CHAPTER II:

FIXED INCOME SECURITIES AND MARKETS

FIXED INCOME PORTFOLIO MANAGEMENT

A: TYPES OF FIXED INCOME SECURITIES

In terms of dollar volume, the U.S. markets for debt instruments are larger than for any other type of security. Debt markets, including markets for mortgages, are more than twice as large as stock markets. Debt securities are IOUs issued by a variety of types of organizations, including federal, state and local governments and their agencies as well as by corporations and other institutions. Debt securities are sold for the purpose of raising money. A debt security represents a claim on the issuer for a fixed series of future payments. For example, a debt security might specify for its owner to receive a stated series of interest payments until the instrument matures. The instrument may also provide for principal repayment when the instrument matures. These securities are issued in *primary markets* and then traded in *secondary markets*, just as are other financial instruments. Debt securities will normally specify terms of payment, including amounts and dates, collateral and priority (who is to paid first in the event of issuer financial distress). Among the various types of debt securities are bonds, notes, mortgages and treasury instruments. Many of the fixed income securities with shorter terms to maturity are considered to be money market instruments.

Treasury Securities and Markets

The United States Treasury is the largest issuer of debt securities in the world. The federal government raises (borrows) money through the sale of U.S. Treasury issues including Treasury Bills (T-Bills), Treasury Notes and Treasury Bonds. By purchasing Treasury issues, an investor is loaning the government money. The United States government has proven to be an extremely reliable debtor (at least it makes good on all of its Treasury obligations). Treasury issues are fully backed by the full faith and credit of the U.S. government which has substantial resources due to its ability to tax citizens and create money. Thus, these securities have lower default risk than the safest of corporate bonds or short term notes.

The treasury obligations with the shortest terms to maturity are *Treasury Bills*. They typically mature in less than one year (13, 26 or 52 weeks). These issues are sold as pure discount debt securities, meaning that their purchasers receive no explicit interest payments. Such pure discount instruments are also known as zero coupon issues.

In primary markets, T-Bills are sold to the public through an auction process managed by district Federal Reserve banks. Auctions of three and six month bills are typically announced on Tuesdays, conducted the following Monday and settled on the following Thursday. There are two ways to purchase T-Bills. The first is to enter a competitive bid at the auction where the bidding institution competes for a given dollar amount of the new issue based on how much it is willing to pay. Secondly, non-competitive bids can be tendered where the prospective purchaser states how many bills he would like to purchase at the average price of accepted competitive bids. Bidders for T-Bills generally enter their bids just before the deadline to participate in the auction. Non-competitive bids are satisfied at the average price of successful competitive bids. The treasury determines the dollar amount of competitive bids that it wishes to satisfy by subtracting the face values of the non-competitive bids from the level of bills that the Treasury wishes to sell. Successful competitive bids are selected by ranking them, starting with the highest bid. Successful bidders competitors obtain their bills at the prices that they bid; the lowest bid is referred to as the stop-out price. Highly liquid secondary markets exist for Treasury Bills. An investor can easily purchase and sell Treasury Bills through a broker in the Over The Counter Markets.

One variation of a T-Bill issue is a so-called *Strip Issue*. Strips are portfolios of T-Bills sold by the Treasury in blocks with varying maturities. For example, a block of five strips maturing at the end of a given year in a five year period may provide for a payment of $1,000 at the end of a given period. The individual strips can be "stripped" from the block and sold in secondary markets.

In addition to the short-term pure discount instrument issues discussed above, the Treasury also offers a number of longer term coupon issues. For example, *Treasury Notes* (T-Notes) have maturities ranging from one to ten years and make semi-annual interest payments. Similarly, *Treasury Bonds* (T-Bonds) typically range in maturity from ten to thirty years and make semi-annual interest payments. These T-Bonds are frequently callable, meaning that the Treasury maintains an option to repurchase them from investors at a stated price.

The United States Treasury also offers non-marketable issues such as *Series EE U.S. Savings Bonds* and *Series H U.S. Savings Bonds*. These savings bonds are normally issued only to individuals and cannot be traded among investors. Such issues are often subjected to certain restrictions (such as a $15,000 maximum level of purchases per year). These bonds can be purchased through most banks and savings institutions, and many businesses maintain plans through which their employees can purchase savings bonds through payroll deduction programs.

