Building curves on a good basis

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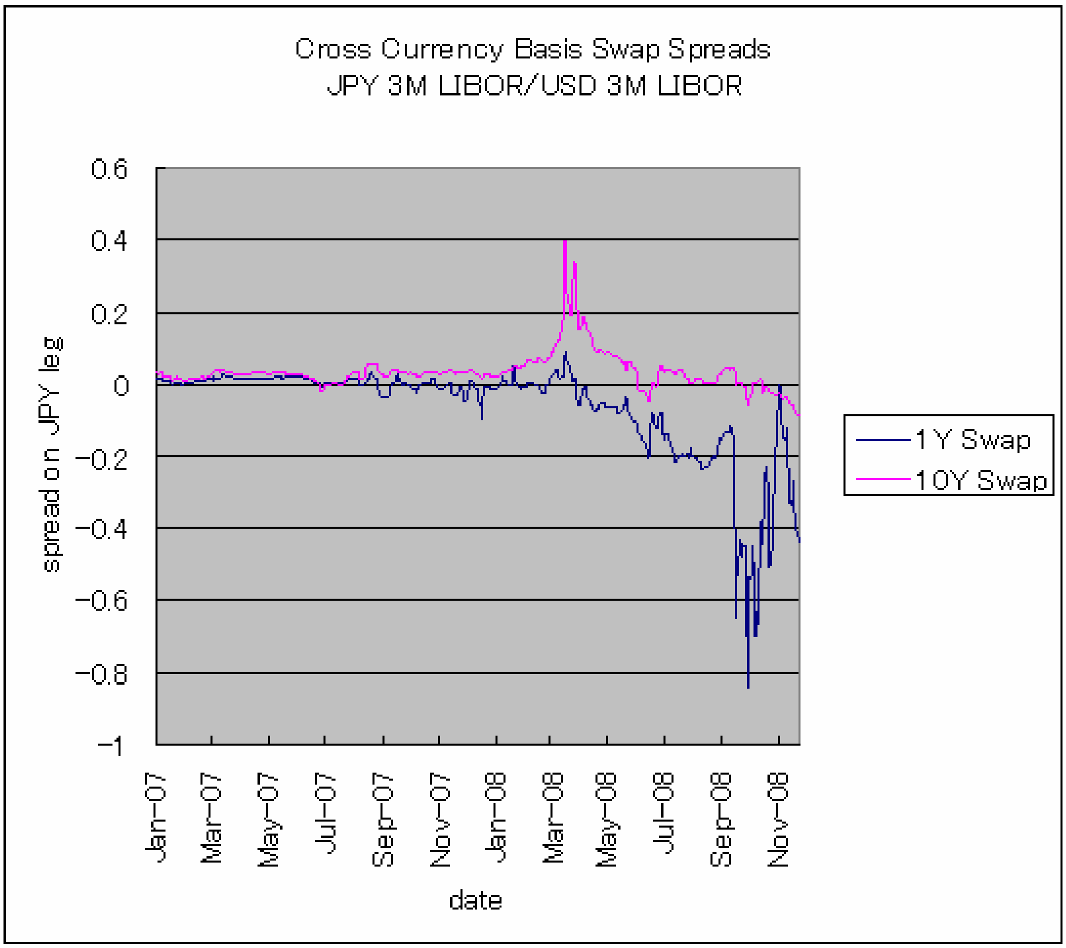
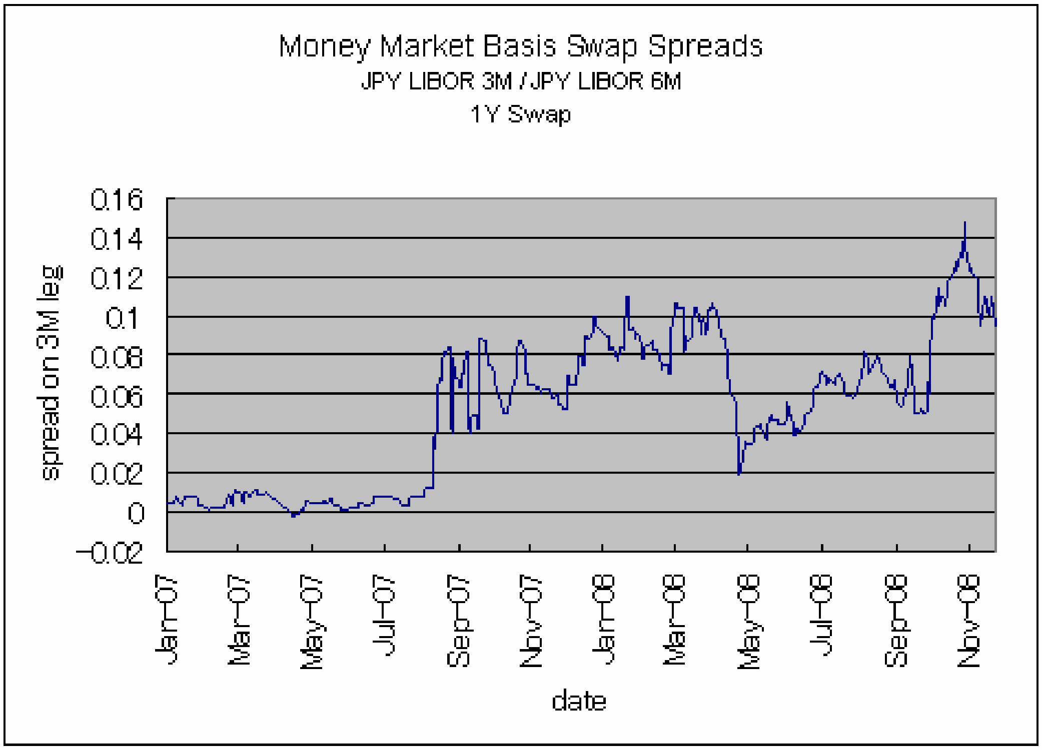
In this article we gather a few facts about discount curve construction used for derivatives pricing and its current state of the art and we address issues surrounding basis swap adjustments.

