

Curve Building and Swap Pricing in the Presence of Collateral and Basis Spreads

S I M O N G U N N A R S S O N

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Abstract

The eruption of the financial crisis in 2008 caused immense widening of both domestic and cross currency basis spreads. Also, as a majority of all fixed income contracts are now collateralized the funding cost of a financial institution may deviate substantially from the domestic Libor. In this thesis, a framework for pricing of collateralized interest rate derivatives that accounts for the existence of non-negligible basis spreads is implemented. It is found that losses corresponding to several percent of the outstanding notional may arise as a consequence of not adapting to the new market conditions.

Keywords: Curve building, swap, basis spread, cross currency, collateral

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1 INTRODUCTION

The global financial meltdown during 2008 inevitably caused a lot of change on the financial markets. Companies were faced with increased credit and liquidity problems and for banks this situation affected their trading abilities. Henceforth it became vital to account for credit and liquidity premia when pricing financial products. The effects were particularly apparent in the market for interest rate products, i.e. FRAs, swaps, swaptions etc., and as a consequence professionals started developing new pricing frameworks that would correctly account for the increased credit and liquidity premia. More specifically, basis spreads between different tenors and currencies that were negligible (typically smaller than the bid/ask spread) before the crisis were now much wider. A new pricing framework would have to account for the magnitudes of these spreads and produce consistent prices that are arbitrage-free. In practice this entails that one should estimate one forward curve for each tenor, instead of using one universal forward curve for all tenors. Also, as most over-the-counter interest rate products are nowadays collateralized the question of how to correctly discount future cash flows must be raised.

In light of this, the purpose of this thesis is to *implement a pricing framework that accounts for non-negligible basis spreads between tenors and currencies that is also able to price collateralized products in a desirable manner*. The approach will be empirical, i.e. forward rates and discount factors will be extracted from available market quotes and we will not develop and implement a framework that models the term structures of basis spreads.

We will assume basic knowledge of stochastic calculus as covered in Øksendal (2003) [22]. Martingale pricing of financial derivatives is also assumed a prerequisite and an introduction is given in Björk (2009) [3]. Geman et. al. (1995) [12] provides an extensive discussion on the important technique of changing the numéraire. Also, Friedman (1983) [8] rigorously presents various essential concepts of analysis, such as fundamental measure theory and Radon-Nikodym derivatives.

The rest of this paper is structured as follows. The remainder of this chapter provides a summary of the various spreads that widened during the financial crisis and concludes with an introduction to swap pricing in the absence of basis spreads. Chapter 2 presents a framework that accounts for the prevailing basis spreads between tenors and currencies, with and without the presence of collateral. This framework is later implemented in Chapter 3, where technical details are covered to a greater extent. Results are discussed in Chapter 4 and Chapter 5 concludes.

1.1 THE LIBOR-OIS AND TED SPREADS

The USD London Interbank Offered Rate (Libor from now on) is an average of the rates at which banks think they can obtain unsecured funding. It is managed by the British Bankers' Association (BBA) to which the participating banks¹ submit their estimated funding costs. The European equivalent to the Libor is the European Interbank Offered Rate (Euribor), which is managed by the European Banking Federation. While the Libor is an average of the perceived funding costs of the participating banks, the Euribor is an average of the rates at which banks believe a *prime bank* can get unsecured funding. Both rates are quoted for a range of tenors, where the 3m and 6m are the most widely monitored.

An overnight indexed swap is a contract between two parties in which one party pays a fixed rate (the OIS rate) against receiving the geometric average of the (compound) overnight rate over the term of the contract. In the US, the overnight rate is the effective Federal Funds rate whereas in Europe it is the Euro Overnight Index Average (Eonia) rate.² The OIS rate can now be viewed upon as a measure of the market's expectation on the overnight rate until maturity (Thornton, 2009) [29]. Because no principal is exchanged and since funds typically are exchanged only at maturity there is very little default risk inherent in the OIS market.

Due to the low risk of default associated with the OIS rate the spread between the Libor and the OIS rate should give an indication of the default risk in the interbank market. In fact, the Libor-OIS spread is considered a much wider measure of the health of the banking system, for example Morini (2009) [21] emphasizes that liquidity risk³ also is explanatory for the Libor-OIS spread. Cui et. al. (2012) [6] moreover suggests that increased overall market volatility and industry-specific problems may cause the Libor-OIS spread to widen. A general flight to safety whereby banks are reluctant to tie-up liquidity over longer periods of time may also cause the spread to increase, as mentioned in the Swedish Riksbank's survey of the Swedish financial markets (2012) [28]. Recently, Filipovic and Trolle (2012) [7] suggested an approach of decomposing the Libor-OIS spread into default and non-default components.

Figure 1.1 depicts the Libor-OIS and Euribor-OIS spreads for 3m rates. As mentioned in Sengupta (2008) [27], the Libor-OIS spread spiked at 365 basis points on October 10th 2008, presumably due to the broad "illiquidity wave" that followed the bankruptcy of

¹A list of the banks contributing to the Libor fixing is found at <http://www.bbalibor.com/panels/usd>.

²The effective Federal Funds rate is computed as a transaction-weighted average of the rates on overnight unsecured loans that banks make between each other. The banks in question do not entirely coincide with the Libor panel, which is the case for the Eonia rate.

³Liquidity risk is defined as the risk that banks cannot convert their assets into cash.

Lehman Brothers on September 15th. Both the Libor-OIS and Euribor-OIS spreads then narrowed to a relatively stable level below 40 basis points. Until mid-2011 the spreads were highly correlated and tended to follow each other, however in the second half of that year the situation in Europe deteriorated and the Euribor-OIS spread peaked at 100 basis points. As of today, the spreads have yet again narrowed and lie between 10-15 basis points.

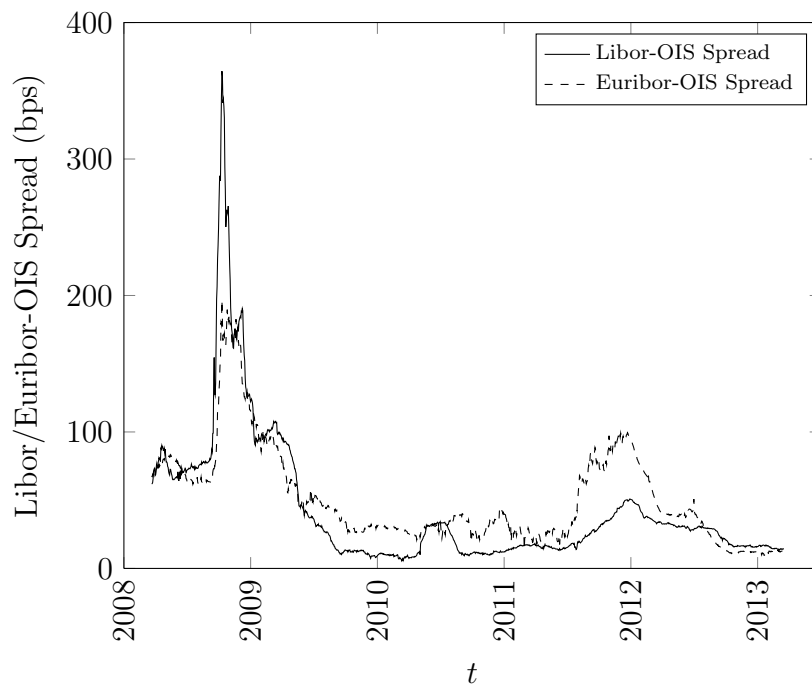


Figure 1.1: The 3m Libor-OIS and 3m Euribor-OIS spreads over a 5 year period (Source: Datastream).

Another, and complementary to the Libor-OIS spread, measure of credit risk is the TED-spread. In the US, it is defined as the difference of the 3m Libor and the 3m T-bill rate whereas in Europe it equals the difference of the 3m EUR Libor (not to be confused with the Euribor) and the average 3m spot rate on AAA-rated European government bonds. The TED spread is thus a measure of the risk premium required by banks for lending to other banks instead of to the government. Hence, when the TED spread widens it is a sign of higher perceived counterparty risk, causing Libor rates to increase and government yields to decrease (flight to safety). Figure 1.2 pictures the American and European TED spread over a 5 year interval. By comparing with Figure 1.1, it is seen that the TED spreads and Libor-OIS spreads are tightly correlated.

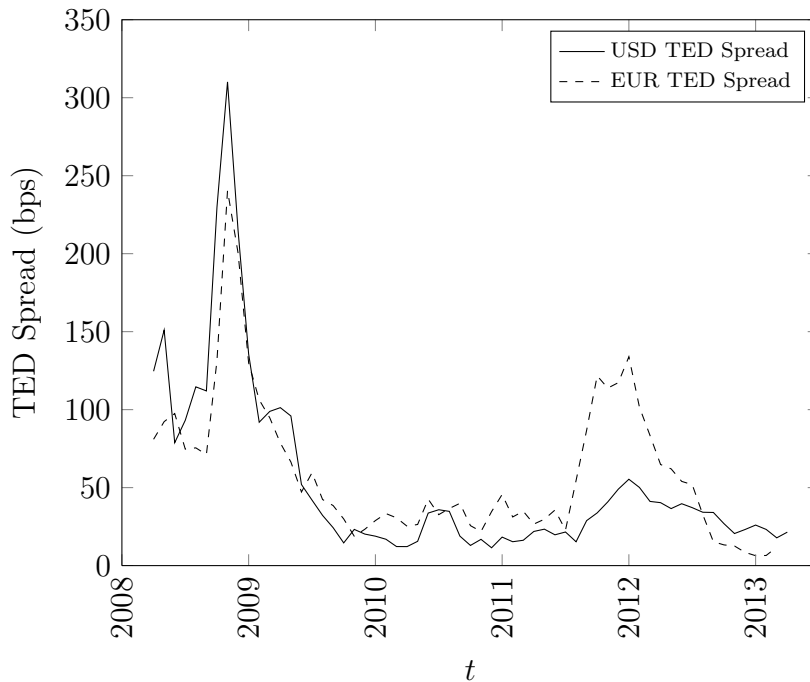


Figure 1.2: The American and European TED spreads over a 5 year period (Source: Datastream).

1.2 TENOR BASIS SPREADS

A tenor basis swap is a floating for floating swap where the payments are linked to indices of different tenors. The payments may for example be 6m Libor semiannually on the first leg and 3m Libor quarterly on the other. Tuckman and Porfirio (2003) [30] shows that in a default-free environment, a tenor basis swap should trade flat. This means that lenders are indifferent between receiving the 6m rate semiannually or the 3m rate rolled over every quarter, and the same goes for other tenors. In reality, the Libor rates have built in credit premia and it is an accepted fact that these premia differ between tenors. For example, lending at 6m Libor is associated with more counterparty risk than rolling lending at 3m Libor. In order to clear markets, the 6m Libor must thus be set higher than the rate implied by the 3m Libor in order to compensate for the higher counterparty risk. However, in a tenor basis swap counterparty risk can be eliminated with collateralization and the advantage of receiving 6m Libor is mitigated by a spread added to the leg paying 3m Libor. Hence, in the presence of credit risk tenor basis swaps do not trade flat, but with a spread added to the leg with the shorter tenor.

Morini (2009) [21] suggests some explanations as to why lending at a longer tenor is associated with more counterparty risk as compared to rolling lending at a shorter tenor. Firstly, in case of default in the 3m-6m period, the 6m lender loses all his interest whereas the 3m roller receives interest for the first 3 months. Even though both lenders lose the

notional, the 3m roller is better off. Also, if the credit conditions of the counterparty worsen during the first 3 months the 3m roller can exit at par and move on to another counterparty. The 6m lender instead has to unwind the position at a cost that incorporates the increased risk of default. Compared to the 3m roller that exits at par, the 6m lender is worse off. However, in the opposite situation, i.e. that the credit conditions for the counterparty improve, the 6m lender may be better off than the 3m roller. This suggests that there is no overall gain for the 3m roller, but since there are commercial reasons for not unwinding a contract when it is convenient for the lender, the 3m roller has an advantage.

Prior to August 2007 the spreads in the tenor basis swap market (tenor basis spreads) were never higher than 10 basis points. The spreads started widening during the fall of 2007 and spiked during the Lehman crash in September 2008. As the tenor basis spreads and Libor/Euribor-OIS spreads to some extent both measure counterparty risk it is not surprising that they are positively correlated. As of today, the USD tenor basis spread is at most ≈ 15 basis points (3m vs. 6m Libor) at short maturities. The tenor basis spreads tend to decrease as the maturity increases and the difference in tenor becomes less important, and for maturities greater than 10 years it rarely exceeds 10 basis points.

1.3 CROSS CURRENCY BASIS SPREADS

A (constant notional) cross currency swap (CCS) exchanges the floating rate in one currency for the floating rate in another currency, plus the notionals at initiation and expiration. On November 14th 2012 one USD was worth 0.787 EUR. A typical CCS could thus look as follows:

- Exchange 1 USD for 0.787 EUR at initiation
- Exchange 3m Libor on 1 USD for 3m Euribor less 21 basis points on 0.787 EUR quarterly for 10 years
- Exchange 1 USD for 0.787 EUR at expiration

In due course it will be evident where the spread of 21 basis points comes from. Imagine a CCS that exchanges the default-free Eonia rate for the default-free Federal Funds rate. Tuckman and Porfirio (2003) [30] shows that such swap should trade flat. Indeed, paying 1 USD today, receiving the default-free Federal Funds rate on 1 USD and finally receiving 1 USD at expiration should be worth 1 USD today. Since a similar argument can be made with the EUR leg the swap should trade without a spread.

However, quoted cross currency swaps exchange Libor rates that are not default-free. One may thus decompose a CCS into a portfolio of three swaps; a cross currency swap

that exchanges the default-free Eonia rate for the default-free Federal Funds rate, a USD tenor basis swap that exchanges the Federal Funds rate for the 3m Libor and a EUR tenor basis swap that exchanges the Eonia for the 3m Euribor. It is now apparent that the cross currency basis spread derives from the difference between local tenor basis spreads. Now assume that the 3m Euribor has more credit risk than the 3m Libor. In a collateralized swap without default risk a stream of 3m Euribor would then be worth more than a stream of 3m Libor. To compensate for this advantage a negative spread is added to the leg paying EUR.

The situation above is exactly what prevails on the markets. Figure 1.3 shows how the 3m USDEUR cross currency basis spread has been negative during the last three years. It is clearly seen that the spread reached -150 basis points in the latter half of 2011, presumably caused by the then worsening situation in the Euro area. As the markets calmed the spread narrowed and it is now less than -30 basis points for all maturities.

