procedure describes the construction of a Euro forward surface assuming the collateral is still posted in US Dollar:

a) *3M curve:* Based on the assumption that the risk free rate equals the collateral rate and that both discounting curves in US Dollar and Euro are determined as shown in the previous section it is possible to back out the 3M Euro forward curve from equation 5.60. Here, the challenge is to estimate the US Dollar leg. For the purpose of transparency, all input parameters are examined in the following:

- The 3M forward Libor rates L_{T_{n-1},T_n} are extracted from the US Dollar collateralized US Dollar surface depicted in figure 5.6.
- The forward exchange rates $FX_{t,T_{n-1}}$ are provided by Nordea Analytics for all necessary maturities.
- The Fed Fund rate volatilities $\sigma_{FB_{n-1,n}}$ are unfortunately not accessible since neither options on 3M OIS futures at the Chicago Mercantile Exchange are traded frequently enough nor is data from the OTC markets available. In order to circumvent this issue the historical 60 days volatilities of Eurodollar futures contracts of different maturities are used as

a proxy. The underlying of a Eurodollar futures is a US Dollar deposit with a maturity of 3 months. Although this rate usually deviates in its level from the needed overnight rate the respective volatilities serve as a good approximation.

- The forward FX volatilities $\sigma_{FX_{n-1}}$ are taken from at-the-money FX options of the cross EUR/USD again provided by Nordea Analytics. However, these volatilities are only liquid up until 5 years and thereafter assumed to be constant.
- The correlations ρ_{n-1} between the forward risk free bonds and the FX forward rates are calculated on historical data. Here, the time series for both variables are estimated applying their respective formulas from 5.58. This requires US Dollar and Euro OIS rates for the period running from June 30th 2009 to June 30th 2010 which need to be interpolated by applying the interpolation scheme of choice, i.e. monotone convex spline, to receive discount factors for all maturities. Having both time series $\{FB\}$ and $\{FX\}$ in place they need to be transformed into continuously compounded log returns since they follow a geometric brownian motion (Campbell, Lo, and MacKinlay, 1997, p.362).
- The cross-currency basis spreads b_N for EUR/USD CCS exchanging payments every 3 months are taken from Bloomberg for all maturities up until 10 years.

Under the condition that both floating legs have a present value of zero, the forward rates can be optimized by using an iterative process such as Excel's solver. The applied data can be found in the attached Excel files.

b) *OIS, 1M, 6M, 12M curves:* The remaining forward curves can be determined by spreading them against the 3M curve similarly to the constant notional case. Here, the tenor basis spreads from table 5.4 are applied.

c) *Interpolation:* Again, after interpolating the jagged forward curves with a polynomial interpolation of 4th order, the resulting curves become smoother.

d) *Turns:* Here, the same historical turn effects are applied as in the constant notional case for the Euro curves.

e) *Stubs:* Interpolating the stub floating forward rates in between the surface spanning tenors OIS, 1M, 3M, 6M and 12M is again done using linear interpolation.