1. Introduction

The derivative industry has been struggling with quite a few paradoxes in the last decade. One of the simplest is how to discount cash flows for a given currency in a consistent framework regardless of which trade it originates from. This recurring issue comes with the joint existence of the cross currency basis swaps and plain vanilla swap markets. In what follows we will show that the approach of valuing both instruments with a single discount curve creates arbitrage between the two markets. Therefore there is a need even more than before to set up a framework where both instruments can be valued to par without inconsistencies. This need is even more pronounced at the date of writing since in the recent months basis swap spreads have reached dramatic levels and their effect on valuing and hedging interest rate derivatives can not be neglected anymore. In the graph below we show the evolution between january 2007 and mid-november 2008 of JPY/USD cross currency basis swap spreads for the one year and ten year maturity as well as money market basis swap spreads for the one year maturity for JPY Libor 3M/ Libor 6M and USD Libor 3M/Libor 6M. In all cases we can see that in the first half of 2007 both types of basis swaps vary very little but in the middle of summer 2007, when the credit crunch started, their start to vary at an increasing speed in huge magnitude from then to mid-november 2008. This alone justfies a more through analysis of whether it is relevant or not to include these instruments in constructing a consistent discount curve methodology.




Money Market Basis Swap Spreads
JPY LIBOR 3M / JPY LIBOR 6M
1Y Swap
spread on 3M leg
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
Jan-07
Mar-07
May-07
Jul-07
Sep-07
Nov-07
Jan-08
Mar-08
May-08
Jul-08
Sep-08
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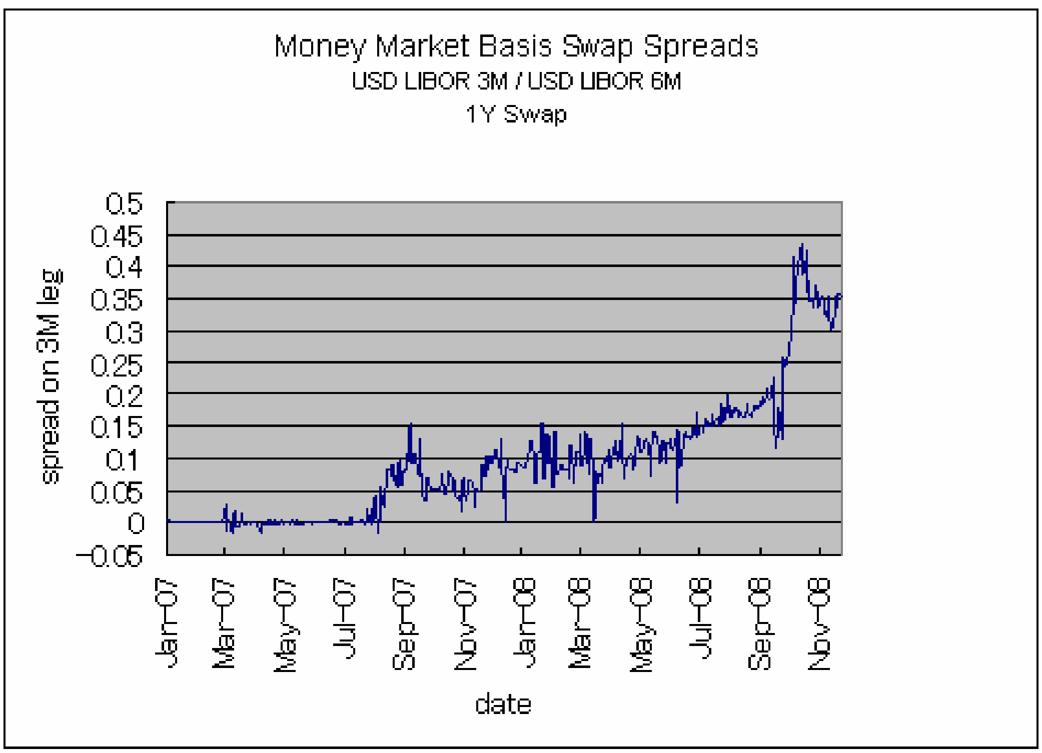
Money Market Basis Swap Spreads
USD LIBOR 3M / USD LIBOR 6M
1Y Swap
spread on 3M leg
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
Jan-07
Mar-07
May-07
Jul-07
Sep-07
Nov-07
Jan-08
Mar-08
May-08
Jul-08
Sep-08
Nov-08
date

This paper is organized as follows. In the first section of this document we will describe both plain vanilla swaps and basis swaps and provide some justification for their existence. In the second part we will describe the problem generated by the one curve valuation framework. The third part will describe a general self consistent framework for valuing both types of swaps. Then we will describe a general bootstrapping methodology that can be applied to build both curves simultaneously. The last part will extend the framework to the joint existence of cross currency basis swaps, plain vanilla single currency swaps and money market basis swaps. We also provide one appendice where we describe how to derive forward rates from futures prices, i.e. the so called futures FRA convexity adjustment.

2. Plain vanilla single currency swaps and basis swaps

A plain vanilla swap is an agreement between two counterparties to exchange fixed rate payments over floating rate payments over a given length of time. Floating rate payment would most typically be based on short tenored Libors (3 month, 6 month). For each currency there is a prefered tenor: that would be 3 months for USD and 6 months for JPY. The floating leg would generally pay at a frequency which agrees with the underlying tenor i.e. 3 month Libor would be paid quaterly, 6 month Libor semi-annually. The fixed rate can be paid on a different frequency. In the rest of this paper we will assume without loss of generality that fixed and floating rate payment frequencies coincide. Because both legs of a plain vanilla swap are denominated in the same currency there is no initial and final exchange of notionals. Plain vanilla swaps enable their user to hedge their floating rate exposure.