Agency Issues

The United States federal government has created and sponsored a number of institutions known as *agencies*. These agencies enable the government to make funds available for a number of functions. Among the oldest of these agencies is *The Federal National Mortgage Association* (FNMA or Fannie Mae) which is currently a federally sponsored private corporation with shareholders and other security holders. FNMA creates mortgage backed securities by purchasing residential mortgages from banks and thrift institutions. In effect, FNMA purchases the mortgage obligations held by banks and thrifts, repackages them as debt security portfolios, insures them and re-sells them to the general public. These portfolios of mortgage backed securities are also pass- through securities. FNMA can obtain money directly from the U.S. Treasury should it need to do so. The *Government National Mortgage Association* (GNMA or Ginnie Mae) and *Federal Home Loan Mortgage Corporation* (FHLMC or Freddie Mac) also create, insure and sell pass-through securities related to residential mortgages. The *Student Loan Marketing Association* (SLMA or Sallie Mae creates, insures and sells pass-through securities related to student loans.

Municipal Securities and Markets

The municipal securities markets owe much of their success to U.S. federal taxation code which permits investors in municipal instruments to omit from their taxable income any interest payments received on these issues. Thus, interest received on municipal bonds need not be declared as part of income subject to federal income taxation. This feature makes municipal bonds more attractive to investors, enabling issuers to offer these bonds at a reduced interest rate.

Several types of municipal issues are offered by state and local governments. The first, so-called *General Obligation Bonds* are full faith and credit bonds. This means that the issuer backs the bonds to the fullest extent possible, given its assets and other obligations. *Limited Obligation Bonds* provide for the issue to be backed only by specific resources or assets. For example, a revenue bonds may be backed only be the cash flows generated by a specific asset such as a toll bridge.

Some municipal issues are insured by private insurance institutions. This insurance is intended to reduce the default risk associated with the issue, and make them more attractive to investors. This reduced risk would enable issuers to offer bonds with reduced interest rates to the public. Among the larger insurers of municipal bonds are The American Municipal Bond Insurance Association and the Municipal Bond Insurance Association.

Financial Institution Issues

Non-government financial institutions are also important participants in primary markets for debt instruments. For example, the *Federal Funds* markets allow banks and other depository institutions to lend to one another to meet federal reserve requirements. Essentially, this market provides that excess reserves of one bank may be loaned to other banks for satisfaction of reserve requirements. The rate at which these loans occur is referred to as the Federal Funds Rate.

Normally, bank accounts are not regarded as marketable securities. One exception to this are *Negotiable Certificates of Deposit* (also known as Jumbo C.D.'s). These are depository institution certificate of deposit accounts with denominations exceeding $100,000. The amounts by which these jumbo C.D.s exceed $100,000 are not subject to FDIC insurance. *Money Market Mutual Funds* are created by banks and investment institutions for the purpose of pooling together depositor or investor funds for the purchase of money market instruments (short term, highly liquid low risk debt securities).

Banker's Acceptances are originated when a bank accepts responsibility for paying a client's loan. Because the bank is likely to be regarded as a good credit risk, these acceptances are usually easily marketable as securities. *Repurchase Agreements* (Repos) are issued by financial institutions (usually securities firms) acknowledging the sale of assets and a subsequent agreement to re-purchase at a higher price in the near term. This agreement is essentially the same as a collateralized short term loan. The counterparty institution buying the securities with the agreement to resell them is said to be taking a *reverse repo*.

Corporate Bonds and Markets

Corporations are also important issuers of debt securities. Large, well-known, credit-worthy firms needing to borrow for a short period of time may issue large denomination short notes frequently referred to as *Commercial Paper*. Well-developed markets exist for these short-term promissory notes. Firms requiring funds for longer periods of time may issue *corporate bonds*. These longer term instruments are often issued with a variety of features, including callability, convertibility, sinking fund provisions, etc. There are a large number of different types of corporate bonds. The terms of the bond will be specified in a contract known as a bond indenture. In addition, firms may make bank commercial loans, though secondary markets for bank loans tend to be limited in size and scope.