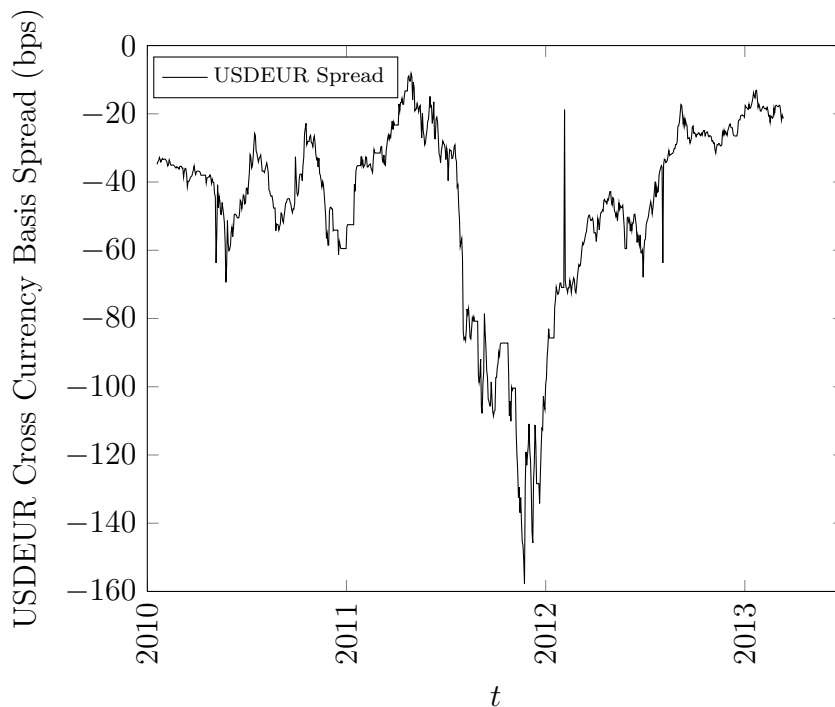


Figure 1.3: The 3m USDEUR cross currency basis spread over a 3 year period (Source: Datastream).

1.4 PREVIOUS RESEARCH

The works of Hull (2011) [17] and Ron (2000) [26] cover how to price interest rate swaps in a market absent of basis spreads. The focus lies on bootstrapping a single yield curve that is used both for discounting and extracting forward rates. This approach is briefly

covered in Section 1.5. Once equipped with a discrete set of yields, various interpolation techniques for obtaining a continuous yield curve are discussed in Hagan and West (2006) [13] and Hagan and West (2008) [14]. To avoid arbitrage, the interpolation scheme needs to produce positive forward rates. It is also desired that the obtained forward rates are stable and that the interpolation function only changes nearby if an input is changed (i.e. it is local).

Henrard (2007) [15] takes one step towards refining the conventional pricing framework by addressing the effects from changing the discounting curve. In Henrard (2010) [16], he further proposes a valuation framework where one forward curve is built for each Libor tenor. Similar work is done in Ametrano and Bianchetti (2009) [1], where a scheme that is able to recover the market swap rates is developed. However, as multiple discount rates exist within the same currency, their model is subject to arbitrage. The arbitrage-free model proposed in Bianchetti (2008) [2] is in one sense an improvement, but as noted in Fujii et. al. (2009a) [9] curve calibration cannot be separated from option calibration, which makes the model somewhat impractical. Mercurio (2009) [20] introduces a new Libor market model that is based on modeling the joint evolution of implied forward rates and FRA rates, where the log-normal case with and without stochastic volatility is analyzed. Johannes and Sundaresan (2009) [19] and Whittall (2010a) [32] further develop the multi-curve pricing framework by considering the impact of collateralization on swap rates, whereas Whittall (2010b) [31] discusses which discount rate to use in an uncollateralized agreement. Other works on the same topic include Morini (2009) [21] and Chibane et. al. (2009) [5].

Fujii et. al. (2010) [11] presents a method that consistently treats interest rate swaps, tenor basis swaps, overnight indexed swaps and cross currency basis swaps, where the effects from collateralization are explicitly addressed. This framework is refined in Fujii et. al. (2009a) [9], where a model of dynamic basis spreads is introduced. Finally, Filipovic and Trolle (2012) [7] proposes a term structure of interbank risk that is derived from observed basis spreads. Moreover, the term structure is decomposed into default (credit) and non-default components. It is shown that default risk increases with maturity whereas the non-default component is more dominant in the short term.

1.5 FRA AND SWAP PRICING BEFORE THE FINANCIAL CRISIS

The forward Libor rate contracted at time t for $[T_{n-1}, T_n]$ is defined by

$$L(t, T_{n-1}, T_n) = \frac{1}{\delta_{n-1, n}} \left(\frac{Z(t, T_{n-1})}{Z(t, T_n)} - 1 \right),$$

where $\delta_{n-1,n}$ is the day count factor and $Z(t, T_i)$ the time- t price of a default-free zero coupon bond maturing at T_i . Similarly, the spot Libor for $[T_{n-1}, T_n]$ is given by

$$L(T_{n-1}, T_n) = \frac{1}{\delta_{n-1,n}} \left(\frac{1}{Z(t, T_n)} - 1 \right).$$

Let Q^{T_n} be the forward measure with $Z(t, T_n)$ as numéraire (an introduction to the forward measure is given in Appendix A) and let $E_t^{T_n}[\cdot] = E^{Q^{T_n}}[\cdot | \mathcal{F}_t]$. It now holds that

$$\begin{aligned} E_t^{T_n}[L(T_{n-1}, T_n)] &= \frac{1}{\delta_{n-1,n}} E_t^{T_n} \left[\frac{Z(T_{n-1}, T_{n-1})}{Z(T_{n-1}, T_n)} - 1 \right] = \frac{1}{\delta_{n-1,n}} \left(\frac{Z(t, T_{n-1})}{Z(t, T_n)} - 1 \right) \\ &= L(t, T_{n-1}, T_n). \end{aligned}$$

A forward rate agreement (FRA) is a contract in which one party receives $L(T_{n-1}, T_n)$ at T_n whereas the counterparty receives a fixed rate K simultaneously. The net payoff at T_n is thus

$$V_{T_n} = \delta_{n-1,n}(L(T_{n-1}, T_n) - K).$$

Hence, the value at some arbitrary $t < T_n$ equals

$$\begin{aligned} V_t &= E_t^Q[\delta_{n-1,n}(L(T_{n-1}, T_n) - K)Z(t, T_n)] = \delta_{n-1,n}(E_t^{T_n}[L(T_{n-1}, T_n)] - K)Z(t, T_n) \\ &= \delta_{n-1,n}(L(t, T_{n-1}, T_n) - K)Z(t, T_n). \end{aligned}$$

At initiation it must hold that $V_0 = 0$ and we can then solve for the fixed rate K . An interest rate swap (IRS) is no more than a portfolio of forward rate agreements and can therefore be priced similarly. Since the present value at initiation has to equal zero we get

$$C(t, T_N) \sum_{m=1}^N \delta_{m-1,m}^{\text{ff}} Z(t, T_m) = \sum_{n=1}^N \delta_{n-1,n}^{\text{ff}} E_t^Q[L(T_{n-1}, T_n)Z(t, T_n)],$$

where $C(t, T_N)$ is the time- t fixed rate for a swap maturing at T_N . By changing the numéraire to $Z(t, T_i)$ if $n = i$ we get

$$\begin{aligned} C(t, T_N) \sum_{m=1}^N \delta_{m-1,m}^{\text{ff}} Z(t, T_m) &= \sum_{n=1}^N \delta_{n-1,n}^{\text{ff}} E_t^{T_n}[L(T_{n-1}, T_n)]Z(t, T_n) \\ &= \sum_{n=1}^N \delta_{n-1,n}^{\text{ff}} L(t, T_{n-1}, T_n)Z(t, T_n). \end{aligned}$$

By the definition of the forward Libor rate we arrive at

$$C(t, T_N) \sum_{m=1}^N \delta_{m-1, m}^{\text{fi}} Z(t, T_m) = \sum_{n=1}^N (Z(t, T_{n-1}) - Z(t, T_n)) = Z(t, T_0) - Z(t, T_N),$$

and it is now a simple matter to determine the swap rate $C(t, T_N)$. A more in-depth introduction to the conventional way of pricing swaps using only one forward curve is given in Björk (2009) [3].

2 THEORETICAL BACKGROUND

In this section it is described how to price interest rate swaps (IRS), tenor basis swaps (TS) and cross currency basis swaps (CCS) consistently with each other in a multi currency setup, both with and without collateralization. The theory is primarily based on Fujii et. al. (2010) [11], Fujii et. al. (2009a) [9] and Fujii et. al. (2009b) [10].

2.1 CURVE CONSTRUCTION WITHOUT COLLATERAL

Using observable quotes on the swap market we derive a discounting curve as well as several (index-linked) forward curves under the assumption that no collateral agreement is in place. Available instruments include IRS, TS and the traditional CCS where the notional is constant until maturity (an introduction to the newer kind of CCS, the mark-to-market CCS, is found in Appendix B). We assume a Libor that accurately reflects the funding cost of the institution at hand as discounting rate, for simplicity the USD 3m Libor. The result will be a set of curves that can price any uncollateralized swap and that is consistent with observed market quotes.

2.1.1 A SINGLE IRS MARKET

At first we consider a single currency (USD) market where only one kind of USD IRS is available. At initiation it holds that

$$C(t, T_N) \sum_{m=1}^N \delta_{m-1,m}^{\text{fi}} Z(t, T_m) = \sum_{n=1}^N \delta_{n-1,n}^{\text{ff}} E_t[L(T_{n-1}, T_n)] Z(t, T_n),$$

where $C(t, T_N)$ is the time- t fair swap rate for an IRS of length T_N , $\delta_{m-1,m}^{\text{fi}}$ and $\delta_{n-1,n}^{\text{ff}}$ are day count factors of the fixed and floating legs, respectively. $Z(t, T_n)$ is the time- t price of a default free discount bond maturing at T_n and $L(T_{n-1}, T_n)$ is the USD 3m Libor from T_{n-1} to T_n . Surveys of day count and swap conventions are found in Appendices C and D, respectively. Unless mentioned otherwise, $E_t[\cdot]$ is assumed to be taken under the appropriate forward measure.

Since the available swaps have floating legs linked to the USD 3m Libor and since the same rate is used for discounting, a simple no-arbitrage argument gives that

$$E_t[L(T_{n-1}, T_n)] = \frac{1}{\delta_{n-1,n}^{\text{ff}}} \left(\frac{Z(t, T_{n-1})}{Z(t, T_n)} - 1 \right).$$

Using this relation, the swap market condition becomes

$$C(t, T_N) \sum_{m=1}^N \delta_{m-1,m}^{\text{fi}} Z(t, T_m) = Z(t, T_0) - Z(t, T_N),$$

where $Z(t, T_0)$ is the discounting factor from time- t to the first fixing date (and can be determined by the ON-rate). The discounting factors can now be uniquely determined by sequentially solving

$$Z(t, T_m) = \frac{Z(t, T_0) - C(t, T_m) \sum_{i=1}^{m-1} \delta_{i-1,i}^{\text{fi}} Z(t, T_i)}{1 + C(t, T_m) \delta_{m-1,m}^{\text{fi}}}.$$

This procedure requires that all necessary maturities are in fact traded and the difficulty that arises when this is not the case is further treated in Section 3. Also, interpolation has to be carried out in order get a continuous curve of discounting factors and corresponding forward USD 3m-Libor rates. This topic is further covered in Section 3 and more deeply in Appendix E.

2.1.2 AN IRS AND TS MARKET

We now consider a (still single currency) market where TS as well as IRS with floating legs linked to USD Libor rates of varying tenor are available. To price an IRS with a floating leg linked to, for example, the USD 1m Libor, we cannot due to the existence of tenor basis spreads use the USD 3m Libor forward curve. It is hence necessary to determine a set of USD 1m Libor forward rates. This can be done by using the quoted USD 1m/3m TS, where one party pays USD 1m Libor plus a spread monthly and receives USD 3m Libor quarterly. The resulting conditions become

$$C(t, T_N) \sum_{m=1}^N \delta_{m-1,m}^{\text{fi}} Z(t, T_m) = \sum_{n=1}^N \delta_{n-1,n}^{3\text{m}} E_t[L^{3\text{m}}(T_{n-1}, T_n)] Z(t, T_n),$$

$$\sum_{k=1}^N \delta_{k-1,k}^{1\text{m}} (E_t[L^{1\text{m}}(T_{k-1}, T_k)] + TS(t, T_N)) Z(t, T_k) = \sum_{n=1}^N \delta_{n-1,n}^{3\text{m}} E_t[L^{3\text{m}}(T_{n-1}, T_n)] Z(t, T_n),$$

where $TS(t, T_N)$ is the time- t 1m/3m tenor basis spread at maturity T_N . The discount factors and corresponding USD 3m Libor rates are computed as in Section 2.1.1. Through the basis swaps and proper interpolation it is then possible to compute a continuous set of USD 1m Libor rates. It is also straightforward to derive forward curves of different tenors (6m, 1y for example) by adding more TS.