The term basis swap describes a general class of swaps where both legs are floating. They can be single currency swaps where two conterparties will for example exchange a floating rate index against a different floating rate index. For example some basis swaps will exchange a T-bill rate versus a 3M Libor rate. Others will exchange 3 month Libor against 6 month Libor. The latter are called money market basis swaps. Cross currency basis swaps consist of exchanging floating rate payments denominated in one currency against floating rate payments in an another currency. A typical example would be a USD / JPY basis swap exchanging 3 month JPY Libor + spread against 3 months USD Libor + spread. Typically the JPY floating rate payer would borrow a notional amount in JPY and lend to the USD floating rate payer the equivalent USD amount converted at the FX rate prevailing at the date of trade inception. On final payment date, the two couterparties would exchange back the same exact notionals. Cross currrency basis swaps enables couterparties to swap their floating liabilities and assets from one currency to another one. Generally the most liquid cross currency basis swaps involve a base currency (most commonly USD). A fair cross currency basis swap trades flat on the base currency (no spread over 3 month USD Libor) against the foreign floating index plus spread. This spread reflects both the liquidity premium there is for the foreign currency over base currency as well as credit worthiness of foreign banks over base currency banks. Similarly money market basis swaps will have the short tenored floating leg trade at a spread over the longer tenor (6 months Libor flat over 3 month Libor + positive spread). This spread reflects the higher probability of default for a six month loan based on 6 month libor which is built in the 6 month Libor quote compared to a 3 month loan rolled over for another 3 months based on 3 months Libor. For a money market basis swap to be fair the 3 months Libor leg has to trade at a spread over the 6 month libor leg.

For a more detailed treatment on swap conventions we refer the reader to [Plat]. For deeper analysis of the meaning of money market basis swaps and cross currency basis swap we refer the reader to respectively [Tuck] and [BIS].

3. Pricing framework

Firstly we will denote a default free domestic discount factor at time *t* in currency *k* and delivery T by $P^k(t,T)$. When there is no ambiguity about the currency we will just drop the superscript *k*. We also define the spot Libor rate at time T_0 for payment date T_1 as the rate of return of the following investment: at time T_0 buy 1 unit of the discount bond with maturity T_1 and get back 1 at time T_1. By definition we have:

$$L(T_0,T_1)=\frac{1}{T_1-T_0}\left(\frac{1}{P(T_0,T_1)}-1\right)$$

We now consider the instruments called Forward Rate Agreements (FRAs). They consist of an agreement to exchange at a future date T_1 a fixed rate payment K against a the spot Libor rate fixing a future time T_0 and covering the period $[T_0,T_1]$. Assuming a notional of 1 unit of domestic currency the payoff at time T_1 of this contract to the floating rate receiver is:

$$V(T_1)=\left(L(T_0,T_1)-K\right)(T_1-T_0)$$

Relying on the usual non arbitrage arguments, the value of this contract at time t is:

$$\begin{aligned}V(t) &= P(t,T_1)E_t^{T_1}\left[V(T_1)\right] = P(t,T_1)E_t^{T_1}\left[\left(L(T_0,T_1)-K\right)\right](T_1-T_0)\\ &= P(t,T_1)\left(E_t^{T_1}\left[L(T_0,T_1)\right]-K\right)(T_1-T_0)\end{aligned}$$

where $E_t^{T_1}[\ ]$ denotes the expectation operator under the T_1 forward measure. We recall that under the T_1 forward measure, the numeraire is the price of the discount bond which matures at time T_1.

Using the terminology introduced in [Mer], we will call $F(t,T_0,T_1) = E_t^{T_1}\left[L(T_0,T_1)\right]$ the FRA rate.

Assuming no arbitrage,no couterparty risk and no liquidity shortage the floating leg part of the FRA is equivalent to being long a discount bond maturing at time T_0 and being short a discount bond maturing at time T_1. We deduce:

$$P(t,T_1)F(t,T_0,T_1)(T_1-T_0) = P(t,T_0) - P(t,T_1)$$

We use this result to derive the value of a plain vanilla swap. Let us consider a swap rate paying Libor at time $T_1, T_2, ...T_n$ against a fixed rate *K*. Assuming a unit notional the value at time *t* of such a swap to the floating rate receiver at time *t* is:

$$V^S(t) = \sum_{i=1}^{n} P(t,T_i)F(t,T_{i-1},T_i)(T_i-T_{i-1}) - K\sum_{i=1}^{n} P(t,T_i)(T_i-T_{i-1})$$

We define the annuity $A(t)=\sum_{i=1}^{n}P(t,T_i)(T_i-T_{i-1})$. Using the above replication arguments for FRAs we simplify the expression for $V^S(t)$ as:

$$V^S(t)=P(t,T_0)-P(t,T_1)-KA(t)$$

The swap rate is said to be "at Par" at time *t* if its value is 0. In such case the fixed rate *K*, which we denote by $S(t)$ from now on is called the par swap rate at time *t* and is obtained through the following expression:

$$S(t)=\frac{P(t,T_0)-P(t,T_1)}{A(t)}$$

We now consider a cross currency basis swap with the same length as the above plain vanilla swap and where the foreign leg is the same currency and that the foreign Libor tenor is the same in both swaps. However it is not necessary to assume the USD floating leg has the same USD Libor tenor as the foreign tenor but for the sake of notation simplicity we will do so. We assume that the base currency is USD. Futhermore we assume there is no liquidity nor credit issue on the USD leg (okay I admit it this is a fairly arguable assumption these days). From now on we will denote respectively the foreign and USD discount factors as $P^f(t,T)$ and $P^{USD}(t,T)$. We will denote by $S(t)$ the FX spot Libor at time *t* expressed in foreign units per USD unit.