Callable bonds may be *called* by the issuing institution at its option. This means that the issuing institution has the right to pay off the callable bond before its maturity date. The callable bond typically has a *call date* associated with it as well as a *call price*. The call date is the first date (and perhaps only date) that the bond can be repurchased by the issuing institution. The call price is normally set higher than the bond's par value and represents the price that the issuing institution agrees to pay the bond owners. Because the issuing institution retains the option to force early retirement of callable debt, the call provision can be expected to reduce the market value of the callable bond relative to otherwise comparable non-callable bonds.

Convertible bonds can be convertible by bond holders into equity or other securities. This normally means that the convertible bondholder has the right to exchange the convertible bond for a specified number of shares of common stock or some other security. The convertibility provision of such a bond enhances its value relative to otherwise comparable non-convertible bonds.

Debentures are not backed by collateral. Many other bonds are either backed by collateral or have some other device such as *Sinking fund* provisions to provide for additional safety for bond holders. One type of sinking fund provisions provides for the issuing institution to place specified sums of money into a fund at specified dates that will be accumulated over time to ensure full satisfaction of the firm's obligation to bondholders. In some instances, sinking funds will be used to retire associated debt early. *Serial bonds* are issued in series with staggered maturity dates.

Many more innovative bonds have been offered in the market. *Floating rate* bonds have coupon rates that rise and fall with market interest rates; reverse floaters have coupon rates that move in the opposite direction of market interest rates. *Indexed bonds* have coupon rates that are tied to the price level of a particular commodity like oil or some other value like the inflation rate. *Catastrophe bonds* make payments that depend on whether some disaster occurs, like an earthquake in California or a hurricane in Florida. These catastrophe bonds provide a sort of insurance for issuers against the occurrence of the disaster. In some respects, purchasers of these bonds are providing insurance to the issuers.

Most corporate bonds are rated by well-known agencies with respect to anticipated default risk. Corporations pay institutions like Standard & Poor's and Moody's to rate the riskiness of their issues. Other rating agencies include Fitch, A.M. Best, Duff and Phelps and Dun and Bradstreet. Bonds without ratings assigned by these agencies are very difficult to sell; in fact, many institutions face restrictions on purchasing bonds that are either unrated or have ratings below a given level. Standard & Poor's and Moody's use the rating schemes depicted in Table 1.

Description	**Standard & Poor's**	**Moody's**
Least likely to default	AAA	Aaa
High quality	AA	Aa
Medium grade investment quality	A	A
Low grade investment quality	BBB	Baa
High grade speculative quality	BB	Ba
Speculative	B	B
Lower grade speculative	CCC	Caa
Highly speculative	CC	Ca
Likely bankruptcy	C	C
Already in default	D	D

Table 1: Standard & Poor's and Moody's Corporate Bond Ratings

Bonds rated BBB (or Baa) and higher are typically referred to as investment grade bonds while bonds below this level are considered to be of speculative grade. Speculative grade bonds are often called junk bonds. There is significant evidence that these bond ratings are highly correlated with incidence of default, suggesting that these agencies are least reasonably in forecasting default and measuring default risk. Furthermore, it is fairly unusual for ratings provided by these agencies to differ by more than one grade. Bond markets seem to agree with these statistical findings, pricing bonds such that their yields are strongly inversely correlated with bond ratings.

Bond rating agencies make extensive use of financial statement and ratio analysis to compute their ratings. Such analyses are frequently supplemented by statistical techniques such as *Multi-discriminate Analysis*, *Probit* and *Logit* modeling.

Eurocurrency Instruments and Markets

Eurodollars are freely convertible dollar-denominated time deposits outside the United States. The banks may be non-U.S. banks, overseas branches of U.S. banks or International Banking Facilities (not subject to reserve requirements). Eurodollar markets began after World War II when practically all currencies other than the U.S. dollar were perceived as unstable. Thus, most foreign trade between countries was denominated in U.S. dollars. However, the Soviet Union and Eastern Europeans were concerned that their dollars held in U.S. banks might be attached by U.S. residents in litigation with these countries. Thus, they dealt not with actual U.S. dollars, but merely denominated their debits and credits with dollars. Monies owed to them were simply offset by monies that they owed. In a sense, they dealt with "fake" euro-dollars, but since their trading partners did also, and their accounts tended to "zero out" over time, this did not create significant problems. Their euro-dollars were left in Western European banks. During the 1960s and 1970s, these markets thrived due to regulations imposed by the U.S. government such as Regulation Q (interest ceilings), Regulation M (reserve requirements), the Interest Equalization Tax imposed beginning in 1963 to tax interest payments on foreign debt sold in the U.S. and restrictions placed on the use of domestic dollars outside the U.S. More generally, eurocurrencies are loans or deposits denominated in currencies other than that of the country where the loan or deposit is created. Approximately 65% of eurocurrency loans are denominated in dollars.