2.1.3 INTRODUCING THE CONSTANT NOTIONAL CCS

In this section, we expand the model to allow for multiple currencies and for the existence of a constant notional CCS. More specifically, USD and EUR are the relevant currencies and the USD 3m Libor is still the discounting rate. Curve construction for US-based institutions is done as in Sections 2.1.1 and 2.1.2, however for European institutions one has to account for the cross currency basis spread inherent in the CCS. Thus, the

conditions for the EUR rates (Euribor) become

$$\begin{aligned}
C(t, T_N) \sum_{l=1}^N \delta_{l-1, l}^{\text{fi}, \text{€}} Z^{\text{€}}(t, T_l) &= \sum_{m=1}^N \delta_{m-1, m}^{6\text{m}, \text{€}} E_t^{\text{€}}[L^{6\text{m}, \text{€}}(T_{m-1}, T_m)] Z^{\text{€}}(t, T_m), \\
\sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \text{€}} (E_t^{\text{€}}[L^{3\text{m}, \text{€}}(T_{n-1}, T_n)] + TS(t, T_N)) Z^{\text{€}}(t, T_n) \\
&= \sum_{m=1}^N \delta_{m-1, m}^{6\text{m}, \text{€}} E_t^{\text{€}}[L^{6\text{m}, \text{€}}(T_{m-1}, T_m)] Z^{\text{€}}(t, T_m), \\
N_{\text{€}} \left(-Z^{\text{€}}(t, T_0) + \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \text{€}} (E_t^{\text{€}}[L^{3\text{m}, \text{€}}(T_{n-1}, T_n)] + CCS(t, T_N)) Z^{\text{€}}(t, T_n) + Z^{\text{€}}(t, T_N) \right) \\
&= f(t) \left(-Z^{\text{\$}}(t, T_0) + \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \text{\$}} (E_t^{\text{\$}}[L^{3\text{m}, \text{\$}}(T_{n-1}, T_n)] Z^{\text{\$}}(t, T_n) + Z^{\text{\$}}(t, T_N)) \right),
\end{aligned}$$

where $CCS(t, T_N)$ is the time- t USDEUR cross currency basis spread at maturity T_N , $N_{\text{€}}$ is the EUR notional per USD and $f(t)$ is the time- t USDEUR exchange rate. The €- and \\$-indices indicate that the variable is relevant for EUR and USD, respectively. Since we still treat the USD 3m-Libor as the discounting rate, the USD floating leg of the CCS equals zero and it holds that

$$\begin{aligned}
\sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \text{€}} E_t^{\text{€}}[L^{3\text{m}, \text{€}}(T_{n-1}, T_n)] Z^{\text{€}}(t, T_n) \\
= Z^{\text{€}}(t, T_0) - Z^{\text{€}}(t, T_N) - CCS(t, T_N) \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \text{€}} Z^{\text{€}}(t, T_n).
\end{aligned}$$

After further elimination of floating parts we easily arrive at

$$\begin{aligned}
C(t, T_N) \sum_{l=1}^N \delta_{l-1, l}^{\text{fi}, \text{€}} Z^{\text{€}}(t, T_l) + (CCS(t, T_N) - TS(t, T_N)) \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \text{€}} Z^{\text{€}}(t, T_n) \\
= Z^{\text{€}}(t, T_0) - Z^{\text{€}}(t, T_N),
\end{aligned}$$

and it is now possible to sequentially compute the EUR discounting factors. Using the quoted IRS and TS one can then derive the 3m- and 6m- forward Euribor curves. By adding more TS, it is of course possible to derive forward Euribor curves with other tenors. Evidently, the EUR discounting factors also depend on the tenor basis spreads and cross currency basis spreads, and not only on the swap rates. Therefore, if holding a simple EUR IRS, one also has to hedge for sensitivities inherent in these spreads. Throughout this survey, the USD 3m Libor has been considered the discounting rate. Using another discounting rate poses no problem, as the methodology of deriving discount factors and

forward rates will be analogous to what has been covered herein.

2.2 CURVE CONSTRUCTION WITH COLLATERAL

According to the ISDA Margin Survey [18], close to 80% of all trades with fixed income derivatives during 2012 were collateralized. For large dealers, this number approaches 90%. As the existence of a collateral agreement substantially reduces the credit risk inherent in the trade it becomes questionable to apply standard Libor discounting when pricing a certain product. In this section, it is explained how to price a collateralized product and more specifically how collateralization affects curve construction for swap pricing.

2.2.1 PRICING OF COLLATERALIZED DERIVATIVES

In a collateralized trade, the party whose contract has a positive present value receives collateral from the counterparty. To compensate for this the party has to pay a certain margin called "collateral rate" on the outstanding collateral. In case of cash collateral, the collateral rate is usually the overnight rate for the collateral currency, i.e. the Federal Funds rate for USD or the Eonia rate for EUR. To avoid problems with non-linearity, we assume that mark-to-market and collateral posting is made continuously. Also, the posted cash collateral is assumed to cover 100% of the contract's present value. As collateral posting is commonly done on a daily basis, these simplifications are probably not too far from reality, at least not for liquid currencies. Since counterparty default risk can now be neglected, it is possible to recover a linear relationship among payments.

With collateral posted in domestic currency and collateral rate $c(s)$ at time s , the time- t value $h(t)$ of a derivative h maturing at T is given by the following proposition.

PROPOSITION 2.1.

$$h(t) = E_t^Q \left[e^{-\int_t^T c(s) ds} h(T) \right],$$

where $E_t^Q[\cdot]$ is the expectation with the money-market account as numéraire.

For a proof we refer to Appendix F. If collateral is posted in foreign currency, the value at time t of the derivative is furthermore given by

$$h(t) = E_t^Q \left[e^{-\int_t^T r(s) ds} \left(e^{\int_t^T (r^f(s) - c^f(s)) ds} \right) h(T) \right],$$

where $r(s)$ and $r^f(s)$ are the domestic and foreign risk-free rates, respectively. $c^f(s)$ is the collateral rate on collateral posted in foreign currency. It can now be seen that in a collateralized trade future cash flows should be discounted by the collateral rate. As

the overnight rate can differ significantly from the Libor, it becomes evident that Libor discounting is no longer appropriate.

2.2.2 INTRODUCING THE OIS

Under the assumption that the collateral rate on cash equals the overnight rate one can determine collateralized discounted factors by using quoted overnight indexed swaps (OIS). An OIS exchanges a fixed coupon for a daily compounded overnight rate, where the dates of the two payments typically coincide. Hence, between two payment dates T_{l-1} and T_l the floating leg pays

$$\prod_{s=T_{l-1}}^{T_l} (1 + \delta_s c(s)) - 1$$

multiplied by the notional. Here, δ_s is the daily accrual factor and $c(s)$ is the collateral rate at time s . By approximating daily compounding with continuous compounding, we get

$$\prod_{s=T_{l-1}}^{T_l} (1 + \delta_s c(s)) - 1 \approx e^{\int_{T_{l-1}}^{T_l} c(s) ds} - 1.$$

If we further assume that the OIS is perfectly collateralized with 100% cash it holds that (as shown in Section 2.2.1)

$$S(t, T_N) \sum_{l=1}^N \delta_{l-1,l}^{\text{fi}} E_t^Q \left[e^{-\int_t^{T_l} c(s) ds} \right] = \sum_{l=1}^N E_t^Q \left[e^{-\int_t^{T_l} c(s) ds} \left(e^{\int_{T_{l-1}}^{T_l} c(s) ds} - 1 \right) \right],$$

where $S(t, T_N)$ is the time- t fair swap rate for an OIS of length T_N . By denoting the collateralized discount factors with

$$D(t, T_l) = E_t^Q \left[e^{-\int_t^{T_l} c(s) ds} \right]$$

we arrive at

$$S(t, T_N) \sum_{l=1}^N \delta_{l-1,l}^{\text{fi}} D(t, T_l) = D(t, T_0) - D(t, T_N).$$

It is now a simple matter to sequentially derive the discount factors by

$$D(t, T_l) = \frac{D(t, T_0) - S(t, T_l) \sum_{i=1}^{l-1} \delta_{i-1,i}^{\text{fi}} D(t, T_i)}{1 + S(t, T_l) \delta_{l-1,l}^{\text{fi}}},$$

and a continuous discount curve is obtained by appropriate splining. Information on common market conventions for overnight indexed swaps is found in Appendix D.

2.2.3 CURVE CONSTRUCTION IN A SINGLE CURRENCY

In a single currency, the construction of forward Libor curves of different tenors is very similar to that of Section 2.1.2. After deriving the collateralized discount curve as in Section 2.2.2, one can compute, let's say, 1m and 3m Libor forward rates through the conditions

$$C(t, T_N) \sum_{m=1}^N \delta_{m-1, m}^{\text{fi}} D(t, T_m) = \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}} E_t^c[L^{3\text{m}}(T_{n-1}, T_n)] D(t, T_n),$$

$$\sum_{k=1}^N \delta_{k-1, k}^{1\text{m}} (E_t^c[L^{1\text{m}}(T_{k-1}, T_k)] + TS(t, T_N)) D(t, T_k) = \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}} E_t^c[L^{3\text{m}}(T_{n-1}, T_n)] D(t, T_n),$$

where $E_t^c[\cdot]$ is the expectation with the appropriate $D(t, T_n)$ as numéraire. It is of course possible to add more TS to derive forward curves with other tenors.

2.2.4 CURVE CONSTRUCTION IN MULTIPLE CURRENCIES

Unlike the single-currency setup, where collateral and swap payments are in the same currency, we must now allow for collateral and swap payments to be of different currencies. As in Section 2.1.3 the constant notional CCS will be used as calibration instrument (how to use the mark-to-market CCS for curve calibration is covered in Appendix B) and the relevant currencies will be USD and EUR. Also, the Federal Funds rate will be treated as the risk-free rate. Since it is also the collateral rate for USD, it now holds that

$$D^{\$}(t, T) = E_t^{Q^{\$}} \left[e^{-\int_t^T c^{\$(s)} ds} \right] = E_t^{Q^{\$}} \left[e^{-\int_t^T r^{\$(s)} ds} \right] = Z^{\$(t, T)}.$$

Conditions for USD-collateralized USD swaps (with the USD 1m/3m TS) are thus

$$S^{\$(t, T_N)} \sum_{l=1}^N \delta_{l-1, l}^{\text{fi}, \$} Z^{\$(t, T_l)} = Z^{\$(t, T_0)} - Z^{\$(t, T_N)},$$

$$C^{\$(t, T_N)} \sum_{m=1}^N \delta_{m-1, m}^{\text{fi}, \$} Z^{\$(t, T_m)} = \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \$} E_t^{\$}[L^{3\text{m}, \$}(T_{n-1}, T_n)] Z^{\$(t, T_n)},$$

$$\sum_{k=1}^N \delta_{k-1, k}^{1\text{m}, \$} (E_t^{\$}[L^{1\text{m}, \$}(T_{k-1}, T_k)] + TS^{\$(t, T_N)}) Z^{\$(t, T_k)}$$

$$= \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \$} E_t^{\$}[L^{3\text{m}, \$}(T_{n-1}, T_n)] Z^{\$(t, T_n)}.$$

Similarly, conditions for EUR-collateralized EUR swaps (with the EUR 3m/6m TS) are

$$\begin{aligned}
S^\epsilon(t, T_N) \sum_{l=1}^N \delta_{l-1, l}^{\text{fi}, \epsilon} D^\epsilon(t, T_l) &= D^\epsilon(t, T_0) - D^\epsilon(t, T_N), \\
C^\epsilon(t, T_N) \sum_{l=1}^N \delta_{l-1, l}^{\text{fi}, \epsilon} D^\epsilon(t, T_l) &= \sum_{m=1}^N \delta_{m-1, m}^{6\text{m}, \epsilon} E_t^{c, \epsilon} [L^{6\text{m}, \epsilon}(T_{m-1}, T_m)] D^\epsilon(t, T_m), \\
\sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \epsilon} (E_t^{c, \epsilon} [L^{3\text{m}, \epsilon}(T_{n-1}, T_n)] + TS^\epsilon(t, T_N)) D^\epsilon(t, T_n) \\
&= \sum_{m=1}^N \delta_{m-1, m}^{6\text{m}, \epsilon} E_t^{c, \epsilon} [L^{6\text{m}, \$}(T_{m-1}, T_m)] D^\epsilon(t, T_m).
\end{aligned}$$

Of course, more TS conditions can be added if needed.

We now turn our attention to USD-collateralized EUR swaps. Assume the existence of a USD cash-collateralized USDEUR constant notional CCS. With the results of Section 2.2.1 it holds that⁴

$$\begin{aligned}
& -Z^\epsilon(t, T_0) + \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \epsilon} (E_t^\epsilon [L^{3\text{m}, \epsilon}(T_{n-1}, T_n)] + CCS(t, T_N)) Z^\epsilon(t, T_n) + Z^\epsilon(t, T_N) \\
&= N_\$ f(t) \left(-Z^\$(t, T_0) + \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \$} E_t^\$ [L^{3\text{m}, \$}(T_{n-1}, T_n)] Z^\$(t, T_n) + Z^\$(t, T_N) \right),
\end{aligned}$$

where the right-hand side is previously known. It is however not possible to derive both the EUR zero coupon bond prices and the EUR forward rates through this condition only. Ideally quotes for USD-collateralized EUR IRS and TS are available, which would allow us to easily derive the sets of discount factors and forward rates. Alternatively, one could assume that

$$E_t^\epsilon [L^\epsilon(T_{n-1}, T_n)] = E_t^{c, \epsilon} [L^\epsilon(T_{n-1}, T_n)].$$

In this approach we thus neglect the change of numéraire, and the approximation is reasonable if the EUR risk-free and collateral rates have similar dynamic properties. This

⁴The sum in the LHS is given by

$$\begin{aligned}
& \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \epsilon} E_t^{Q^\epsilon} \left[e^{-\int_t^{T_n} r^\epsilon(s) ds} \left(e^{-\int_t^{T_n} (r^\$(s) - c^\$(s)) ds} \right) (L(T_{n-1}, T_n) + CCS(t, T_N)) \right] \\
&= \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \epsilon} E_t^{Q^\epsilon} \left[e^{-\int_t^{T_n} r^\epsilon(s) ds} (L(T_{n-1}, T_n) + CCS(t, T_N)) \right] \\
&= \sum_{n=1}^N \delta_{n-1, n}^{3\text{m}, \epsilon} (E_t^\epsilon [L^{3\text{m}, \epsilon}(T_{n-1}, T_n)] + CCS(t, T_N)) Z^\epsilon(t, T_n).
\end{aligned}$$

enables us to sequentially derive the EUR zero coupon bond prices.

We finally consider the case of EUR-collateralized USD swaps. The conditions for EUR-collateralized USD IRS and constant notional CCS are

$$\begin{aligned}
\overline{C}^{\$}(t, T_N) \sum_{m=1}^N \delta_{m-1, m}^{\text{fi}, \$} Z^{\$}(t, T_m) E_t^{\$} \left[e^{\int_t^{T_m} (r^{\epsilon}(s) - c^{\epsilon}(s)) ds} \right] \\
= \sum_{n=1}^N \delta_{n-1, n}^{3m, \$} Z^{\$}(t, T_n) E_t^{\$} \left[e^{\int_t^{T_n} (r^{\epsilon}(s) - c^{\epsilon}(s)) ds} L^{3m, \$}(T_{n-1}, T_n) \right], \\
\sum_{n=1}^N \delta_{n-1, n}^{3m, \$} Z^{\$}(t, T_n) E_t^{\$} \left[e^{\int_t^{T_n} (r^{\epsilon}(s) - c^{\epsilon}(s)) ds} L^{3m, \$}(T_{n-1}, T_n) \right] \\
= N_{\$} \left(-D^{\epsilon}(t, T_0) + \sum_{n=1}^N \delta_{n-1, n}^{3m, \epsilon} (E_t^{c, \epsilon} [L^{3m, \epsilon}(T_{n-1}, T_n)] + \overline{CCS}(t, T_N)) D^{\epsilon}(t, T_n) + D^{\epsilon}(t, T_N) \right),
\end{aligned}$$

where $\overline{CCS}(t, T_N)$ and $\overline{C}^{\$}(t, T_N)$ are the fair rates for the EUR-collateralized CCS and USD IRS, respectively. With these instruments at hand, it is possible to determine

$$\left[e^{\int_t^{T_m} (r^{\epsilon}(s) - c^{\epsilon}(s)) ds} \right] \quad \text{and} \quad \left[e^{\int_t^{T_n} (r^{\epsilon}(s) - c^{\epsilon}(s)) ds} L^{3m, \$}(T_{n-1}, T_n) \right]$$

for each m and n . By adding EUR-collateralized USD tenor basis swaps, it is also possible to derive Libor curves of other tenors.