To the foreign leg receiver the foreign value $V^{b,S}(t)$ of a the usual basis swap trading flat on the USD leg and carrying a spread *sp* on the foreign leg at time *t* assuming one unit of USD notional is:

$$V^{b,S}(t) = S(t)\left(1 - \sum_{k=1}^{n} P^{USD}(t,T_i) F^{USD}(t,T_{i-1},T_i)(T_i - T_{i-1}) - P^{USD}(t,T_n)\right)$$
$$-\left(1 - \sum_{k=1}^{n} P^{f}(t,T_i)\left(F^{f}(t,T_{i-1},T_i) + sp\right)(T_i - T_{i-1}) - P^{f}(t,T_n)\right)$$

With the above assumptions we know that the USD leg values to Par i.e.:

$$1 - \sum_{k=1}^{n} P^{USD}(t,T_i) F^{USD}(t,T_{i-1},T_i)(T_i - T_{i-1}) - P^{USD}(t,T_n) = 0$$

Therefore the expression for the basis swap value simplifies to:

$$V^{b,S}(t) = \sum_{k=1}^{n} P^{f}(t,T_i)\left(F^{f}(t,T_{i-1},T_i) + sp\right)(T_i - T_{i-1}) + P^{f}(t,T_n) - 1$$

The basis swap is said to value at Par at time *t* if its initial value is 0. In this case:

$$V^{b,S}(t) = \sum_{k=1}^{n} P^{f}(t,T_i)\left(F^{f}(t,T_{i-1},T_i) + sp\right)(T_i - T_{i-1}) + P^{f}(t,T_n) - 1$$

Assuming no arbitrage, if the initial assumptions on no liquidity issue, no couterparty risk were true then we know we would also have the following inequality:

$$\sum_{k=1}^{n} P^f(t,T_i)\left(F^f(t,T_{i-1},T_i)\right)(T_i - T_{i-1}) + P^f(t,T_n) - 1 = 0$$

Therefore the only admissible basis swap spread *sp* would be 0. However the market quotes both par plain vanilla swaps and basis swap with non zero basis spread. We conclude that it is not reasonable to write:

$$P(t,T_1)F(t,T_0,T_1)(T_1 - T_0) = P(t,T_0) - P(t,T_1)$$

where $P(t,.)$ would be a unique discount curve. In what follows we show how to move away from the one curve methodology in a theoretically sound two curve framework.

4. Two Curve methodology

Since we have to divorce the knowledge of proper discount factor $\left(P(t,T)\right)$ from knowledge of FRA rates $\left(F(t,T_0,T_1)\right)$, we could consider having two curves to interpolate from: a discount curve and a FRA rate curve. The latter would have FRA rates as the primary interpolated variable. However since the coverage period $T_1 - T_0$ is not going to be constant due to calendar conventions, by doing so we run the risk of interpolating on rates with non perfectly homogenous tenors. A reasonable way to circumvent this is to mirror the one curve framework and still define FRA rates in terms of artificial discount factors. We call this set of artifical discount factors the "forecast curve" and denote them by $\left(P^*(t,T)\right)$. In [Mer], the FRA rate is related to P^* in a "discount curve" style such that:

$$F(t,T_0,T_1)=\frac{1}{T_1-T_0}\left(\frac{P^*(t,T_0)}{P^*(t,T_1)}-1\right)$$

whereas in [Boe] this relation is of "mixed discount forecast" nature. In this case the rationale is that the value of the floating leg of the FRA should coincide with the value of the strategy long T_0 discount bond short T_1 discount bond if their prices were obtained from the forecast curve. This would lead to:

$$F(t,T_0,T_1)=\frac{1}{T_1-T_0}\frac{P^*(t,T_0)-P^*(t,T_1)}{P(t,T_1)}$$

Whether one should choose a representation is quite arbitrary and will have a real impact only on the price of off market swaps. We decide here without loss of generality to stick to the first formalism.

Now that notation has been set we can engage into the curve construction algorithm per se. The trading date will be identified by the calendar time *t*.

We will start with the following setting. We consider a set of swap fixing dates $(T_i,T_2,...,T_n)$ regularly spaced so that for all indices *k*, $T_k-T_{k-1}=\tau_k$ corrresponds to a constant libor tenor. The case where swap fixing dates are sparse rather than regular will be treated afterwards. We also assume that discount factors $\left(P(t,T_k),P^*(t,T_k)\right)_{0\le k\le i-1}$ are known.

Furthermore we assume that the floating leg on a the foreign plain vanilla swap has the same Libor tenor as the floating leg on the foreign/USD basis swap. This is not necessarily the case but we can always go back to this situation by the use of money market basis swaps.

We also assume that par cross basis swap maturities and par plain vanilla swap maturities coincide perfectly. We will see later how we can relax all of these assumptions.

The market provides quotes for the above plain vanilla swaps and par cross currency basis swaps respectively through the values of par basis swap spreads $(sp_i)_{1 \le i \le T_n}$ and par swap rates $(S_i)_{1 \le i \le n}$.