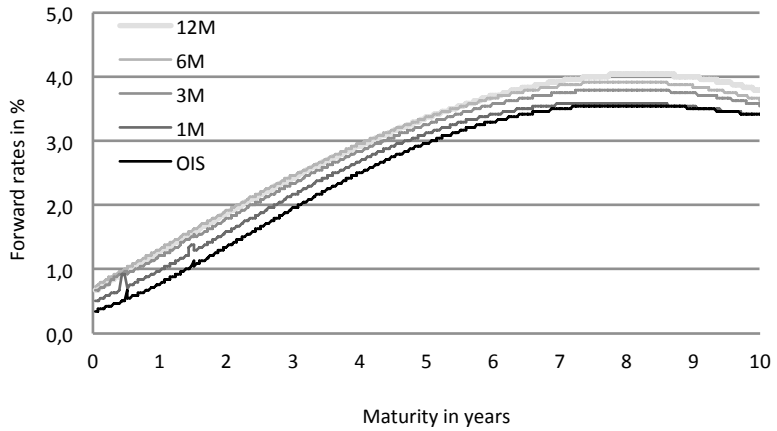


Figure 5.10: Euro forward curves including turn effects for mark-to-market swaps

Having constructed the Euro forward surface depicted in figure 5.11 it enables the pricing of mark-to-market derivatives of different tenor and maturities that are collateralized in US Dollar. Comparing this surface with the one constructed in the constant notional case it can be inferred that both are very similar up to a maturity of approximately 5 years. Afterwards the mark-to-market surface becomes slightly steeper across all tenors. This is especially caused by correlations that are positive and large for longer maturities. A more intuitive explanation is the reduced credit default risk inherent in mark-to-market swaps.

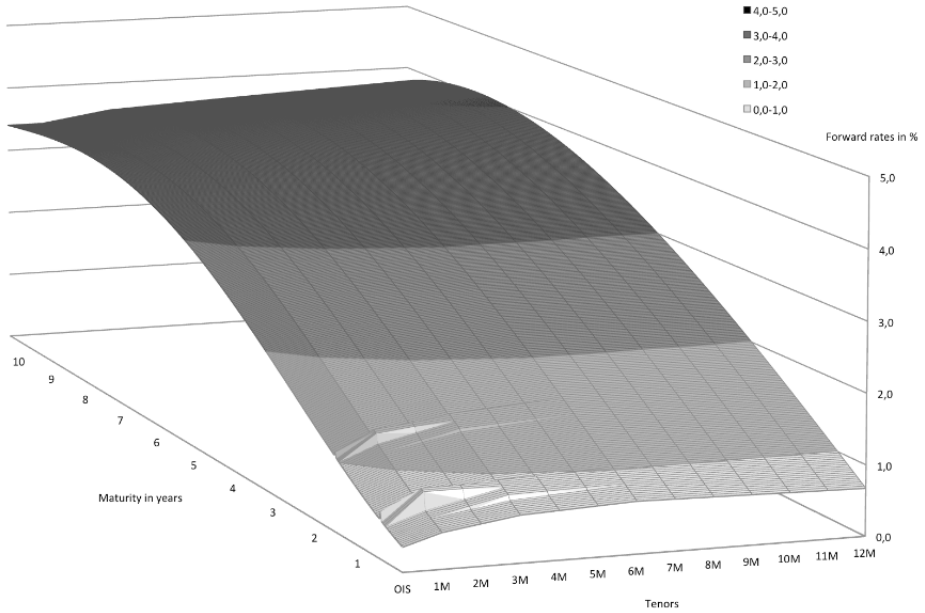


Figure 5.11: US Dollar collateralized Euro forward surface for mark-to-market swaps

Inspired by Linderstrøm and Scavenius (2010)

By reconstructing the input data the methodology is validated and hence applicable for pricing purposes. Figure 5.12 compares the input cross-currency basis spreads for EUR/USD as they are quoted in the market with the basis spreads backed out from the Euro surface for mark-to-market swaps. It can be concluded that the post-crisis framework is able to recover the input spreads reasonably. The maximum difference of 3 basis points occurs at a maturity of 5 years and exceeds the usual bid/ask spread of approximately 1 basis point.