3 IMPLEMENTATION

This section describes how the various discounting and forward curves are derived. We deal with data on collateralized (in domestic currency) swaps as found in Appendix G and consider the USD and EUR markets separately. The implementation is done in Python, where the SWIG-bindings for the C++ library QuantLib⁵ are used for calendar and day counter classes.

3.1 BUILDING THE USD CURVES

The discounting curve is first built through the USD OIS market. Being equipped with the the relevant discounting factors we are allowed to extract the USD 3m forward rates through IRS quotes. Tenor basis spreads are then used to derive the USD 1m and 6m forward curves.

3.1.1 THE USD DISCOUNTING CURVE

The relevant data is found in Table G.3. For swaps of length less than one year there is only one payment at maturity. The discount factors of shortest maturities are hence given by

$$D^{\$}(0, i) = \frac{1}{1 + S^{\$}(0, i)\delta_{0,i}^{\text{fi},\$}}, \quad i = 1\text{d}, 1\text{w}, 2\text{w}, 3\text{w}, 1\text{m}, 2\text{m}, \dots, 11\text{m}.$$

For maturities greater than 1 year, there is one payment at the end of each year. As shown in Section 2.2.2, the relevant condition is

$$S^{\$}(t, T_N) \sum_{l=1}^N \delta_{t-1,l}^{\text{fi},\$} D^{\$}(t, T_l) + D^{\$}(t, T_N) = 1,$$

where we have assumed that $D^{\$}(t, T_0) = 1$. To be able to solve for each discount factor requires that there is a liquid market in yearly maturities for 1 year up to 50 years. Since this is not the case we make an approximation by using interpolation with cubic splines to estimate the necessary quotes. Another way of dealing with this issue is covered in Hagan and West (2006) [13], where instead a method of iteration is applied. Having estimated

⁵For documentation see <http://quantlib.org/docs.shtml>.

all required OIS rates, we are able to solve the following system of equations:

$$\begin{pmatrix} S^{\$}(0, 1y)\delta_{0,1y}^{\text{fi},\$} + 1 & 0 & \dots & 0 \\ S^{\$}(0, 2y)\delta_{0,1y}^{\text{fi},\$} & S^{\$}(0, 2y)\delta_{1y,2y}^{\text{fi},\$} + 1 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ S^{\$}(0, 50y)\delta_{0,1y}^{\text{fi},\$} & \dots & \dots & S^{\$}(0, 50y)\delta_{49y,50y}^{\text{fi},\$} + 1 \end{pmatrix} \begin{pmatrix} D^{\$}(0, 1y) \\ D^{\$}(0, 2y) \\ \vdots \\ \vdots \\ D^{\$}(0, 50y) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}.$$

We are now supplied with estimates of yearly discount factors from 1 year up to 50 years. However, we only use those with maturities corresponding to quoted overnight index swaps. To obtain a continuous set of discount factors, interpolation with cubic splines is applied to this subset.

3.1.2 THE USD 3M FORWARD CURVE

To build the USD 3m forward curve we use the 3m spot Libor in Table G.1 together with the IRS quotes in Table G.3. The relevant condition is now

$$C^{\$}(t, T_N) \sum_{m=1}^N \delta_{m-1,m}^{\text{fi},\$} D^{\$}(t, T_m) = \sum_{n=1}^N \delta_{n-1,n}^{3m,\$} E_t^{c,\$}[L^{3m,\$(T_{n-1}, T_n)] D^{\$}(t, T_n),$$

and is previously known from Section 2.2.4. As we have already built the discounting curve it is now possible to extract the 3m forward rates. However, just as in Section 3.1.1 we need to interpolate the swap curve to obtain estimates of all necessary swap rates. Also, since the fixed leg pays semiannually and the floating leg quarterly the resulting system of equations would become underdetermined. To mitigate this problem we assume that the forward rates are piecewise flat, i.e. that

$$\begin{aligned} E_t^{c,\$}[L^{3m,\$(6m, 9m)] &= E_t^{c,\$}[L^{3m,\$(9m, 12m)]], \\ E_t^{c,\$}[L^{3m,\$(12m, 15m)] &= E_t^{c,\$}[L^{3m,\$(15m, 18m)] \end{aligned}$$

and so on. Since we already know the 3m spot Libor, the resulting system of equations is

$$\begin{pmatrix} \delta_{3m,6m}^{3m,\$} D^{\$}(0,6m) & 0 & \cdots & 0 \\ \delta_{3m,6m}^{3m,\$} D^{\$}(0,6m) & \sum_{i=9m}^{12m} \delta_{i-3m,i}^{3m,\$} D^{\$}(0,i) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ \delta_{3m,6m}^{3m,\$} D^{\$}(0,6m) & \sum_{i=9m}^{12m} \delta_{i-3m,i}^{3m,\$} D^{\$}(0,i) & \cdots & \sum_{i=597m}^{600m} \delta_{i-3m,i}^{3m,\$} D^{\$}(0,i) \end{pmatrix} \begin{pmatrix} E_t^{c,\$}[L^{3m,\$(3m,6m)}] \\ E_t^{c,\$}[L^{3m,\$(9m,12m)}] \\ \vdots \\ \vdots \\ E_t^{c,\$}[L^{3m,\$(597m,600m)}] \end{pmatrix} \\ = \begin{pmatrix} C^{\$(0,6m)} \sum_{n=6m}^{6m} \delta_{n-6m,n}^{fi,\$} D^{\$(0,n)} - \delta_{0,3m}^{3m,\$} E_t^{c,\$}[L^{3m,\$(0,3m)}] D^{\$(0,3m)} \\ C^{\$(0,1y)} \sum_{n=6m}^{12m} \delta_{n-6m,n}^{fi,\$} D^{\$(0,n)} - \delta_{0,3m}^{3m,\$} E_t^{c,\$}[L^{3m,\$(0,3m)}] D^{\$(0,3m)} \\ \vdots \\ \vdots \\ C^{\$(0,50y)} \sum_{n=6m}^{600m} \delta_{n-6m,n}^{fi,\$} D^{\$(0,n)} - \delta_{0,3m}^{3m,\$} E_t^{c,\$}[L^{3m,\$(0,3m)}] D^{\$(0,3m)} \end{pmatrix}.$$

By solving this system we obtain an array of USD 3m forward Libors, however we choose to discard those with maturities that do not coincide with the maturities of quoted interest rate swaps. Interpolation with cubic splines on the remainder then gives us the continuous 3m forward curve.

3.1.3 THE USD 1M FORWARD CURVE

To construct the USD 1m forward curve we use the 1m spot Libor in Table G.1, the quoted tenor basis spreads in Table G.2 and the quoted 1m IRS in Table G.3. The 1m implied swap rate is first computed by

$$C^{\$(0,1m)} = \frac{\delta_{0,1m}^{1m,\$}}{\delta_{0,1m}^{fi,\$}} E_t^{c,\$}[L^{1m,\$(0,1m)}],$$

and is added to the array of quoted IRS with maturities up to 12 months. We extract the forward rates with maturities $\leq 12m$ through the condition

$$C^{\$(t, T_N)} \sum_{k=1}^N \delta_{k-1,k}^{fi,\$} D^{\$(t, T_k)} = \sum_{k=1}^N \delta_{k-1,k}^{1m,\$} E_t^{c,\$}[L^{1m,\$(T_{k-1}, T_k)}] D^{\$(t, T_k)},$$

and arrive at the following system of equations:

$$\begin{pmatrix}
\delta_{0,1m}^{1m,\$} D^{\$(0,1m)} & 0 & \dots & 0 \\
\delta_{0,1m}^{1m,\$} D^{\$(0,1m)} & \delta_{1m,2m}^{1m,\$} D^{\$(0,2m)} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 \\
\delta_{0,1m}^{1m,\$} D^{\$(0,1m)} & \delta_{1m,2m}^{1m,\$} D^{\$(0,2m)} & \dots & \delta_{11m,12m}^{1m,\$} D^{\$(0,12m)}
\end{pmatrix}
\begin{pmatrix}
E_t^{c,\$}[L^{1m,\$(0,1m)}] \\
E_t^{c,\$}[L^{1m,\$(1m,2m)}] \\
\vdots \\
\vdots \\
E_t^{c,\$}[L^{1m,\$(11m,12m)}]
\end{pmatrix}
=
\begin{pmatrix}
C^{\$(0,1m)} \sum_{n=1m}^{1m} \delta_{n-1m,n}^{fi,\$} D^{\$(0,n)} \\
C^{\$(0,2m)} \sum_{n=1m}^{2m} \delta_{n-1m,n}^{fi,\$} D^{\$(0,n)} \\
\vdots \\
\vdots \\
C^{\$(0,12m)} \sum_{n=1m}^{12m} \delta_{n-1m,n}^{fi,\$} D^{\$(0,n)}
\end{pmatrix}.$$

Note that we do not have to interpolate the swap curve in this case. Next, we compute the implied tenor basis spreads for maturities 3m, 6m, 9m and 12m by using the derived short-end of the forward curve. These spreads are then added to the array of quoted spreads.

Since we have already obtained the discounting curve and the 3m forward curve, we can now use the tenor basis spreads and extract the 1m forward curve through

$$\begin{aligned}
\sum_{k=1}^N \delta_{k-1,k}^{1m,\$} (E_t^{c,\$}[L^{1m,\$(T_{k-1}, T_k)] + TS^{\$(t, T_N)}) D^{\$(t, T_k)} \\
= \sum_{n=1}^N \delta_{n-1,n}^{3m,\$} E_t^{c,\$}[L^{3m,\$(T_{n-1}, T_n)] D^{\$(t, T_n)},
\end{aligned}$$

where interpolation of the basis spreads will be necessary. Due to the same issue as in Section 3.1.2 we assume piecewise flat forward rates, i.e.

$$\begin{aligned}
E_t^{c,\$}[L^{1m,\$(12m, 13m)}] &= E_t^{c,\$}[L^{1m,\$(13m, 14m)}] = E_t^{c,\$}[L^{1m,\$(14m, 15m)}], \\
E_t^{c,\$}[L^{1m,\$(15m, 16m)}] &= E_t^{c,\$}[L^{1m,\$(16m, 17m)}] = E_t^{c,\$}[L^{1m,\$(17m, 18m)}], \\
&\vdots \\
&\vdots
\end{aligned}$$

which allows us to build a solvable system of equations.

With

$$A = \sum_{i=1m}^{12m} \delta_{i-1m,i}^{1m,\$} E_t^{c,\$} [L^{1m,\$(i-1m,i)}] D^{\$(0,i)}$$

we arrive at

$$\begin{pmatrix} \sum_{i=13m}^{15m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} & 0 & \cdots & 0 \\ \sum_{i=13m}^{15m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} & \sum_{i=16m}^{18m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ \sum_{i=13m}^{15m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} & \sum_{i=16m}^{18m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} & \cdots & \sum_{i=598m}^{600m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} \end{pmatrix} \begin{pmatrix} E_t^{c,\$} [L^{1m,\$(14m,15m)}] \\ E_t^{c,\$} [L^{1m,\$(17m,18m)}] \\ \vdots \\ \vdots \\ E_t^{c,\$} [L^{1m,\$(599m,600m)}] \end{pmatrix} \\ = \begin{pmatrix} \sum_{n=3m}^{15m} \delta_{n-3m,n}^{3m,\$} E_t^{c,\$} [L^{3m,\$(n-3m,n)}] D^{\$(0,n)} - TS^{\$(0,15m)} \sum_{i=1m}^{15m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} - A \\ \sum_{n=3m}^{18m} \delta_{n-3m,n}^{3m,\$} E_t^{c,\$} [L^{3m,\$(n-3m,n)}] D^{\$(0,n)} - TS^{\$(0,18m)} \sum_{i=1m}^{18m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} - A \\ \vdots \\ \vdots \\ \sum_{n=3m}^{600m} \delta_{n-3m,n}^{3m,\$} E_t^{c,\$} [L^{3m,\$(n-3m,n)}] D^{\$(0,n)} - TS^{\$(0,600m)} \sum_{i=1m}^{600m} \delta_{i-1m,i}^{1m,\$} D^{\$(0,i)} - A \end{pmatrix}.$$

By solving this and applying interpolation with cubic splines we have successfully derived the continuous USD 1m forward curve.

3.1.4 THE USD 6M FORWARD CURVE

We first compute the implied 6m 3m/6m tenor basis spread by using the 6m spot Libor in Table G.1 and the derived USD 3m forward curve, and then add this spread to the array of quoted spreads in Table G.2. After interpolating the basis spreads to estimate semiannual quotes we can successfully solve

$$\begin{pmatrix} \delta_{0,6m}^{6m,\$} D^{\$(0,6m)} & 0 & \cdots & 0 \\ \delta_{0,6m}^{6m,\$} D^{\$(0,6)} & \delta_{6m,12m}^{6m,\$} D^{\$(0,12m)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ \delta_{0,6m}^{6m,\$} D^{\$(0,6)} & \delta_{6m,12m}^{6m,\$} D^{\$(0,12m)} & \cdots & \delta_{594m,600m}^{6m,\$} D^{\$(0,600m)} \end{pmatrix} \begin{pmatrix} E_t^{c,\$} [L^{6m,\$(0,6m)}] \\ E_t^{c,\$} [L^{6m,\$(6m,12m)}] \\ \vdots \\ \vdots \\ E_t^{c,\$} [L^{6m,\$(594m,600m)}] \end{pmatrix} \\ = \begin{pmatrix} \sum_{n=3m}^{6m} \delta_{n-3m,n}^{3m,\$} (E_t^{c,\$} [L^{3m,\$(n-3m,n)}] + TS^{\$(0,6m)}) D^{\$(0,n)} \\ \sum_{n=3m}^{12m} \delta_{n-3m,n}^{3m,\$} (E_t^{c,\$} [L^{3m,\$(n-3m,n)}] + TS^{\$(0,12m)}) D^{\$(0,n)} \\ \vdots \\ \vdots \\ \sum_{n=3m}^{600m} \delta_{n-3m,n}^{3m,\$} (E_t^{c,\$} [L^{3m,\$(n-3m,n)}] + TS^{\$(0,600m)}) D^{\$(0,n)} \end{pmatrix}.$$

Next, we filter out the forward rates with maturities that correspond to maturities of quoted instruments. After performing interpolation (again with cubic splines) on this subset we have successfully completed the construction of the USD 6m forward curve.