Using our two curves framework and the market quotes we get the following equalities

$$\forall j \in [i,...,n] \quad S_j \sum_{k=1}^{j} P(t,T_k)(T_k - T_{k-1}) = \sum_{k=1}^{j} P(t,T_k) F(t,T_{k-1},T_k)(T_k - T_{k-1})$$
$$\forall j \in [i,...,n] \quad \sum_{k=1}^{j} P(t,T_k)\left(F(t,T_{k-1},T_k) + sp_j\right)(T_k - T_{k-1}) = 1 - P\left(t,T_j\right)$$

We can then deduce the whole set of discount and forecast factors for both curves using the following bootstrapping algorithm:

$$For\ all\ j\ in[i,...,n]$$
$$P\left(t,T_j\right) = \frac{1-\left(S_j + sp_j\right)\sum_{k=1}^{j-1} P(t,T_k)(T_{k-1} - T_k)}{1+\left(S_j + sp_j\right)\left(T_j - T_{j-1}\right)}$$
$$F(t,T_{j-1},T_j) = \frac{1}{T_j - T_{j-1}}\left(\frac{P\left(t,T_{j-1}\right)}{P\left(t,T_j\right)} - 1\right) + \frac{1}{P\left(t,T_j\right)\left(T_j - T_{j-1}\right)}\left(Sp_{j-1}\sum_{k=1}^{j-1} P(t,T_k)(T_k - T_{k-1}) - Sp_j \sum_{k=1}^{j} P(t,T_k)(T_k - T_{k-1})\right)$$
$$P^*\left(t,T_j\right) = P^*(t,T_{j-1})\frac{1}{1+\left(T_j - T_{j-1}\right)F\left(t,T_{j-1},T_j\right)}$$

From the above recursive expression we note the presence of basis swap spread sp_{i-1} which is not quoted by the market. However if we define it from the already known discount factors:

$$\sum_{k=1}^{i-1} P(t,T_k)\left(F(t,T_{k-1},T_k)+sp_{i-1}\right)(T_k-T_{k-1}) = 1-P(t,T_{i-1})$$

Then the bootstrapping algorithm described above is perfectly determined.

Now let us consider the case of sparse maturities. In other words we find ourselves in a situation where the market does not provide liquid quotes for all the maturities between T_j and $T_{j+\delta}$ where $\delta > 1$. We assume that all both set of discount factors for times $(T_k)_{1\le k\le j}$ are known. Then the above bootstrapping methodology can not be applied. However we can generalize it by assuming that all intermediate discount factors can be interpolated from the set of known discount factors respectively: $\left(P(t,T_k)\right)_{1\le k\le j}, \left(P^*(t,T_k)\right)_{1\le k\le j}$ and the one we wish to solve for $P\left(t,T_{j+\delta}\right), P^*\left(t,T_{j+\delta}\right)$. We call f, f^* these interpolating function. We can write:

$$\begin{aligned} \forall k\in[j+1, j+\delta-1] \quad & P(t,T_k) = f\left(T_k, \left(P(t,T_l)\right)_{1\le l\le j}, P\left(t,T_{j+\delta}\right)\right) \\ & P^*(t,T_k) = f^*\left(T_k, \left(P^*(t,T_l)\right)_{1\le l\le j}, P^*\left(t,T_{j+\delta}\right)\right) \end{aligned}$$

The bootstrapping algorithm becomes:

Knowing $\left(P(t,T_k)\right)_{1\le k\le j}, \left(P^*(t,T_k)\right)_{1\le k\le j}$

Find $P\left(t,T_{j+\delta}\right), P^*\left(t,T_{j+\delta}\right)$ such that:

$$S_j\sum_{k=1}^{j+\delta}P\left(t,T_k\right)\left(T_k-T_{k-1}\right)=\sum_{k=1}^{j+\delta}P\left(t,T_k\right)F\left(t,T_{k-1},T_k\right)\left(T_k-T_{k-1}\right)$$

$$\sum_{k=1}^{j+\delta}P\left(t,T_k\right)\left(F\left(t,T_{k-1},T_k\right)+sp_i\right)\left(T_k-T_{k-1}\right)=1-P\left(t,T_{j+\delta}\right)$$

$$F\left(t,T_{k-1},T_k\right)=\frac{1}{T_k-T_{k-1}}\left(\frac{P^*\left(t,T_{k-1}\right)}{P^*\left(t,T_k\right)}-1\right)$$

$$\forall k\in\left[j+1,j+\delta-1\right]\quad P\left(t,T_k\right)=f\left(T_k,\left(P\left(t,T_l\right)\right)_{1\le l\le j},P\left(t,T_{j+\delta}\right)\right)$$

$$P^*\left(t,T_k\right)=f^*\left(T_k,\left(P^*\left(t,T_l\right)\right)_{1\le l\le j},P^*\left(t,T_{j+\delta}\right)\right)$$

5. Bootstraping with money market basis swaps

In the general case vanilla swaps and cross currency swaps will share the same Libor tenor. For example JPY plain vanilla swap assume a 6M Libor tenor whereas USD/JPY cross currency basis swap assume 3M for both USD and JPY Libors. However this is not necessarily an issue since there also is a market for single currency JPY 3M/6M basis swaps which enables to reexpress the floating leg of 6M plain vanilla swap w.r.t.
as a 3M Libor floating leg. We now define repectively the 3M and 6M forecast curves $P^{*,3M}\left(t,.\right), P^{*,6M}\left(t,.\right)$ and their respective FRA rates as $F^{3M}\left(t,.\right), F^{6M}\left(t,.\right)$. Note here that we dropped the pay date in the definition of the FRA rates since the superscripted tenor already contains the information on the pay date. Let 's consider a par 3M/6M basis swaps with maturity T_n and trading at a spread $sp_n^{3M,6M}$ and assuming that the 3M leg pays on the schedule $\left(T_1,T_2,...,T_n\right)$

By definition we have:

$$\sum_{k=1}^{n} P(t,T_k)\left(F^{3M}(t,T_{k-1}) + sp_n^{3M,6M}\right)(T_k - T_{k-1}) = \sum_{k=1}^{n/2} P(t,T_{2k}) F^{6M}(t,T_{2k-2})(T_{2k} - T_{2k-2})$$

Changing the tenor of plain vanilla swap will create a set of parse swap maturities for plain vanilla swap maturities. However we saw how to bootstrap the forward curve in this context.