Eurocredits (e.g: Eurodollar Credits) are bank loans denominated in currencies other than that of the country where the loan is extended. They are attractive due to very low interest rate spreads which are possible due to the large size of the loans and the lack of reserve, FDIC and other requirements directly or indirectly with domestic loans and deposits. Their rates are generally tied to LIBOR (the London Interbank Offered Rate) and U.S. rates. Loan terms are usually less than five years, typically for six months. Euro-Commercial paper are short-term (usually less than six months) notes issued by large, particularly "credit-worthy" institutions. Most commercial paper is not underwritten. The notes are generally very liquid and most are denominated in dollars. Euro-Medium Term Notes (EMTN's), unlike Eurobonds, are usually issued in installments. Again, most are not underwritten. Eurobonds are generally underwritten, bearer bonds denominated in currencies other than that of the country where the loan is extended. Eurobonds often have call and sinking fund provisions as well as other features found in bonds publicly traded in American markets.

Euro-Commercial paper is the term given to short-term (usually less than six months) notes issued by large, particularly "credit-worthy" institutions. Most commercial paper is not underwritten. The notes are generally very liquid (much more so than Syndicated loans) and most are denominated in dollars. They are usually pure discount instruments. Euro-Medium Term Notes (EMTN's) interest-bearing instruments usually issued in installments. Most are not underwritten. Eurobonds are generally underwritten, bearer bonds denominated in currencies other than that of the country where the loan is extended. Eurobonds typically make annual coupon payments and often have call and sinking fund provisions.

B. Bond Yields, Rates and Sources of Risk

Assume that we wish to analyze a bond maturing in n periods with a **face value** (or principle amount) equal to F paying interest annually at a rate of c. The annual interest payment is rate c multiplied by face value F (or cF) and the bond makes a single payment in time n equal to F. Using a standard **present value** model discounting cash flows at a rate of k, the bond is evaluated as follows:

$$PV = \sum_{t=1}^{n} \frac{cF_t}{(1+k)^t} + \frac{F}{(1+k)^n}$$

(1)

For example, let c equal .10, F equal $1000, k equal .12 and n equal 2. The present value of this bond is computed as follows:

$$PV = \frac{100}{(1+.12)^1} + \frac{100}{(1+.12)^2} + \frac{1000}{(1+.12)^2} = 966.20$$

(2)

Present Value is used to determine the economic worth of a bond; the return of a bond measures the profit relative to the investment of a bond. There are several measures of bond return including current yield and yield to maturity. The more simple measurement, **current yield**, is concerned with annual interest payments relative to the initial investment required by the bond and is measured as follows:

(3)

$$\frac{cF}{P_0}$$

If the bond used in the example above may be purchased for $986.48, its current yield is simply 10.14%.

Financial pages in newspapers frequently quote yields for treasury bills and other pure discount instruments using the **bank discount method**:

$$\frac{F - P_0}{F} \times \frac{360}{n}$$

where n is the number of days before the bond matures. This measure of the bond's economic efficiency is rather odd for several reasons. First, it is based on simple rather than compound interest. Second, it assumes a 360 day yield. Finally, it measures return as a proportion of face value rather than the sum invested.