3.2 BUILDING THE EUR CURVES

In this section it is described how the EUR discounting curve is constructed through the OIS market, how EUR 1m, 3m, 6m and 1y forward curves are built using quoted interest rate and tenor basis swaps and, finally, how to derive the discounting curve for USD-collateralized EUR swaps.

3.2.1 THE EUR DISCOUNTING CURVE

The construction of the EUR discounting curve is analogous to that of the USD discounting curve. For maturities $< 1y$ we get the discount factors by

$$D^{\epsilon}(0, i) = \frac{1}{1 + S^{\epsilon}(0, i)\delta_{0,i}^{\text{fi},\epsilon}}, \quad i = 1d, 1w, 2w, 3w, 1m, 2m, \dots, 11m.$$

For maturities $\geq 1y$ the condition

$$S^{\epsilon}(t, T_N) \sum_{l=1}^N \delta_{l-1,l}^{\text{fi},\epsilon} D^{\epsilon}(t, T_l) + D^{\epsilon}(t, T_N) = 1$$

holds, and we thus end up with the following set of equations:

$$\begin{pmatrix} S^{\epsilon}(0, 1y)\delta_{0,1y}^{\text{fi},\epsilon} + 1 & 0 & \dots & 0 \\ S^{\epsilon}(0, 2y)\delta_{0,1y}^{\text{fi},\epsilon} & S^{\epsilon}(0, 2y)\delta_{1y,2y}^{\text{fi},\epsilon} + 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ S^{\epsilon}(0, 50y)\delta_{0,1y}^{\text{fi},\epsilon} & \dots & \dots & S^{\epsilon}(0, 50y)\delta_{49y,50y}^{\text{fi},\epsilon} + 1 \end{pmatrix} \begin{pmatrix} D^{\epsilon}(0, 1y) \\ D^{\epsilon}(0, 2y) \\ \vdots \\ \vdots \\ D^{\epsilon}(0, 50y) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}.$$

To obtain a continuous set of discount factors we apply the same procedure as in Section 3.1.1. The input data is found in Table G.7.

3.2.2 THE EUR 6M FORWARD CURVE

The EUR 6m forward curve is built in a similar manner as the USD 3m forward curve, only that the interest rate swaps now pay fixed annually and floating semiannually. The relevant condition is thus

$$C^{\epsilon}(t, T_N) \sum_{l=1}^N \delta_{l-1,l}^{\text{fi},\epsilon} D^{\epsilon}(t, T_l) = \sum_{m=1}^N \delta_{m-1,m}^{6m,\epsilon} E_t^{c,\epsilon}[L^{6m,\epsilon}(T_{m-1}, T_m)] D^{\epsilon}(t, T_m),$$

where the input swap rates and 6m Euribor are found in Tables G.7 and G.4, respectively. Having determined the appropriate discount factors and assuming that (compare with Section 3.1.2)

$$\begin{aligned}
E_t^{c,\epsilon}[L^{6m,\epsilon}(12m, 18m)] &= E_t^{c,\epsilon}[L^{3m,\epsilon}(18m, 24m)], \\
E_t^{c,\epsilon}[L^{6m,\epsilon}(24m, 30m)] &= E_t^{c,\epsilon}[L^{6m,\epsilon}(30m, 36m)], \\
&\vdots \\
&\vdots
\end{aligned}$$

the 6m forward rates are computed by solving

$$\begin{pmatrix}
\delta_{6m,12m}^{6m,\epsilon} D^{\epsilon}(0, 12m) & 0 & \cdots & 0 \\
\delta_{6m,12m}^{6m,\epsilon} D^{\epsilon}(0, 12m) & \sum_{i=18m}^{24m} \delta_{i-6m,i}^{6m,\epsilon} D^{\epsilon}(0, i) & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 \\
\delta_{6m,12m}^{6m,\epsilon} D^{\epsilon}(0, 12m) & \sum_{i=18m}^{24m} \delta_{i-6m,i}^{6m,\epsilon} D^{\epsilon}(0, i) & \cdots & \sum_{i=594m}^{600m} \delta_{i-6m,i}^{6m,\epsilon} D^{\epsilon}(0, i)
\end{pmatrix}
\begin{pmatrix}
E_t^{c,\epsilon}[L^{6m,\epsilon}(6m, 12m)] \\
E_t^{c,\epsilon}[L^{6m,\epsilon}(18m, 24m)] \\
\vdots \\
\vdots \\
E_t^{c,\epsilon}[L^{6m,\epsilon}(594m, 600m)]
\end{pmatrix}
=
\begin{pmatrix}
C^{\epsilon}(0, 1y) \sum_{n=12m}^{12m} \delta_{n-12m,n}^{fi,\epsilon} D^{\epsilon}(0, n) - \delta_{0,6m}^{6m,\epsilon} E_t^{c,\epsilon}[L^{6m,\epsilon}(0, 6m)] D^{\epsilon}(0, 6m) \\
C^{\epsilon}(0, 2y) \sum_{n=12m}^{24m} \delta_{n-12m,n}^{fi,\epsilon} D^{\epsilon}(0, n) - \delta_{0,6m}^{6m,\epsilon} E_t^{c,\epsilon}[L^{6m,\epsilon}(0, 6m)] D^{\epsilon}(0, 6m) \\
\vdots \\
\vdots \\
C^{\epsilon}(0, 50y) \sum_{n=12m}^{600m} \delta_{n-12m,n}^{fi,\epsilon} D^{\epsilon}(0, n) - \delta_{0,6m}^{6m,\epsilon} E_t^{c,\epsilon}[L^{6m,\epsilon}(0, 6m)] D^{\epsilon}(0, 6m)
\end{pmatrix}.$$

Appropriate splining (a procedure familiar by now) renders the continuous EUR 6m forward curve.

3.2.3 THE EUR 1M FORWARD CURVE

To build the EUR 1m forward curve we use the 1m Euribor spot rate in Table G.4, the quoted 1m/6m tenor basis spreads in Table G.6 and the quoted 1m interest rate swaps in Table G.7. The procedure is very similar to that of the USD 1m forward curve in Section 3.1.3, apart from a few differences. This time quoted 1m IRS of maturities up to two years are available, but since they are not so on a monthly basis interpolation of the swap curve becomes necessary. With the interpolated 1m swap curve we can bootstrap the short-end of the forward curve (i.e. with maturities ≤ 2 years). Subsequently, the implied 6m, 1y, 18m and 2y tenor basis spreads are computed with the aid of the bootstrapped 1m forward curve. By initially assuming that the long-end of the 1m forward curve is piecewise flat in intervals of 6 months (compare with Section 3.1.3) we can successfully compute forward rates for maturities > 2 years. As usual, forward rates that correspond to maturities

of quoted instruments are filtered out before the continuous EUR 1m forward curve is obtained through interpolation with cubic splines.

3.2.4 THE EUR 3M FORWARD CURVE

The short-end of the EUR 3m forward curve is built using the 3m Euribor in Table G.4 and the Euribor futures in Table G.5. As Euribor contracts are available up to, and including, MAR2015⁶ we can successfully estimate forward rates with maturities up to 30 months using interpolation. After computing the implied 6m, 1y, 18m, 2y and 30m tenor basis spreads and adding these to the quoted 3m/6m spreads in Table G.6 we end up with (after customary interpolation of the basis spreads and assuming piecewise flat forward rates in intervals of 6 months)

$$\begin{pmatrix}
\sum_{i=33m}^{36m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0,i) & 0 & \cdots & 0 \\
\sum_{i=33m}^{36m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0,i) & \sum_{i=39m}^{42m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0,i) & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & 0 \\
\sum_{i=33m}^{36m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0,i) & \sum_{i=39m}^{42m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0,i) & \cdots & \sum_{i=597m}^{600m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0,i)
\end{pmatrix}
\begin{pmatrix}
E_t^{c,\epsilon}[L^{3m,\epsilon}(33m, 36m)] \\
E_t^{c,\epsilon}[L^{3m,\epsilon}(39m, 42m)] \\
\vdots \\
\vdots \\
E_t^{c,\epsilon}[L^{3m,\epsilon}(597m, 600m)]
\end{pmatrix}
=
\begin{pmatrix}
\sum_{n=6m}^{36m} \delta_{n-6m,n}^{6m,\epsilon} E_t^{c,\epsilon}[L^{6m,\epsilon}(n-6m, n)] D^\epsilon(0, n) - TS^\epsilon(0, 36m) \sum_{i=3m}^{36m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0, i) - A \\
\sum_{n=6m}^{42m} \delta_{n-6m,n}^{6m,\epsilon} E_t^{c,\epsilon}[L^{6m,\epsilon}(n-6m, n)] D^\epsilon(0, n) - TS^\epsilon(0, 42m) \sum_{i=3m}^{42m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0, i) - A \\
\vdots \\
\vdots \\
\sum_{n=6m}^{600m} \delta_{n-6m,n}^{6m,\epsilon} E_t^{c,\epsilon}[L^{6m,\epsilon}(n-6m, n)] D^\epsilon(0, n) - TS^\epsilon(0, 600m) \sum_{i=3m}^{600m} \delta_{i-3m,i}^{3m,\epsilon} D^\epsilon(0, i) - A
\end{pmatrix},$$

where

$$A = \sum_{i=3m}^{30m} \delta_{i-3m,i}^{3m,\epsilon} E_t^{c,\epsilon}[L^{3m,\epsilon}(i-3m, i)] D^\epsilon(0, i).$$

Now that both the short- and the long-end are estimated we have finished the construction of the EUR 3m forward curve.

3.2.5 THE EUR 1Y FORWARD CURVE

The EUR 1y forward curve is constructed using the 1y Euribor spot rate in Table G.4 and the 6m/1y tenor basis spreads in Table G.6. Apart from all intervals in time being twice as long, this curve is built in exactly the same way as the USD 6m forward curve. We therefore refer to Section 3.1.4 for more details.

⁶This refers to a contract starting on the IMM date in March 2015 and maturing on the IMM date in June 2015. The IMM (International Monetary Market) dates are the third Wednesday of March, June, September and December.

3.2.6 THE CASE OF USD COLLATERAL

In this section we derive the discounting curve for EUR derivatives that are collateralized in USD, where we use data on USDEUR cross currency basis spreads as found in Table G.8. As the cross currency swaps are only available with maturities ≤ 30 years we assume a constant basis spread for swaps of length ≥ 30 years. Drawing from Section 2.2.4 and assuming that $N_{\$} = 1$ and $Z^{\text{€}}(0, T_0) = 1$ we arrive at

$$\begin{pmatrix} \delta_{0,3m}^{3m,\text{€}}(E_t^{c,\text{€}}[L^{3m,\text{€}}(0, 3m)] + b_{0,3m}) + 1 & 0 & \cdots \\ \delta_{0,3m}^{3m,\text{€}}(E_t^{c,\text{€}}[L^{3m,\text{€}}(0, 3m)] + b_{0,6m}) & \delta_{3m,6m}^{3m,\text{€}}(E_t^{c,\text{€}}[L^{3m,\text{€}}(3m, 6m)] + b_{0,6m}) + 1 & \cdots \\ \vdots & \ddots & \\ \vdots & & \ddots \\ \delta_{0,3m}^{3m,\text{€}}(E_t^{c,\text{€}}[L^{3m,\text{€}}(0, 3m)] + b_{0,600m}) & \cdots & \cdots \end{pmatrix} \begin{pmatrix} Z^{\text{€}}(0, 3m) \\ Z^{\text{€}}(0, 6m) \\ \vdots \\ \vdots \\ Z^{\text{€}}(0, 600m) \end{pmatrix} \\ = \begin{pmatrix} V_{3m} + 1 \\ V_{6m} + 1 \\ \vdots \\ \vdots \\ V_{600} + 1 \end{pmatrix},$$

where $b_{0,i} = CCS(0, i)$ and

$$V_i = f(0) \left(-1 + \sum_{n=3m}^i \delta_{n-3m,n}^{3m,\$} E_t^{c,\$} [L^{3m,\$}(n-3m, n)] D^{\$}(0, n) + D^{\$}(0, i) \right),$$

where an exchange rate of $f(0) = 0.7867$ is used. Having derived the discounting curve for EUR instruments that are collateralized in USD we have successfully constructed a set of discounting and forward curves that are able to price USD-collateralized USD swaps, EUR-collateralized EUR swaps and USD-collateralized EUR swaps. Section 4 presents some of the results that arise from incorporating basis spreads in the pricing of interest rate derivatives.

4 RESULTS

In this section we present the discounting and forward curves as derived in Section 3, where a discussion on how well these curves are able to replicate the prices of quoted instruments is included. Moreover, we address the impact basis spreads have on swap pricing and the importance of correctly adjusting for the existence of such spreads. The cases of USD and EUR are dealt with separately, where a comparison between the two currencies concludes the chapter.

4.1 THE CASE OF USD

The discounting and forward curves are shown in Figures 4.1 and 4.2. To be able to compare the different forward curves we show the implied 6m forward rates for each tenor. For example, the 6m implied forward rate with basis 1m is computed as

$$E_t^c[L^{1m}(i, i + 6m)] = \frac{\prod_{j=1}^6 \left(1 + \delta_{i+(j-1)m, i+jm}^{1m} E_t^c[L^{1m}(i + (j-1)m, i + jm)]\right) - 1}{\delta_{i, i+6m}^{1m}}.$$

It is seen that the 6m forward curve lies above the 3m forward curve (and the 3m curve above the 1m curve). This is supposedly due to the higher liquidity and credit risk associated with lending at 6m Libor as compared to rolling lending at 3m Libor, and is what one should expect.

With these curves at hand we can price a wide array of USD-collateralized USD swaps, where the most interesting question is how important it is to adjust for basis spreads when pricing such swaps. In Figure 4.3 the computed 1m, 3m and 6m swap curves are pictured along with the market quotes for 3m swaps. The 3m curve seems fairly good at replicating the quoted swap rates, however as further seen in Table 4.1 there are some discrepancies.

Maturity	Δ (bps)
1y	0.48
2y	1.12
3y	1.77
5y	3.73
10y	4.13
20y	2.79

Table 4.1: The difference between quoted and estimated USD 3m swap rates (a selection).

That the 3m swap curve on average underestimates the quoted 3m swap rate by 3 basis points is explained by the assumption we made when constructing the forward curve

in Section 3.1.2. As the USD IRS pays floating quarterly but fixed only semiannually we assumed that the forward curve was piecewise flat in intervals of 6 months. This assumption was further disregarded when interpolating the bootstrapped forward rates (since only forward rates that corresponded to maturities of quoted instruments were used for interpolation) and thus some of the estimated forward rates are lower than implied by our assumption and the quoted swap rates. Consequently, the value of the floating leg, and in turn the swap rate, is slightly underestimated. Observe that this would not be an issue for uncollateralized derivatives, where we could use the relationship between the discounting factors and forward rates to (almost) perfectly replicate the swap curve.