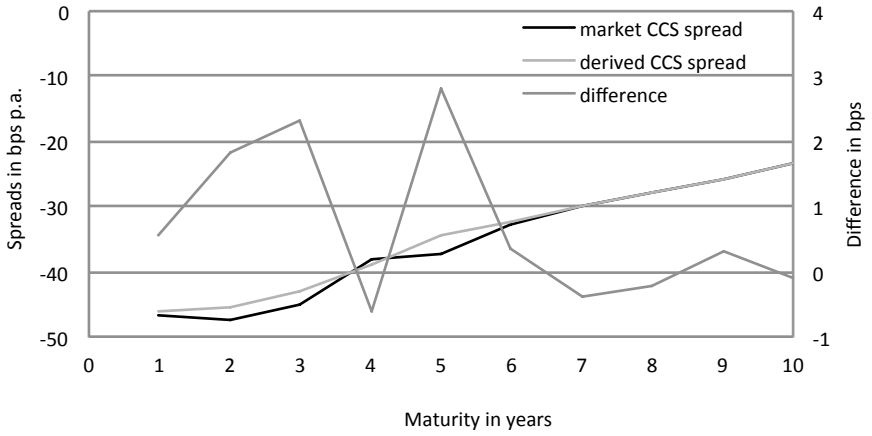


Figure 5.12: Reconstruction of the EUR/USD cross-currency basis spreads

5.5 The impact of basis spreads on swap pricing

Having demonstrated the construction of the forward surfaces, it remains to answer the question about the impact of basis spreads on swap pricing. A common misperception is that basis spreads affect the present value of interest rate derivatives by the proportion *spread size* to the *level of interest rate*. However, the profit or loss of such a position solely depends on the *change of interest rate*. Counterparties that entered a swap contract but are unable to reflect basis spreads in their present values might be exposed to disastrous consequences. Following the example in Fujii, Shimada, and Takahashi (2009b, p.17) the existence of basis spreads affects the mark-to-market of derivatives through two paths:

1) Change of the forward expectation of Libor rates

2) Change of the discounting rate

In the following, the importance of appropriate surface construction will be clarified by examining both paths.

The first effect shall be discussed exemplarily for a bank that offers structured products without taking basis spreads into consideration. The bank pays the payoff of their structured products to its clients in exchange for Libor, meaning the bank invests the notional, it receives by selling structured products, in the interbank market. Here, it is assumed that the bank's funding sources are 3M and 6M US Dollar Libor in equal measure, reflecting the different demands among their clients. If the bank still applies the pre-crisis pricing framework and thus derives both the 3M and the 6M forward curve from the same spot curve as outlined in chapter 3, then the implied 3M vs 6M tenor spread is zero. Moreover, using US Dollar IRS with semiannually payments as bootstrapping instruments implies that the value of the 3M funding leg is significantly overestimated, remembering formula 5.42 for the TS. The impact can be quantified by converting this overvalued leg into a stream of 6M Libor rates via entering a tenor swap where the bank pays 3M Libor plus spread. Hence, the bank experiences a loss that can be approximated as

$$\text{Loss} \simeq \text{Oustanding Notional} \times \text{PV01(Average Duration)} \times \text{3M vs 6M spread}$$

where PV01 denotes the annuity of the corresponding swap, i.e. the sum of the discounting factors times daycount fractions. Assuming an average duration of 10 years and a tenor spread of 10 basis points, the loss would be around one percent of the notional outstanding which would be far from negligible. If the same bank is active also in the IRS market, traders would have an incentive to enter swaps as 3M receivers since they can offer very competitive swap rates due the bank's faulty system. When disregarding basis spreads the forward rates become too high leading to the floating leg being overvalued and thus resulting in higher swap rates.

The impact from the second effect is mostly significant when taking collateralization properly into account. Changes in the discounting curve are especially significant for pricing cross-currency swaps as the notional is exchanged at maturity. Assuming a Libor-OIS spread of 10 basis points the present value of a notional payment in 10 years time would be different by approximately one percent point of its notional. Again, if the bank trades CCS in many currencies then the impact from the differences between various Libor and collateral rates of each currency would be tremendous. In principle, the value of products that depend on discounting are strongly influenced when the basis spreads are large which results in a significant difference between Libor and risk free discounting factors. Such products are for instance forward starting swaps (Linderstrøm and Scavenius, 2010, p.20).

The aforementioned example illustrates the difference between the pre- and post-

crisis pricing framework where the former applies one common curve for all tenors. Triggered by the distress in the financial markets each tenor now carries its own credit and liquidity premia which is reflected in the post-crisis pricing framework. It is worth remembering that an IRS is in fact a portfolio of FRAs which means that they must also be adjusted for spreads even though they are not directly exposed to these.