The bootstraping equations becomes for the 3M cross currency swaps and plain vanilla swaps:

$$S_j \sum_{k=1}^{\frac{j+\delta}{2}} P(t,T_k^*)(T_k^* - T_{k-1}^*) = \sum_{k=1}^{j+\delta} P(t,T_k)\left(F^{3M}(t,T_{k-1}) + sp_{j+\delta}^{3M,6M}\right)(T_k - T_{k-1})$$

$$\sum_{k=1}^{j+\delta} P(t,T_k)\left(F^{3M}(t,T_{k-1}) + sp_i\right)(T_k - T_{k-1}) = 1 - P(t,T_{j+\delta})$$

$$F^{3M}(t,T_{k-1}) = \frac{1}{T_k - T_{k-1}}\left(\frac{P^{*,3M}(t,T_{k-1})}{P^{*,3M}(t,T_k)} - 1\right)$$

$$\forall k \in [j+1, j+\delta-1] \quad P(t,T_k) = f\left(T_k, \left(P(t,T_l)\right)_{1\le l\le j}, P(t,T_{j+\delta})\right)$$

$$P^{*,3M}(t,T_k) = f^*\left(T_k, \left(P^{*,3M}(t,T_l)\right)_{1\le l\le j}, P^{*,3M}(t,T_{j+\delta})\right)$$

$$\forall i \in \left[0, \frac{j+\delta}{2}\right] \qquad T_k^* = T_{2k}$$

Once the discount curve and 3M curve have been computed we can then bootstrap the 6M forecast curve using:

$$F^{6M}(t,T_{n-2})(T_n - T_{n-2}) = \frac{1}{P(t,T_n)} \begin{pmatrix} \sum_{k=1}^{n} P(t,T_k)\left(F^{3M}(t,T_{k-1}) + sp_n^{3M,6M}\right)(T_k - T_{k-1}) - \\ \sum_{k=1}^{n/2-1} P(t,T_{2k}) F^{6M}(t,T_{2k-2})(T_{2k} - T_{2k-2}) \end{pmatrix}$$

$$P^{*,6M}(t,T_n) = P^{*,6M}(t,T_{n-2}) \frac{1}{1 + F^{6M}(t,T_{n-2})(T_n - T_{n-2})}$$

6. Solving for the short end of the curve

Throughout this document we assumed that before the first swap maturities we knew values of both discount and forecast curves for dates $(T_1, T_2, ..., T_k)$. In this section we show how these were obtained.

Firstly in curve construction it is customary to use liquid instrument for the short end of the curve like short dated cash deposits overnight (O/N), tom/next (T/N), one week (1W), one month (1M) and 3 months (3M) as well as a strip of futures prices.

Treatment of the cash intruments

From the cash instruments it is easy to derive discount factors for the very short end of the curve using some basic non arbitrage arguments. Let us call $C_{O/N}, C_{T/N}, C_{1W}, C_{1M}, C_{3M}$ the market quoted rates for the above cash deposit instruments and $\tau_{O/N}, \tau_{T/N}, \tau_{1W}, \tau_{1M}, \tau_{3M}$ their respective coverage periods. We must have:

$$P(t,T_{O/N}) = \frac{1}{1+C_{O/N}\tau_{O/N}}$$

$$P(t,T_{T/N}) = P(t,T_{O/N})\frac{1}{1+C_{T/N}\tau_{T/N}}$$

$$P(t,T_{1W}) = P(t,T_{T/N})\frac{1}{1+C_{1W}\tau_{1W}}$$

$$P(t,T_{1M}) = P(t,T_{T/N})\frac{1}{1+C_{1M}\tau_{1M}}$$

$$P(t,T_{3M}) = P(t,T_{T/N})\frac{1}{1+C_{3M}\tau_{3M}}$$

Furthermore each forecast curve should be able to reprice the cash intrument with the underlying tenor. So using the previously defined notation we should have:

$$P^{*,3M}(t,T_{3M}) = \frac{1}{1+C_{3M}\tau_{3M}}$$

We should add that sometimes cash deposits are quoted for longer tenors but they are not liquid enough to be used to derive valid discount factors. They can just be used as a realized value for an index. This is the case of the JPY 6M cash deposit. However the 6M forward curve should reprice the 6M deposit. This is garanteed by imposing:

$$P^{*,6M}(t,T_{6M}) = \frac{1}{1+C_{6M}\tau_{6M}}$$

Other liquid instruments are interest rate futures. In JPY they are based on the 3M Libor. (In reality they are based on 3M Tibor but we neglect that issue for the time being) Futures rates give us some information about the forward rates underlying the same period. Let us consider the Libor rate covering the period $\left[T, T+3M\right]$, denoted by $L\left(T, T+3M\right)$. As shown in the appendix we can define respectively the futures rate $\hat{F}^{3M}\left(t,T\right)$ and FRA rate $F^{3M}\left(t,T\right)$ for the period $\left[T, T+3M\right]$ as:

$$\begin{aligned} \hat{F}^{3M}\left(t,T\right) &= E_t\left[L\left(T,T+3M\right)\right] \\ F^{3M}\left(t,T\right) &= E_t^{T+3M}\left[L\left(T,T+3M\right)\right] \end{aligned}$$

Where $E_t\left[\ \right], E_t^{T+3M}\left[\ \right]$ respectively denote the risk-neutral and *T+3M*-forward measure expectation operators conditional on time *t*. The difference $\hat{F}^{3M}\left(t,T\right) - F^{3M}\left(t,T\right) = adj$ is called the Futures FRA-Convexity adjustment. This quantity is model dependent and a model for the yield curve is necessary to compute it. We can guess the sign of this adjustment just by writing:

$$\begin{aligned} adj &= E_t\left[L\left(T,T+3M\right)\right] - E_t^{T+3M}\left[L\left(T,T+3M\right)\right] = E_t\left[L\left(T,T+3M\right)\right] - \frac{E_t\left[D\left(T\right)L\left(T,T+3M\right)\right]}{P\left(t,T\right)} \\ &= \frac{1}{P\left(t,T+3M\right)}\left(P\left(t,T+3M\right)E_t\left[L\left(T,T+3M\right)\right] - E_t\left[D\left(T+3M\right)L\left(T,T+3M\right)\right]\right) \\ &= \frac{1}{P\left(t,T+3M\right)}\left(E_t\left[D\left(T+3M\right)\right]E_t\left[L\left(T,T+3M\right)\right] - E_t\left[D\left(T\right)L\left(T,T+3M\right)\right]\right) \\ &= -\frac{1}{P\left(t,T\right)}\operatorname{cov}\left(D\left(T\right), L\left(T,T+3M\right)\right) \end{aligned}$$

Where $D\left(T\right) = \exp\left(-\int_t^T r\left(s\right)ds\right)$ is the stochastic discount factor. It is negatively correlated with rates and thereofer we should find that $adj > 0$.