Figure 4.3 displays how there is a positive spread between the 6m and 3m swap curves (and similarly for the 3m and 1m curves). This result agrees with theory, since by the definition of the tenor basis spread we get that

$$C^{6m}(t, T) - C^{3m}(t, T) \approx TS^{3m,6m}(t, T) > 0,$$

and so on. The reason for this not being an equality is that day count conventions and/or payment frequencies might differ between the fixed legs of the interest rate swaps. The fact that the swap curves do not coincide raises the question of how important it is to account for basis spreads when pricing interest rate swaps. Assume for example that a party uses a single (3m forward) curve when pricing derivatives. This party would thus be willing to pay a higher fixed rate in an IRS with a floating leg linked to the 1m Libor than what is justified by the basis spreads. This loss can be approximated with

$$\text{Loss} = (C^{3m}(t, T) - C^{1m}(t, T)) \cdot PV(\text{Discount Factors}) \cdot N,$$

where N is the notional and $PV(\text{Discount Factors})$ is the present value of all discount factors times year fractions. Table 4.2 displays the percentage loss/gain of the notional that would incur if basis spreads in the USD market are not accounted for properly. For maturities > 10 years the loss is close to/above 1% for the 1m and 6m swaps, respectively. This amount is clearly significant and underlines the importance of a pricing framework that correctly and accurately accounts for basis spreads. That the loss is greater for 6m swaps is in this sense natural, since the 3m/6m basis spreads are greater than the corresponding 1m/3m spreads.

Maturity	Loss (%)	
	1m Tenor	6m Tenor
1y	0.09	0.21
2y	0.19	0.33
3y	0.28	0.42
4y	0.38	0.51
5y	0.49	0.59
6y	0.58	0.66
7y	0.65	0.73
8y	0.71	0.82
9y	0.76	0.90
10y	0.80	0.98
12y	0.86	1.14
15y	0.91	1.36
20y	0.97	1.68

Table 4.2: The loss as a percentage of notional from not accounting for basis spreads in the USD market.

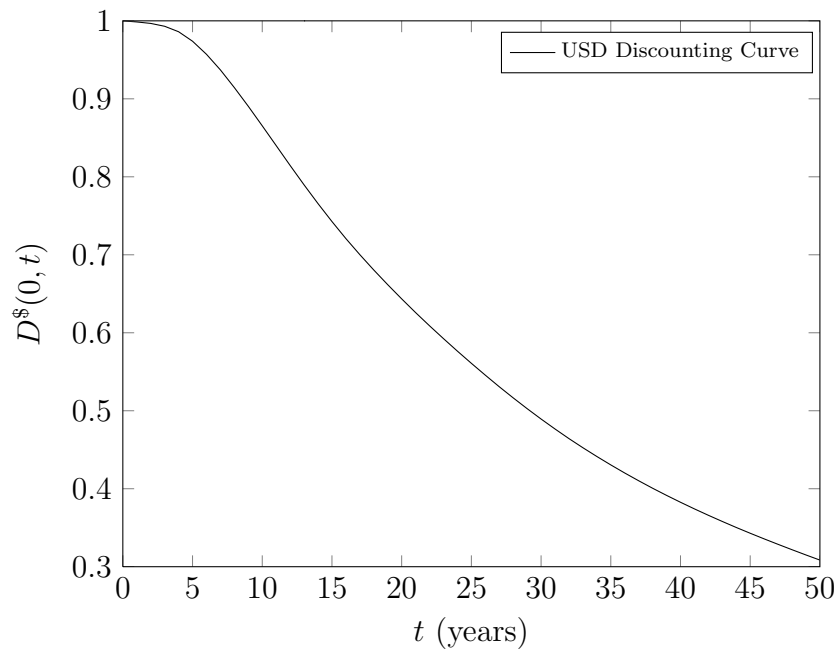


Figure 4.1: The USD discounting curve.

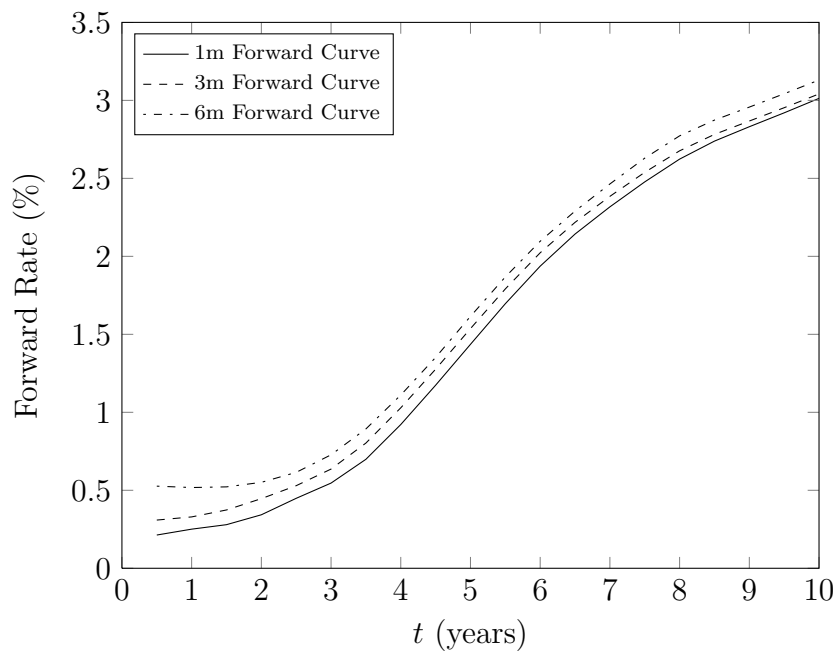


Figure 4.2: The USD forward curves.

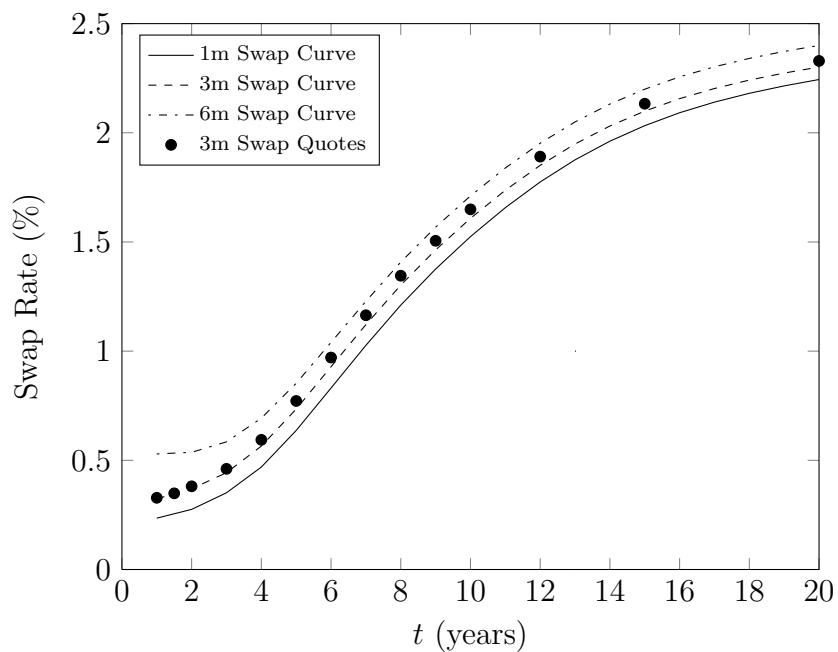


Figure 4.3: The USD swap curves along with quotes for 3m USD IRS.

4.2 THE CASE OF EUR

The discounting and forward curves are pictured in Figures 4.4 and 4.5, where the implied 12m forward rates are shown for each tenor. Figure 4.4 also includes the relevant discount factors for USD-collateralized EUR instruments, where the spread in part is due to the cross currency basis spreads in Table G.8. As in the case with USD in Section 4.1, the relative placement of the EUR forward curves agrees well with what is predicted by theory. The discounting and forward curves are used to construct the EUR-collateralized EUR swap curves in Figure 4.6, where the quoted 6m swap rates are also displayed. Table 4.3 shows the difference between the quoted rates and the rates computed using the set of curves. With an average difference of 4.1 basis points it is clear that the pricing framework

Maturity	Δ (bps)
1y	-0.27
2y	2.22
3y	4.02
5y	7.28
10y	6.70
20y	3.25

Table 4.3: The difference between quoted and estimated EUR 6m swap rates (a selection).

underestimates the true swap rate somewhat, where the difference gets smaller as the swap curve flattens out. The reason for this is the same as in Section 4.1 and was elaborated further upon there.

Furthermore, Table 4.4 highlights the importance of accounting for basis spreads by showing the percentage loss (of the notional) that would follow from only using the 6m forward curve when pricing interest rate swaps, regardless of what tenor the floating leg is linked to. The loss evidently approaches 4% of the notional for 1m swaps as the maturity increases. Even for short maturities and independent of tenor, the loss is close to 1%. For deals with notionals in the ballpark of 10 million EUR or more, the magnitude of the potential loss is substantial. These numbers thus exemplify how important it could be to correctly account for basis spreads on a day-to-day basis.

Finally, Figure 4.7 displays the difference in swap rate between USD-collateralized and EUR-collateralized EUR 6m interest rate swaps. For maturities less than 20 years the difference is at most around 1 basis point, which obviously is a consequence of the same curve being used for discounting both the fixed and the floating leg. It is worth remembering that we assumed unchanged forward rates when moving from EUR-collateralized to USD-collateralized instruments (to derive the relevant discounting curve).

Maturity	Loss (%) 1m Tenor	Loss (%) 3m Tenor	Loss (%) 1y Tenor
1y	0.26	0.17	0.25
2y	0.48	0.27	0.41
3y	0.79	0.44	0.50
4y	1.18	0.72	0.57
5y	1.52	0.87	0.65
6y	1.85	1.06	0.71
7y	2.13	1.21	0.78
8y	2.39	1.35	0.84
9y	2.62	1.46	0.90
10y	2.83	1.57	0.96
12y	3.18	1.74	1.07
15y	3.56	1.90	1.15
20y	3.98	2.06	1.24

Table 4.4: The loss as a percentage of notional from not accounting for basis spreads in the EUR market.

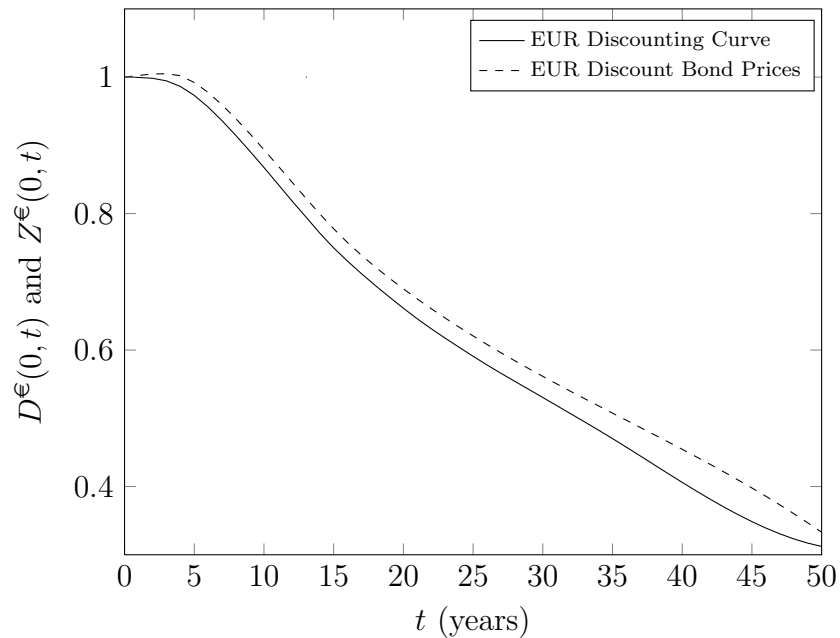


Figure 4.4: The EUR discounting curve together with the EUR zero coupon bond prices.

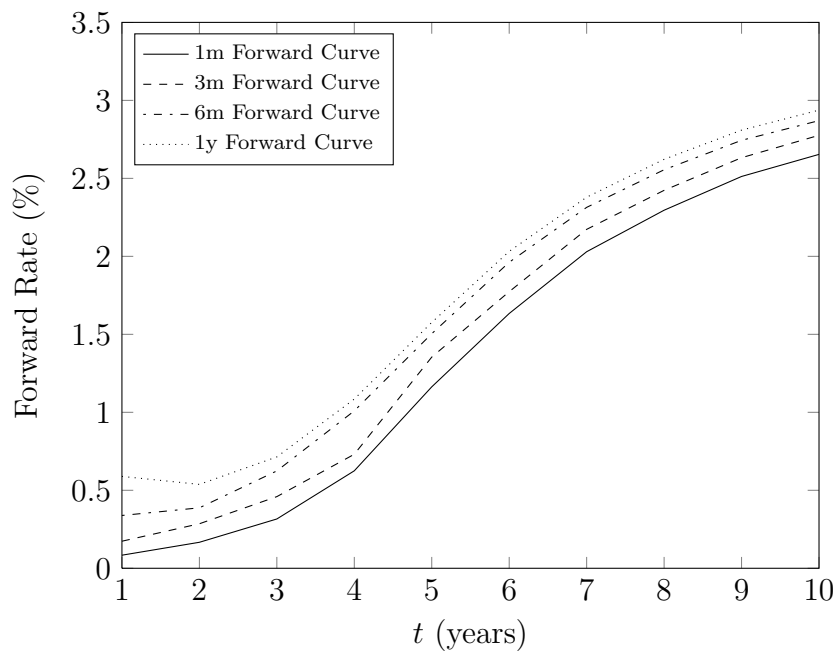


Figure 4.5: The EUR forward curves.