5.6 The impact of collateralization on swap pricing

A crucial point with respect to pricing interest rate swaps is collateralization. Meanwhile, the majority of banks agree on using OIS rates to discount future cashflows on collateralized swaps as opposed to the bank's own funding cost which theoretically should be used for pricing uncollateralized swaps. This is in fact far from easy and makes standardisation nearly impossible. If each bank discounts cashflows at its own funding rate, the same swap will deviate in value from one bank to the other. In fact, a bank has to move away from marking-to-market towards marking-to-model (Whittall, 2010a, p.1).

In theory, higher quality banks with a lower cost of funding will be able to offer a better rate to clients that receive fixed through an uncollateralized swap and vice versa. Consequently, the new pricing framework might exacerbate this problem if clients prefer to use high-quality banks when receiving fixed and banks with a poorer rating when paying fixed. On the other hand, it could have a self-regulating effect. Market participants might be forced to reduce the mismatch between collateralized and uncollateralized counterparties and thus establish a more balanced client portfolio (Whittall, 2010a, p.2).

Many CSA trades include thresholds which require a counterparty to post collateral once a predetermined level has been breached, see section 5.2.2. This optionality element complicates the pricing of interest rate swaps since a bank needs to use its own funding rate up until the moment the market value of the swap crosses the threshold and from this point onwards it must use the OIS rate as collateral rate (Whittall, 2010a, p.3). Table 5.6 summaries swap rates on a semiannual 10-year British Pound IRS based on different assumptions.

In other words, the lower the funding rate is for the bank, the better is the rate that will be offered to clients that receive fixed. Likewise, a smaller threshold goes hand in hand with a higher swap rate (Whittall, 2010a, p.3). Due to the fact that discounting is done at the funding cost of the bank up until the threshold limit, a higher threshold limit clearly increases the total funding cost and hence lowers the swap rate.

Table 5.6: Swap rates in comparison June 9th 2010

Assumption	Swap rate (%)
uncollateralized - funding: Libor flat	3.454
uncollateralized - funding: Libor+100 bps	3.420
collateralized - threshold: 50 Mio GBP	3.437
collateralized - threshold: 5 Mio GBP	3.452

Source: Barclays Capital

Another issue that challenges practitioners more and more is the so called one-way CSA trade. Often sovereigns or supra-national institutions with a high credit rating require from their counterparties with a lower credit rating to sign such a contract. Here, the counterparty has to post collateral if the market value of the swap is negative but does not receive collateral in return in the case the swap value moves in its favour. Also smaller market participants experience these unequal conditions if they want to trade with big international banks. Naturally, one-way CSA swaps create asymmetric default risks and thus complicate swap pricing significantly.

Lastly, Johannes and Sundaresan (2007, p.385) elaborate on the effect of mark-to-market and cost of collateral on swap rates. They consider a swap from the perspective of the fixed receiver and assume that collateral and interest rates are positively related with each other which seems very intuitive. If floating rates fall, the swap will have a positive mark-to-market value and the fixed receiver receives collateral which yields a lower return since interest rates decreased. Conversely, if rates increase, the swap will have a negative mark-to-market value and the fixed receiver will have to post collateral. However, financing this collateral is now more costly due to increased interest rates. As a consequence the fixed receiver will demand a higher swap rate to get compensated for the acceleration of costs implied by mark-to-market effect and costly collateral.

Summarizing the impact of collateralization on swap rates it can be stated that there exist several factors that complicate the pricing of swaps. Choice of collateral, minimum threshold limit and other CSA options may lead to varying swap rates.

6

Reflections

The purpose of this chapter is to put the thesis into perspective. Therefore, this chapter is divided into three sections. The first part seeks to critically reflect on the assumptions applied to construct forward surfaces. The second part discusses the implications of the post-crisis pricing framework in practice. The last part outlines ideas for further research within derivatives pricing.

6.1 Critique

The thesis contributes to a very actual research area within derivative pricing which there certainly will be drawn further attention to in the nearer future. Critique with respect to the introduced pricing framework can be stated from two perspectives.