Once this adustment has been computed we can move on to bootsrapping the 3M forecast curve at the short end. Let us assume that the last known point on the forecast curve is is at date T_k and the next futures maturing at time T is such that $T+3M > T_k$. Also according to our formulation of the forecast curve we can write:

$$P^{*,3M}\left(t,T+3M\right)=P^{*,3M}\left(t,T\right)\frac{1}{1+F^{3M}\left(t,T\right)\tau_{3M}}$$

There are two bootstrapping cases:

<u>First case</u>: $T_k \le T$

Boostrapping consist in finding $P^{*,3M}\left(t,T+3M\right)$ such that:

$$P^{*,3M}\left(t,T\right)=f\left(\left(P^{*,3M}\left(t,T_i\right)\right)_{1\le i\le k},P^{*,3M}\left(t,T+3M\right)\right)$$
$$P^{*,3M}\left(t,T+3M\right)=P^{*,3M}\left(t,T\right)\frac{1}{1+F^{3M}\left(t,T\right)\tau_{3M}}$$

<u>Second case:</u> $T \le T_k$

Bootstrapping consists in interpolating $P^{*,3M}\left(t,T\right)$ knowing all the forecast curve for points $\left(T_i\right)_{1\le i\le k}$. Then $P^{*,3M}\left(t,T+3M\right)$ is determined by:

$$P^{*,3M}\left(t,T+3M\right)=P^{*,3M}\left(t,T\right)\frac{1}{1+F\left(t,T\right)\tau_{3M}}$$
$$P^{*,3M}\left(t,T\right)=f^{*,3M}\left(\left(P\left(t,T_i\right)\right)_{1\le i\le k}\right)$$

7. Appendix 1: Futures-Fra convexity adjustment

First we recall that the Futures rate is a martingale under the Risk-Neutral measure. Using the notation introduced in paragraph 6 we can write:

$$\hat{F}^{3M}(t,T) = E\left[L(T,T+3M)\right]$$

A quick proof can be derived as such. Let us consider a general futures like a commodity futures with delivery date T which does not have all the fiddly quoting conventions associated with IR futures. Let us call $F(t)$ the price of this futures at time t. Let us also denote the stochastic discount factor by $D(t) = \exp\left(-\int_0^t r(s)\,ds\right)$ where $r(s)$ denotes the short rate prevailing at time *s*.

We know that entering into a futures contract at any time t prior to the delivery date is costless. That translates into saying that $\forall t \in [0,T] \quad E_t\left[\int_t^T D(s)\,dF(s)\right] = 0$

In this case we can deduce that $\forall t \in [0,T]\,\forall s \in [t,T] \quad E_t\left[D(s)\,dF(s)\right] = 0$. In particular for $s = t$ we have $E_t\left[D(t)\,dF(t)\right] = D(t)E_t\left[dF(t)\right] = 0$. We deduce that for all t $E_t\left[dF(t)\right] = 0$ which means the futures price process has no drift under the risk-neutral measure. Therefore it is a martingale and $F(0) = E\left[F(T)\right]$. However we recall that $F(T)$ is the value of the quantity to be delivered at time T which in our case is the Libor rate fixing at time *T*, $L(T,T+3M)$. Below we derive the calculation of the Futures-Fra convexity adjustment in the time dependent Hull-White model.

We postulate the following dynamics for the short rate model:

$$dr(t) = \kappa\big(\theta(t) - r(t)\big)dt + \sigma(t)\,dW(t)$$

Where W is the brownian motion under the risk-neutral measure.

Under this model it can be shown that the dynamics of discount bonds can be expressed as:

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt - \Sigma(t,T)\,dW(t)$$
$$\Sigma(t,T) = \sigma(t)\int_t^T \exp\big(-\kappa(s-t)\big)ds$$

We also assume that forecast curves are driven by the same brownian motion and that the forecast discount curves moves as:

$$\frac{dP^{*,3M}(t,T)}{P^{*,3M}(t,T)} = r^*(t)\,dt + \Sigma(t,T)\,dW(t)$$

This means that our model assumes forward basis swap spread are constant through time. This is a rather gross approximation but any more complex assumption would complicate the curve building hugely.

The actual knowledge of the forecast short rate $r^*(t)$ is not needed for the purpose of the Futures-Fra convexity adjustment. Using the definition of the futures rate we can write:

$$\hat{F}^{3M}(t) = E\left[L(T, T+3M)\right]$$

We know that $E_t^{T+3M}\left[L(T, T+3M)\right] = F^{3M}(t, T)$ is a martingale under the $T+3M$ forward measure. Using the assumed dynamics for the forecast curve we get

$$F^{3M}(t, T) = \frac{1}{\delta}\left(\frac{P^{*,3M}(0, T)}{P^{*,3M}(0, T+\delta)}\exp\left(\int_0^t \left(\Sigma(s, T+\delta) - \Sigma(s, T)\right) dW^{T+3M}(s) - \frac{1}{2}\int_0^t \left(\Sigma(s, T+\delta) - \Sigma(s, T)\right)^2 ds\right) - 1\right)$$
$$\delta = 3M$$

where W^{T+3M} is a browinan motion under the $T+3M$ measure. Therfore we can also write:

$$L(T, T+\delta) = \frac{1}{\delta}\left(\frac{P^{*}(0, T)}{P^{*}(0, T+\delta)}\exp\left(\int_0^T \left(\Sigma(s, T+\delta) - \Sigma(s, T)\right) dW^{T+3M}(s) - \frac{1}{2}\int_0^T \left(\Sigma(s, T+\delta) - \Sigma(s, T)\right)^2 ds\right) - 1\right)$$
$$\delta = 3M$$