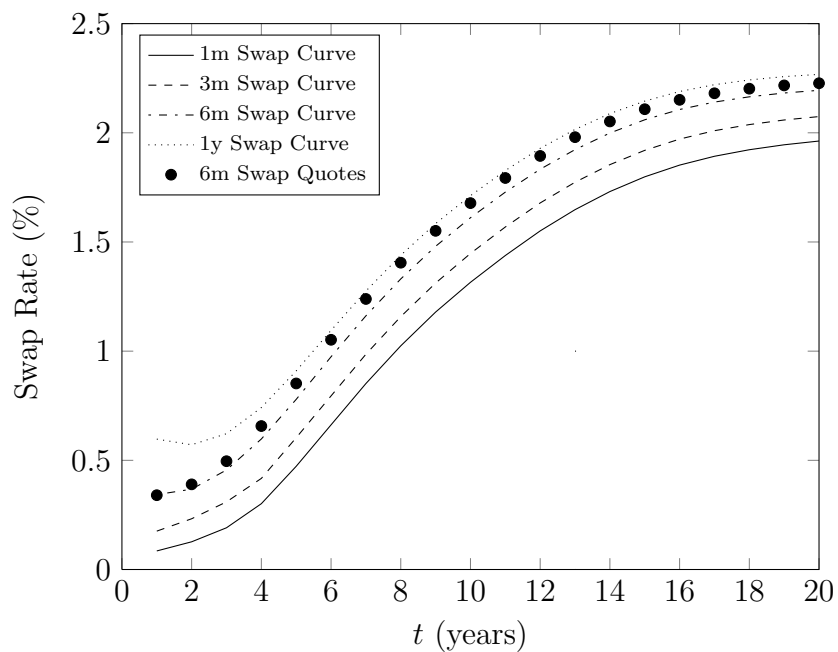


Figure 4.6: The EUR swap curves along with quotes for 6m EUR IRS.

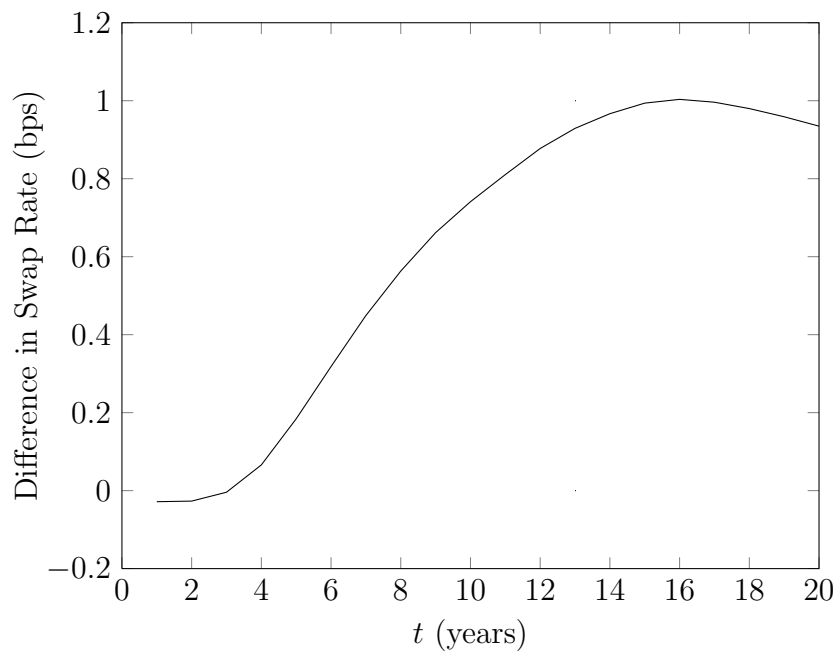


Figure 4.7: The difference in swap rate between USD-collateralized EUR 6m swaps and EUR-collateralized EUR 6m swaps.

4.3 COMPARING THE CURRENCIES

By comparing Tables 4.1 and 4.3 it is seen that both the USD 3m and EUR 6m swap rates are underestimated by the respective pricing frameworks. With an average difference between the quoted and estimated swap rate of ≈ 4 bps, the effect is more substantial for EUR swaps. The effect is even greater when considering the steep part of the swap curve, where the difference amounts to 7-8 bps for some maturities. For USD swaps, on the other hand, the quoted swap rate is never underestimated by more than 5 bps. As previously mentioned, this effect is due to the assumption that forward rates are piecewise flat in intervals of 6 months (12 months for EUR) in the bootstrapping procedure. In turn, we made this assumption since the fixed legs pay semiannually and annually for USD and EUR interest rate swaps, respectively (as compared to quarterly and semiannual floating payments). To minimize this effect one could instead assume that the dates of the fixed and floating payments coincide, i.e. that EUR swaps pay fixed semiannually whereas USD swaps pay fixed quarterly.

Moreover, Tables 4.2 and 4.4 highlight the importance of accounting for basis spreads in the respective currencies. Whereas the USD loss is at most around 1.7% (20 year 6m swap), the EUR loss approaches 4% for similar maturities. Obviously, the larger basis spreads in the EUR market as compared to the USD market are to blame for this. This is especially true when considering the EUR 1m swaps, where the loss amounts to 2% even for a swap maturing in 7 years. To incur a loss of similar magnitude in the USD market, one needs to consider a 6m swap maturing in 26 years from now (for 1m swaps of maturities less than 50 years, losses of similar magnitude are never possible). Evidently, it is currently more important to account for basis spreads when pricing European instruments as compared to American. This result is hardly surprising when considering the withholding state of the European financial climate, where the perceived liquidity/credit risks remain high.

5 CONCLUSIONS

To recapitulate, the purpose of this thesis was to implement a pricing framework that accounts for the basis spreads between different tenors and currencies, with and without collateralization. Distinct forward curves were built for each available Libor/Euribor tenor using quoted tenor basis swaps. Furthermore, the quoted overnight indexed swaps allowed us to build discount curves suitable for pricing collateralized contracts. The cross currency basis spread between USD and EUR was then used to construct a discount curve applicable on EUR-denominated instruments that are collateralized with USD cash. The resulting set of discount and forward curves made it possible to price USD-collateralized USD instruments as well as USD- and EUR-collateralized EUR instruments.

Section 4 presented the derived discount and forward curves as well as the corresponding swap curves. In particular, two aspects were discussed more deeply. Firstly, the method of constructing the curves was evaluated by how well it managed to replicate quoted swap rates. It was seen that the pricing framework underestimates quoted rates by a few basis points, where the effect was most significant for EUR swaps. Moreover, we computed the incurred loss from not accounting for basis spreads appropriately. Even now when basis spreads have narrowed considerably the effect was significant and could lead to a loss of several percent of the notional, especially for EUR denominated swaps. Therefore, we can stress the importance for institutions to implement a framework where basis spreads are not assumed negligible. Mispricing of interest rate products (and of other asset classes for that matter) could lead to substantial losses, especially if the credit climate worsens and basis spreads widen.

The rest of this Section is devoted to discussing potential improvements to the curve construction procedure as well as relevant topics for further research.

5.1 CRITIQUE

One drawback is that the derived swap curves underestimate the quoted swap spread by a few basis points, and the reason for this was explained in Section 4.1. Note that since not all relevant maturities are available it is not possible to perfectly replicate all swap rates, however one could try to reach better estimates by for example assuming that the fixed and floating legs have equal payment frequencies.

Moreover, the interpolation method could be reviewed. We have used interpolation with cubic splines consistently throughout the curve construction procedure, but as mentioned in Hagan and West (2006) [13] this method implicitly treats discrete forward rates as a property of one of the endpoints of the interval, and not as a property of the entire interval. It could therefore be an idea to implement interpolation with forward monotone

convex splines, as discussed in Hagan and West (2006) [13]. This method also makes sure that the resulting continuous curve is convex and locally monotone given that the discrete inputs have the corresponding discrete properties.

Lastly, the quoted cross currency basis swaps are in fact of the mark-to-market type and do not have constant notionals. One could account for this by adapting the implementation procedure, but this will not have significant impact on the results.

5.2 SUGGESTIONS FOR FURTHER RESEARCH

A natural extension to the empirical approach in this paper is to implement a model with dynamic basis spreads as proposed in Fujii et. al. (2009) [9] or in Filipovic and Trolle (2012) [7]. Such model could further be used to price more exotic interest rate derivatives, for example interest rate swaptions or options on tenor basis swaps.

The implementation procedure could further be extended by computing appropriate hedging ratios. Of most interest is probably the delta sensitivity of a portfolio consisting of instruments with a variety of different underlying tenors. Of course, the convexity and other Greeks could also be of interest.

Finally, the framework implemented in this thesis only produces basis consistent prices of interest rate products. It is certainly necessary to develop similar frameworks for other asset classes, for example equity and commodity derivatives.

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A THE FORWARD MEASURE

We begin with a definition of the forward measure.

DEFINITION A.1. *The T_n -forward measure Q^{T_n} is defined as the martingale measure for the numéraire process $Z(t, T_n) = e^{-\int_t^{T_n} r(s)ds}$, where $r(s)$ is the short rate process.*

This measure should not be confused with the conventional martingale measure Q with the money market account $B = e^{\int_0^t r(s)ds}$ as numéraire. It can however be shown that the Radon-Nikodym derivative $\frac{dQ^{T_n}}{dQ}$ is given by

$$\frac{dQ^{T_n}}{dQ} = \frac{Z(t, T_n)}{B(t)Z(0, T)},$$

and if assuming that the zero coupon bond dynamics under Q are Wiener driven it holds that the Girsanov kernel for the transition from Q to Q^{T_n} equals the bond volatility $\sigma(t, T_n)$. The following theorem is now rather evident.

THEOREM A.1. *The relation $Q = Q^{T_n}$ holds if and only if r is deterministic.*

Finally, let X_{T_n} be the value of an arbitrary claim at T_n . Now, $\frac{X_{T_n}}{Z(t, T_n)} = X_{T_n}$ is a martingale under Q^{T_n} and we arrive at

THEOREM A.2. *For any claim X_{T_n} it holds that*

$$X_t = Z(t, T_n)E^{T_n}[X_{T_n}|\mathcal{F}_t],$$

where $E^{T_n}[\]$ denotes expectation under Q^{T_n} .

Bear in mind that $Z(t, T_n)$ can be observed directly on the market at time t (or bootstrapped from other quoted instruments).

B MARK-TO-MARKET CROSS CURRENCY SWAPS

The mark-to-market cross currency swap and the constant notional cross currency swap are different in one aspect. In the mark-to-market cross currency swap the notional on the leg paying Libor flat (i.e. the USD leg) is adjusted at all Libor fixing times, where the calculation is based on the prevailing spot exchange rate. The difference between the old and the new notional amount is then paid or received. Conceptually, the USD leg can be seen as a portfolio of one period constant notional cross currency swaps, where the terminal and initial exchanges correspond to the transfer of notional.

We first consider a USD-collateralized USDEUR mark-to-market cross currency swap (MtMCCS). Since the notional on the EUR leg remains constant it still holds that

$$PV_{\text{€}} = -Z^{\text{€}}(t, T_0) + \sum_{n=1}^N \delta_{n-1,n}^{3m,\text{€}} (E_t^{\text{€}}[L^{3m,\text{€}}(T_{n-1}, T_n)] + CCS(t, T_N)) Z^{\text{€}}(t, T_n) + Z^{\text{€}}(t, T_N).$$

However, the present value of the USD leg is now given by

$$PV_{\text{₹}} = - \sum_{n=1}^N E_t^{Q^{\text{₹}}} \left[\frac{e^{-\int_t^{T_{n-1}} r^{\text{₹}}(s) ds}}{f(T_{n-1})} \right] + \sum_{n=1}^N E_t^{Q^{\text{₹}}} \left[\frac{e^{-\int_t^{T_n} r^{\text{₹}}(s) ds} (1 + \delta_{n-1,n}^{3m,\text{₹}} L^{3m,\text{₹}}(T_{n-1}, T_n))}{f(T_{n-1})} \right].$$

Under the assumption that the Libor is the risk-free rate it holds that $PV_{\text{₹}} = 0$ and uncollateralized swaps with 3m Libor discounting would thus not be affected by the change from constant notional to mark-to-market cross currency swaps. As we here deal with collateralized derivatives where the risk-free rate is the Federal Funds rate it gets necessary to decompose the Libor forward rate into a risk-free part plus a spread. Hence, assume that

$$L^{3m,\text{₹}}(T_{n-1}, T_n) = \frac{1}{\delta_{n-1,n}^{3m,\text{₹}}} \left(\frac{1}{Z^{\text{₹}}(T_{n-1}, T_n)} - 1 \right) + S(T_{n-1}, T_n),$$

where $S(T_{n-1}, T_n)$ is the spread at T_{n-1} . We then get

$$\begin{aligned} PV_{\text{₹}} &= - \sum_{n=1}^N E_t^{Q^{\text{₹}}} \left[\frac{e^{-\int_t^{T_{n-1}} r^{\text{₹}}(s) ds}}{f(T_{n-1})} \right] + \sum_{n=1}^N E_t^{Q^{\text{₹}}} \left[\frac{e^{-\int_t^{T_n} r^{\text{₹}}(s) ds} \delta_{n-1,n}^{3m,\text{₹}} S(T_{n-1}, T_n)}{f(T_{n-1})} \right] \\ &+ \sum_{n=1}^N E_t^{Q^{\text{₹}}} \left[\frac{1}{f(T_{n-1})} \frac{e^{-\int_t^{T_n} r^{\text{₹}}(s) ds}}{e^{-\int_{T_{n-1}}^{T_n} r^{\text{₹}}(s) ds}} \right] = \sum_{n=1}^N E_t^{Q^{\text{₹}}} \left[\frac{e^{-\int_t^{T_n} r^{\text{₹}}(s) ds} \delta_{n-1,n}^{3m,\text{₹}} S(T_{n-1}, T_n)}{f(T_{n-1})} \right]. \end{aligned}$$

This expression depends on the correlation between risk-free discount bonds and the exchange rate, even with a deterministic spread S . One way to proceed is to assume a deterministic spread S and that the forward exchange rate and the forward risk-free bond both follow geometric Brownian motions. Let $f(t, T_{n-1})$ and $Z(t, T_{n-1}, T_n)$ be the time- t forward exchange rate maturing at T_{n-1} and the time- t value of a discount bond on

$[T_{n-1}, T_n]$, respectively. Furthermore, $\sigma_{f_{n-1}}(t)$ and $\sigma_{Z_{n-1}}(t)$ are the log-normal volatilities and $\rho_{n-1}(t)$ is the correlation coefficient. It can now be shown that the present value of the USD leg is given by

$$PV_{\$} = \sum_{n=1}^N \frac{Z^{\$}(t, T_n) \delta_{n-1, n}^{3m, \$} S(T_{n-1}, T_n)}{f(t, T_{n-1})} e^{\int_t^{T_{n-1}} \sigma_{f_{n-1}}(s) \sigma_{Z_{n-1}}(s) \rho_{n-1}(s) ds}.$$

To calibrate this model the spread S can be computed as the difference between the Federal Funds rate curve and the collateralized Libor curve. Forward exchange rates are further observed directly on the market. Moreover, the exchange rate volatility is observed on the FX option market. Similarly, the volatility of the Federal Funds rate can be extracted from the overnight indexed swaption market. The correlation can lastly be estimated from historical data or perhaps be calibrated using quanto products. Having completed these steps, the necessary curves can be built as prescribed in Section 2.2.4.