On the theoretical front, the focus was to provide academics and practitioners with a broad and accessible overview of the main formulas and issues for the pricing of swaps in the aftermath of the financial crisis. Certainly, some topics and details have been left out which others may find important. For example, the construction of a US Dollar forward surface collateralized in Euro or a toolkit for the change of numeraire, just to mention a few. However, all topics that the authors found essential when dealing with interest rate swaps were covered in the thesis.

On the practical front, there certainly can be brought up many arguments against the implementation of the pricing framework in the thesis. Firstly, constructing surfaces is basically a 3D interpolation problem and therefore strongly depending on the interpolation technique of choice. There exist several alternatives to interpolating linearly across tenors and using a 4th order polynomial to interpolate across maturities. Secondly, the availability and accessibility of data might limit the analysis of pricing swaps. In order to set-up a comprehensive pricing framework, a broad range of input data is needed. For example, the resulting surfaces have a limited maturity up to 10 years as US Dollar OIS rates are not quoted for longer maturities, although this problem could be fixed by applying the observed OIS vs 3M basis spreads reaching out to 30 years. Likewise, there is not distinguished between the mark-to-market and constant notional cross-currency basis spreads which are

supposed to be slightly different in theory. Moreover, the assessment of whether market instruments are liquid enough is difficult due to the fact that most derivatives are traded OTC and thus information provided by the clearing houses is scarce. Thirdly, additional criticism might rise with regards to the applied methodology. For instance, when constructing the Euro forward surfaces collateralized in US Dollar the set of Euro discount factors $\{P^f_{t,T_n}\}$ is bootstrapped directly from OIS rates. Others might argue that this set needs to be backed out from FX forward rates using the interest rate parity. But this example just outlines the main challenge when constructing a consistent pricing framework. Generally speaking, some curves are bootstrapped from the market whereas others are backed out and yet others are calibrated from models. In the end, all curves are interconnected and that makes swap pricing a demanding but interesting topic. Lastly, one could have applied the resulting surfaces to price collateralized swaps with respect to varying thresholds and cheapest-to-deliver options between different collateral currencies. However, this would require non-trivial Monte Carlo simulations to model the swap market value over time.

6.2 Applicability in practice

After reviewing the post-crisis framework, market participants are faced with the challenge to implement pricing schemes for both uncollateralized and collateralized contracts. Firstly, it is now clear that the existing single-curve system can be disregarded. Instead, discounting and forward curves need to be separated to implement a full surface for pricing trades in a multi product trading environment.

For pricing uncollateralized products, future cashflows should be funded at the rate the bank's treasury desk is able to borrow money for in the market. Therefore it is up to each bank to determine one discounting curve per currency while defining only one anchor discounting curve in one anchor currency which is particular important to avoid arbitrage within the bank's own systems. As outlined in section 5.6, this also involves the necessity to determine one discounting curve per currency for each counterparty. Here, credit risk models come into play if spreads cannot be observed in the CDS or bond markets.

As a consequence the market is shifting towards a new market standard in pricing derivatives backed by collateral. As the collateral currency can vary Whittall (2010b) suggests to either set up separate books for each currency or divide the collateralized book into buckets. For instance, counterparties with CSAs only allowing to post collateral in US Dollar will go into the Fed fund rate bucket. Up until this point the main issues associated with setting up the simulation scheme

have been discussed under the assumption of collateral being posted in cash and in one currency. Challenges emerge if there exists an optionality in the currency or type of collateral, such as equities or bonds, that can be posted. Incorporating cheapest-to-deliver switch options might make the pricing dependent on the repo markets if one assumes that the counterparty is going to use whichever collateral is cheapest to post. Alternatively, looking at what collateral a counterparty has historically posted can be a meaningful consideration if ones assumes that some market participants do not post collateral efficiently (Whittall, 2010b, p.5). These features next to thresholds and one-way CSAs as introduced in section 5.6, complicate curve calibration tremendously as most of them require some more advanced models.