We now want to derive an expression for $L(T, T+\delta)$ that is convenient to work with under the risk-neutral measure. We introduce the Radon-Nykodim derivative relating the risk-neutral measure and the *T+3M* forward measure defined by:

$$E_t\left[\frac{dQ^{T+\delta}}{dQ}\right] = D(t)P(t, T+3M) = P(0, T+\delta)\exp\left(-\int_0^t \Sigma(t, T+\delta)\, dW(t) - \frac{1}{2}\int_0^t \Sigma(t, T+\delta)^2\, dt\right)$$

Using the the Girsanov theorem we deduce the new expression for the spot Libor rate:

$$
\begin{aligned}
L(T,T+\delta) &= \frac{1}{\delta}\left(\frac{P^*(0,T)}{P^*(0,T+\delta)}\exp\left(\int_0^T \left(\Sigma(s,T+\delta)-\Sigma(s,T)\right)\left(dW(s)+\Sigma(s,T+\delta)ds\right)-\frac{1}{2}\int_0^T \left(\Sigma(s,T+\delta)-\Sigma(s,T)\right)^2 ds\right)-1\right)\\
&= \frac{1}{\delta}\left(\frac{P^*(0,T)}{P^*(0,T+\delta)}\exp\left(\int_0^T \left(\Sigma(s,T+\delta)-\Sigma(s,T)\right)dW(s)+\frac{1}{2}\int_0^T \left(\Sigma(s,T+\delta)^2-\Sigma^2(s,T)\right)ds\right)-1\right)
\end{aligned}
$$

Where W is now a brownian motion under the risk-neutral measure.

We can now deduce the value of the Futures rate;

$$
\begin{aligned}
\hat{F}^{3M}(t,T) &= E\left[L(T,T+\delta)\right]\\
&= E\left[\frac{1}{\delta}\left(\frac{P^*(0,T)}{P^*(0,T+\delta)}\exp\left(\int_0^T \left(\Sigma(s,T+\delta)-\Sigma(s,T)\right)dW(s)+\frac{1}{2}\int_0^T \left(\Sigma(s,T+\delta)^2-\Sigma(s,T)^2\right)ds\right)-1\right)\right]\\
&= \frac{1}{\delta}\left(\frac{P^*(0,T)}{P^*(0,T+\delta)}\exp\left(\int_0^T \left(\Sigma(s,T+\delta)-\Sigma(s,T)\right)\Sigma(s,T+\delta)ds\right)-1\right)
\end{aligned}
$$

By using today's forward rate we get:

$$
\hat{F}^{3M}(t,T) = \frac{1}{\delta}\left(\left(1+\delta F^{3M}(0,T)\right)\exp\left(\int_0^T \left(\Sigma(s,T+\delta)-\Sigma(s,T)\right)\Sigma(s,T+\delta)ds\right)-1\right)
$$

We see immediately that we get what we predicted i.e.:

$$
F^{3M}(t,T) > F^{*,3M}(0,T)
$$

However so far we have not stated how we obtained the volatilities curve $\left(\sigma(t)\right)$. First of all we assume that this curve is piece-wise flat. Then we assume that the market provides us with a set of caplet volatilities λ_i^{caplet} for expiries T_i and strikes K_i. A caplet can be priced analytically in the Hull and White model since using the following reasoning:

$$
\begin{aligned}
C^i &= E\left[D(T_{i+1})\left(L(T_i,T_i+3M)-K_i\right)^+\delta_i\right] \\
&= P(0,T_{i+1})E^{T_{i+1}}\left[\left(L(T_i,T_i+3M)-K_i\right)^+\delta_i\right] \\
&= P(0,T_{i+1})E^{T_{i+1}}\left[\left(\frac{1}{P^{*,3M}(T_i,T_{i+1})}-(1+K_i\delta)\right)^+\right] \\
x^+ &= \max(x,0) \\
\delta_i &= T_{i+1}-T_i
\end{aligned}
$$

Under the Hull and White model the inverse discount factor $\frac{1}{P^{*,3M}(T_i,T_{i+1})}$ is lognormally distributed with forward $\frac{P^{*,3M}(0,T_i)}{P^{*,3M}(0,T_{i+1})}$ and volatility $\frac{1}{T_i}\sqrt{\int_0^{T_i}\left(\Sigma(t,T_i+\delta)-\Sigma(t,T_i)\right)^2 ds}$. Therefore $E^{T_{i+1}}\left[\left(\frac{1}{P^{*,3M}(T_i,T_{i+1})}-(1+K_i\delta)\right)^+\right]$ can be obtained using the undiscounted Black-Scholes formulae with forward $\frac{P^{*,3M}(0,T_i)}{P^{*,3M}(0,T_{i+1})}$, volatility $\frac{1}{T_i}\sqrt{\int_0^{T_i}\left(\Sigma(t,T_i+\delta)-\Sigma(t,T_i)\right)^2 ds}$ and shifted strike $1+\delta K_i$.

Given the curves $\left(P(t,T)\right),\left(P^{*,3M}(t,T)\right)$ and a curve of caplet volatilities $\left(T_i,\lambda_i^{caplet}\right)$ we can solve for the piece-wise flat Hull-White volatility curve $\left(T_i,\sigma_i\right)$. So that we match market implied caplet volatilities.

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