We will not go into the case of the EUR-collateralized USDEUR MtMCCS and instead refer to Fujii et. al. (2010) [11]. More information on mark-to-market cross currency swaps is found in Fujii et. al. (2009a) [9].

C DAY COUNT CONVENTIONS

This appendix lists how to compute day count factors using a few different day count conventions. A deeper survey is found in [23].

- **30/360 methods.** The day count factor between two dates is given by

$$\delta = \frac{360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)}{360},$$

where Y , M and D denote year, month and day, respectively. How to compute these depends on which 30/360 method that is used.

- **ACT/360.** The day count factor is given by

$$\delta = \frac{d_2 - d_1}{360},$$

where $d_2 - d_1$ is the number of days between the dates.

- **ACT/365 Fixed.** In this case,

$$\delta = \frac{d_2 - d_1}{365}.$$

Note that 365 is used also in a leap year.

- **ACT/ACT ISDA.** The day count factor is

$$\delta = \frac{\text{Days in a non-leap year}}{365} + \frac{\text{Days in a leap year}}{366},$$

where the first day is included and the last day is excluded when computing the number of days.

- **Business/252.** The day count factor is

$$\delta = \frac{\text{Business}}{252},$$

where the number of business day is in a given calendar (including the first date and excluding the last date).

D SWAP CONVENTIONS

Market conventions for USD and EUR interest rate- and overnight indexed swaps are listed herein. More information is found in [23].

		Fixed Leg		Floating Leg		
Currency	Spot Lag	Tenor	Convention	Reference	Tenor	Convention
USD	2	6m	30/360	Libor	3m	ACT/360
EUR 1y	2	1y	30/360	Euribor	3m	ACT/360
EUR >1y	2	1y	30/360	Euribor	6m	ACT/360

Table D.1: Market conventions for interest rate swaps. The spot lag is the number of days between the trade date and the first fixing date.

		Fixed Leg		Floating Leg		
Currency	Spot Lag	Tenor	Convention	Reference	Convention	Pay Lag
USD \leq 1y	2	Maturity	ACT/360	Fed Fund	ACT/360	2
USD $>$ 1y	2	1y	ACT/360	Fed Fund	ACT/360	2
EUR \leq 1y	2	Maturity	ACT/360	Eonia	ACT/360	2
EUR $>$ 1y	2	1y	ACT/360	Eonia	ACT/360	2

Table D.2: Market conventions for overnight indexed swaps. The spot lag is the number of days between the trade date and the first fixing date. The pay lag is the number of days between the last fixing date and the payment.

E CUBIC SPLINE INTERPOLATION

As mentioned in Ron (2000) [26] and Hagan and West (2006) [13] it is desired that the output of an interpolation method is a smooth curve that is able to replicate observed market data points reasonably well. A risk with over-smoothing the curve is however that valuable market pricing information may be lost. Also, in order to avoid arbitrage the produced forward rates must be positive. As we mostly perform interpolation immediately on extracted forward rates we need not worry about this issue.

In this thesis we have consistently used interpolation with cubic splines. This method does indeed produce a smooth, continuous curve and it is one of the most prevalent techniques on the market. The method computes $n - 1$ third-order polynomials $f_i(t)$ between the n market observations, over the time interval $[t_1, t_n]$. I.e.,

$$f_i(t) = a_i + b_i(t - t_i) + c_i(t - t_i)^2 + d_i(t - t_i)^3, \quad t_i < t < t_{i+1}, \quad 1 \leq i \leq n - 1.$$

The number of unknowns is thus $4n - 4$, and we need to impose a few constraints to compute all coefficients. First, we require that the function meets the observed market data points f_i , $1 \leq i \leq n$. With $h_i = t_{i+1} - t_i$ we obtain the following n equations:

$$\begin{aligned} a_i &= f_i, \quad i = 1, 2, \dots, n - 1 \\ a_{n-1} + b_{n-1}h_{n-1} + c_{n-1}h_{n-1}^2 + d_{n-1}h_{n-1}^3 &= f_n. \end{aligned}$$

Furthermore, we want the entire function to be continuous and differentiable, which gives rise to another $2n - 4$ equations.

$$\begin{aligned} a_i + b_i h_i + c_i h_i^2 + d_i h_i^3 &= a_{i+1}, \quad i = 1, 2, \dots, n - 2 \\ b_i + 2c_i h_i + 3d_i h_i^2 &= b_{i+1}, \quad i = 1, 2, \dots, n - 2. \end{aligned}$$

We now have a set of $3n - 4$ equations with $4n - 4$ unknowns and hence need n extra conditions. One approach is to use the natural boundary conditions as described in Burden and Faires (1997) [4]. In this set-up we assume that the derivative is differentiable everywhere and that the second derivative at each endpoint equals zero. We then get

$$\begin{aligned} c_i + 3d_i h_i &= c_{i+1}, \quad i = 1, 2, \dots, n - 2 \\ c_{n-1} + 3d_{n-1} h_{n-1} &= 0 \\ c_1 &= 0, \end{aligned}$$

and can now solve for all $4n - 4$ coefficients.

Other, and more advanced, interpolation methods are covered in Hagan and West (2006) [13] and Hagan and West (2008) [14]. These papers also discuss preferable characteristics of an interpolation method for curve construction to a greater extent than what is done here.

F PROOF OF PROPOSITION 2.1

First consider collateral posted in the domestic currency. Let $V(t)$ be the stochastic process of the collateral account and adopt a suitable self-financing trading strategy under the risk-neutral measure. Since the posted collateral can be invested at the risk-free rate but decays by the collateral rate, an SDE for the process is given by

$$dV(s) = (r(s) - c(s))V(s)ds + a(s)dh(s),$$

where $h(s)$ is the time- s value of the derivative maturing at T and $a(s)$ is the number of positions in h at time s . Integration yields

$$V(T) = e^{\int_t^T y(u)du}V(t) + \int_t^T e^{\int_s^T y(u)du}a(s)dh(s),$$

where $y(s)$ is the funding spread defined by $y(s) = r(s) - c(s)$. With the trading strategy

$$\begin{aligned} V(t) &= h(t) \\ a(s) &= e^{\int_t^s y(u)du} \end{aligned}$$

we get

$$\begin{aligned} V(T) &= e^{\int_t^T y(u)du}h(t) + \int_t^T e^{\int_s^T y(u)du}e^{\int_t^s y(u)du}dh(s) \\ &= e^{\int_t^T y(u)du}h(t) + e^{\int_t^T y(u)du}(h(T) - h(t)) = e^{\int_t^T y(u)du}h(T). \end{aligned}$$

The present value of the derivative is then given by

$$h(t) = E_t^Q \left[e^{-\int_t^T r(s)ds} V(T) \right] = E_t^Q \left[e^{-\int_t^T c(s)ds} h(T) \right],$$

where expectation is taken with the money-market account as numéraire.

Next, consider a derivative where the collateral is posted in foreign currency. The process of the foreign currency collateral account is given by

$$dV^f(s) = ((c^f(s) - r^f(s))V^f(s)ds + a(s)d\left(\frac{h(s)}{f(s)}\right)),$$

where $c^f(s)$ and $r^f(s)$ are the time- s foreign collateral and risk-free rates, respectively. $f(s)$ is the foreign exchange rate. Integration yields

$$V^f(T) = e^{\int_t^T y^f(u)du}V^f(t) + \int_t^T e^{\int_s^T y^f(u)du}a(s)d\left(\frac{h(s)}{f(s)}\right),$$

where $y^f(s) = r^f(s) - c^f(s)$ is the foreign funding spread. Adopting the trading strategy

$$V^f(t) = \frac{h(t)}{f(t)}$$

$$a(s) = e^{\int_t^s y^f(u) du}$$

yields

$$V^f(T) = e^{\int_t^T y^f(u) du} \frac{h(T)}{f(T)}.$$

Finally, the price of the derivative (in domestic currency) is given by

$$h(t) = V^f(t)f(t) = E_t^Q \left[e^{-\int_t^T r(s) ds} V^f(T)f(T) \right] = E_t^Q \left[e^{-\int_t^T r(s) ds} \left(e^{\int_t^T (r^f(s) - c^f(s)) ds} \right) \right].$$

Alternative proofs of these results are provided by Piterbarg (2010) [24] and Piterbarg (2012) [25].

G TABLES

G.1 TABLES OF USD DATA

Tenor	Rate
1m	0.2085
3m	0.3100
6m	0.5265

Table G.1: Quoted USD spot rates (%) on November 12th 2012.

Maturity/Basis	1m/3m	3m/6m
18m	0.0900	
2y	0.0880	0.1510
3y	0.0830	0.1320
4y	0.0790	0.1200
5y	0.0750	0.1120
6y	0.0710	0.1060
7y	0.0670	0.1020
8y	0.0630	0.1010
9y	0.0590	0.1000
10y	0.0550	0.0990
12y	0.0480	0.0980
15y	0.0410	0.0970
20y	0.0340	0.0960
25y	0.0290	0.0960
30y	0.0260	0.0950
40y	0.0233	0.0940
50y	0.0209	0.0930

Table G.2: Quoted USD tenor basis spreads (%) on November 12th 2012.

Maturity/Basis	OIS	1m	3m
ON	0.1540		
1w	0.1590		
2w	0.1590		
3w	0.1590		
1m	0.1580		
2m	0.1500	0.2125	
3m	0.1460	0.2136	
4m	0.1430	0.2100	
5m	0.1410	0.2124	
6m	0.1390	0.2142	0.3107
7m	0.1385	0.2170	
8m	0.1387	0.2198	
9m	0.1390	0.2244	
10m	0.1392	0.2289	
11m	0.1396	0.2317	
1y	0.1400	0.2351	0.3284
18m			0.3488
2y	0.1630		0.3810
3y	0.2300		0.4610
4y	0.3460		0.5939
5y	0.5205		0.7719
6y	0.7163		0.9708
7y	0.9070		1.1645
8y	1.0881		1.3454
9y	1.2492		1.5055
10y	1.3943		1.6494
12y	1.6370		1.8901
15y	1.8840		2.1331
20y	2.0828		2.3291
25y	2.1823		2.4263
30y	2.2448		2.4865
40y	2.2798		2.5166
50y	2.2648		2.4750

Table G.3: Quoted USD OIS and IRS rates (%) on November 12th 2012.

G.2 TABLES OF EUR DATA

Tenor	Rate
1m	0.1090
3m	0.1930
6m	0.3660
1y	0.5890

Table G.4: Quoted EUR spot rates (%) on November 12th 2012.

Maturity/Basis	3m
DEC2012F	0.1775
MAR2013F	0.1575
JUN2013F	0.1675
SEP2013F	0.1875
DEC2013F	0.2275
MAR2014F	0.2725
JUN2014F	0.3375
SEP2014F	0.3925
DEC2014F	0.4625
MAR2015	0.5325

Table G.5: Quoted Euribor futures (%) on November 12th 2012.

Maturity/Basis	1m/6m	3m/6m	6m/1y
3y	0.2630	0.1530	0.1600
4y	0.2605	0.1485	0.1390
5y	0.2580	0.1440	0.1260
6y	0.2525	0.1395	0.1170
7y	0.2480	0.1350	0.1100
8y	0.2430	0.1310	0.1050
9y	0.2375	0.1265	0.1010
10y	0.2330	0.1220	0.0980
11y	0.2265	0.1175	0.0960
12y	0.2195	0.1125	0.0930
13y	0.2135	0.1082	0.0898
14y	0.2084	0.1044	0.0864
15y	0.2040	0.1010	0.0830
16y	0.1999	0.0979	0.0800
17y	0.1959	0.0949	0.0774
18y	0.1921	0.0922	0.0750
19y	0.1885	0.0895	0.0729
20y	0.1850	0.0870	0.0710
21y	0.1819	0.0847	0.0693
22y	0.1790	0.0826	0.0678
23y	0.1765	0.0806	0.0664
24y	0.1741	0.0787	0.0651
25y	0.1720	0.0770	0.0640
26y	0.1698	0.0753	0.0630
27y	0.1674	0.0737	0.0621
28y	0.1650	0.0721	0.0613
29y	0.1625	0.0705	0.0606
30y	0.1600	0.0690	0.0600
35y	0.1513	0.0632	0.0572
40y	0.1456	0.0595	0.0551
50y	0.1373	0.0540	0.0524

Table G.6: Quoted EUR tenor basis spreads (%) on November 12th 2012.

Maturity/Basis	OIS	1m	6m
ON	0.0300		
1w	0.0820		
2w	0.0820		
3w	0.0820		
1m	0.0810		
2m	0.0790	0.1090	
3m	0.0720	0.1030	
4m	0.0660	0.0990	
5m	0.0600	0.0940	
6m	0.0550	0.0910	
7m	0.0520	0.0880	
8m	0.0500	0.0870	
9m	0.0490	0.0860	
10m	0.0480	0.0850	
11m	0.0480	0.0850	
1y	0.0480	0.0850	0.3430
18m		0.0945	
2y	0.0830	0.1259	0.3900
3y	0.1790		0.4960
4y	0.3370		0.6570
5y	0.5330		0.8520
6y	0.7340		1.0520
7y	0.9230		1.2390
8y	1.0910		1.4050
9y	1.2400		1.5510
10y	1.3710		1.6780
11y	1.4910		1.7930
12y	1.5980		1.8940
13y	1.6851		1.9800
14y	1.7671		2.0520
15y	1.8280		2.1080
16y	1.8689		2.1510
17y	1.9013		2.1810
18y	1.9271		2.2020
19y	1.9483		2.2170
20y	1.9670		2.2270
21y	1.9818		2.2350
22y	1.9919		2.2420
23y	1.9990		2.2470
24y	2.0045		2.2510
25y	2.0100		2.2530
26y	2.0146		2.2540
27y	2.0175		2.2550
28y	2.0199		2.2560
29y	2.0227		2.2570
30y	2.0270		2.2600
35y	2.0634		2.2930
40y	2.1314		2.3420
50y	2.1912		2.4120

Table G.7: Quoted EUR OIS and IRS rates (%) on November 12th 2012.

Maturity/Basis	EUR/USD
3m	-0.2730
6m	-0.2625
9m	-0.2675
1y	-0.2725
18m	-0.2700
2y	-0.2975
3y	-0.3050
4y	-0.3000
5y	-0.2825
6y	-0.2625
7y	-0.2425
8y	-0.2308
9y	-0.2192
10y	-0.2075
12y	-0.1955
15y	-0.1775
20y	-0.1625
30y	-0.1525

Table G.8: Quoted EURUSD cross currency basis spreads (%) on November 12th 2012.

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