Linderstrøm and Scavenius (2010) suggest to implement a central calibration system that collects various inputs from different trading units and builds one surface model per currency. Hence, this gives consistent pricing across all desks though presenting a cultural revolution and many problems when running across geographically and temporary separated trading floors. Setting up such a system becomes quite demanding when complicated calibration is needed for even plain vanilla instruments as well as implementing a sufficient noise reduction technique to recover the observed swap prices within reasonable calculation time.

Concludingly, many especially smaller institutions might face challenges developing pricing schemes for both uncollateralized and collateralized trades that are able to incorporate as many of the above mentioned features as possible. With a growing tendency to engage in collateralized trades, a higher level of standardization of these contracts can be expected in the future. This would lead to a more common pricing framework as well as an enriched transparency of derivatives pricing.

6.3 Ideas for further research

The simplifying assumptions of constant and time-homogeneous spreads between the discounting curve and the Libor index curves may give rise to potentially important problems. Particularly, when credit conditions are tight, the dynamics of the overnight rate set by the central bank and the Libor index in the market can differ substantially (Fujii, Shimada, and Takahashi, 2009b, p.20). Therefore developing a model which can handle multiple dynamic curves and its practical calibration scheme is an important topic for further research. Recently, this issue has been addressed by Fujii, Shimada, and Takahashi (2009a).

After implementing dynamic basis spreads there remain several interesting topics for the practical implementation of the new framework. Fujii, Shimada, and Takahashi (2009a) acknowledge that the analytical approximation for vanilla options

will be necessary for fast calibration and for the use as regressors for Bermudan and American type of exotics. This also leads to further investigation of how to price other more complicated instruments in this framework. For instance, the implications on the price of convexity products due to the separation of the discounting curve and the Libor-OIS spread. Another important consideration is the method to obtain stable attribution of vega exposure to each vanilla option for generic exotics (Fujii, Shimada, and Takahashi, 2009a, p.23).

Inevitably, looking into hedging possibilities within this framework constitutes an important consideration for the implementation of the framework. Here, especially the calculation of the delta sensitivity of a portfolio consisting of a variety of multiple interest rate derivatives with different underlying tenors is an interesting topic that has recently been addressed by Bianchetti (2010). Furthermore, dealing with issues in regards to which hedging instruments to be used and how the calculation of hedge ratios can be done as well as determining the many sensitivities (the Greeks), remain interesting research topics.

Lastly, adjusting discounting curves for collateralized trades began with interest rate swaps which is why the first research material is conducted. Conversely, the implications of the widening of OIS-Libor spreads on other asset classes such as equity, credit or commodity derivatives have hardly been studied yet.

7

Conclusion

The aim of the thesis has been to revisit the pricing of interest rate swaps before and after the financial crisis. Interest rate swaps constitute a significant part of the OTC traded interest derivatives market. From the late 1990's the OTC traded interest rate derivatives market has grown rapidly and is by far the most dominating asset class compared to FX, CDS, equity and commodity derivatives. The literature review showed that the pricing framework of interest rate swaps has been altered after the financial crisis due to the widening of basis spreads. Addressing the post-crisis framework has been done by reviewing the pre-crisis framework and the impacts on basis spreads from the financial crisis. In the following the main findings and conclusions are summarized.

In the pre-crisis pricing framework interest rate, tenor and cross currency swaps were examined. The construction of the spot curve involved several important steps. Firstly, by applying the most liquid market instruments such as short term deposits, FRAs, futures and swaps, the key spot rates were bootstrapped. For the short end of the curve the deposit rates were directly applied. The middle area was done by backing out the forward rate on each instrument, i.e. FRAs and futures, at its respective maturity to construct the effective key spot rate by combining both spot and forward rate. In the long end, swap rates were used to consecutively bootstrap discounting factors and then determine spot rates. Secondly, due to the importance of interpolation when constructing yield curves, an analysis of the linear, the cubic hermite spline with a Hyman filter and the monotone convex spline was conducted. The analysis found the monotone convex interpolation technique best applicable for bootstrapping the key spot rates as this technique ensures a continuous, non-negative forward curve. Interestingly, interpolation plays an important role even before interpolating the key spot rates. Here, interpolation between short term spot rates was done in order to construct other key spot rates for longer maturities, eventually combining short term spot rates with forward rates. Thirdly, from the spot curve it was then possible to derive the forward curve. Here, the turn of year effect was incorporated into the curves in order to reflect the increased search for liquidity by financial institutions before a periodic balance sheet. These jumps can

be found as one discontinuity in the spot curve and as two discontinuities in the forward curve. In addressing the size of the turn of year effect a historical average based on the previous four years was applied. The pre-crisis framework bases its pricing of swaps on the bootstrapped spot curve, i.e. the zero curve, which consists of several instruments, some with different underlying tenors. Even though the pre-crisis pricing framework allows the pricing of interest rate swaps it does not provide a consistent pricing framework. All forward curves are based on one and the same spot curve although each tenor now contains its own credit and liquidity premia. Consequently, the prevailing forward curve does not represent the risk inherent in each tenor appropriately.

The analysis of the distress in the financial markets was based on several risk indicators. These indicators of financial distress such as the gap between FRA market rates and their replicated forward rate, Libor-OIS spreads, tenor and cross-currency basis spreads all experienced significant changes in levels that were not yet seen in the financial markets. Consequently, Libor rates can no longer be perceived as risk free, basis spreads can no longer be neglected and the standard calculation of implied forward rates no longer holds.

The main part of the thesis was then to present an improved post-crisis pricing framework that would take the consequences of the financial distress into account. Firstly, the discrepancies between the FRA rates and their implied forward rates were addressed where a basis-consistent replication of the FRA rates was provided. Secondly, the post-crisis pricing methodology was introduced, distinguishing between collateralized and uncollateralized contracts. Both approaches seek to determine the set of forward curves for each underlying tenor, i.e. the forward surface, in order to price interest rate swaps. In the uncollateralized case, deriving the discount factors is done sequentially by setting up an equation system using IRS, TS and CCS. From the discount factors it is then straight forward to derive the set of forward curves. However, not all financial institutions find it reasonable to apply the Libor rate as a proxy for their funding cost. Addressing an appropriate discounting rate can be of great challenge for institutions that differ significantly in credit quality as the choice of discounting has great impact on the price of the swap price. Due to this complication the pricing of different contract types were analyzed in the collateralized section. Deriving the forward surface in the collateralized case was done by sequentially deriving the risk free discount factors from the OIS rates for both constant notional and mark-to-market swaps for US Dollars and Euro assuming US Dollar as collateral currency. Once having the risk free discount factors in both currencies, backing out the forward rates for different underlying tenors was done using an optimizer. For the CNCCS two surfaces were constructed, one for US Dollars and Euro respectively. Here, the US Dollar surface generally

showed higher levels of forwards rates for longer maturities. From the MtMCCS an additional Euro forward surface was constructed. The level here deviates only slightly from the surface in the constant notional case. Comparing the pricing of swaps before and after the financial crisis, it can be concluded that basis spreads might have a significant impact on the swap rates, depending on the length of the contract and the change of the underlying interest rate. Moreover, factors such as thresholds, one-way CSAs and other options regarding the posting of collateral considerably complicate the pricing framework due to model dependent parameters and may lead to varying swap rates.

The refined approach to price interest rate swaps can now be done in a more consistent framework where basis spreads are taken into consideration. From this pricing framework it also becomes possible to price other interest rate derivates. Although the methodology proposes a consistent pricing framework, researchers have already come up with improvements with regards to incorporating dynamic spreads and more advanced calibration techniques. Inevitably, pricing collateralized interest rate derivatives will become an even more important topic in the financial industry as collateral contracts have increased over the past years and now constitute 79 percent of all OTC traded interest rate derivatives. A higher level of standardization of these contracts can be expected in the future which will contribute to improve the transparency in derivatives pricing.

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