The bond equivalent yield represents a slight improvement over the bank discount formula:

$$\frac{F - P_0}{P_0} \times \frac{365}{n}$$

One encounters two problems using current yield as a return measure. First, the current yield does not account for any capital gain or loss ($F - P_0$) that may accrue when the bond matures. Second, current yield does not account for the time value of money or compounding of the cash flows associated with the bond. Hence, one may wish to compute the bond's internal rate of return, which is generally referred to as **yield to maturity** (y):

(4) $$P_0 = \sum_{t=1}^{n} \frac{cF_t}{(1+y)^t} + \frac{F}{(1+y)^n}$$

Yield to maturity is that value for y which satisfies Equality (4). Usually, a solution must be obtained through an iterative process. The yield to maturity (or internal rate of return for the bond described above has a yield to maturity of 12%, computed as follows:

(5) $$P_0 = 966.20 = \frac{100}{(1+.12)^1} + \frac{100}{(1+.12)^2} + \frac{1000}{(1+.12)^2}$$

Thus, yield to maturity may be interpreted as that discount which sets the purchase price of a bond equal to its present value.

The **yield to call** for a callable bond differs from yield to maturity in two respects:

1. Cash flows are assumed to cease at the call date rather than the maturity date and
2. The call price is used as the bond's final cash flow rather than the face value of the bond

Bond risk may be categorized as follows:

1. **Default or credit risk**: the bond issuer may not fulfill all of its obligations
2. **Liquidity risk**: there may not exist an efficient market for investors to resell their bonds
3. **Interest rate risk**: market interest rate fluctuations affect values of existing bonds.

United States Treasury Issues are generally regarded as being practically free of default risk. Furthermore, there exists an active market for treasury issues, particularly those maturing within a short period. Thus, treasury issues are regarded as having minimal liquidity risk. However, all bonds are subject to interest rate risk. Longer term bonds are subject to increased interest rate risk due to the increased periods that the yields on longer term bonds are likely to differ from newly issued bonds.

C. The Term Structure of Interest Rates

The Term Structure of Interest Rates is concerned with the change in interest rates on debt securities resulting from varying times to maturity on the debt. For example, it may be concerned with explaining why the interest rate on debt maturing in one year is 4% versus 7% for debt maturing in twenty years. Generally at a given point in time, we observe longer term interest rates exceeding shorter term rates, though this is not always the case (for example, the years 1980-1983). Following are three theories which, either separately or in combination, attempt to explain the relation between long and short term interest rates.

The Pure Expectations Theory states that long term spot rates (interest rates on loans originating now) can be explained as a product of short term spot rates and short term forward rates (interest rates on loans committed to now but actually originating at later dates). Where $y_{t,m}$ is the rate on a loan originated at time t to be repaid at time m, the Pure Expectations Theory defines the relationship between long and short term interest rates as follows:

(1) $$(1+y_{0,n})^n = \prod_{t=1}^{n} (1+y_{t-1,t})$$

Thus, the long term spot rate $y_{0,t}$ is defined as n'th root of the product of the one period spot rate $y_{0,1}$ and a series of one period forward rates $y^t_{-1,t}$ minus one. In other words, the long term spot rate $y_{0,n}$ can be determined based on the short-term spot rate $y_{0,1}$ and a series of one period forward rates $y_{t-1,t}$ as follows:

(2) $$y_{0,n} = \sqrt[n]{\prod_{t=1}^{n} (1+y_{t-1,t})} - 1$$

Consider an example where the one year spot rate $y_{0,1}$ is 5%. Investors are expecting that the one year spot rate one year from now will increase to 6%; thus, the one year forward rate $y_{1,2}$ on a loan originated in one year is 6%. Furthermore, assume that investors are expecting that the one year spot rate two years from now will increase to 7%; thus, the one year forward rate $y_{2,3}$ on a loan originated in two years is 7%. Based on the pure expectations hypothesis, what is the three year spot rate? This is determined with Equation (2) as follows:

$$y_{0,n} = \sqrt[n]{\prod_{t=1}^{n} (1+y_{t-1,t})} - 1 = \sqrt[3]{(1+y_{0,1})(1+y_{1,2})(1+y_{2,3})} - 1$$
$$= \sqrt[3]{(1+.05)(1+.06)(1+.07)} - 1 = .05997$$

One problem with the Pure Expectations Theory is that it does not explain why interest rates on longer term bonds tend to exceed short term rates much more than fifty percent of the time. If one were to use the Pure Expectations Theory to explain this phenomenon, one would conclude that investors consistently expect that interest rates will rise. Since this is probably not true, it may be useful to propose an alternative theory to explain the term structure of interest rates. *The Liquidity Premium Theory* is based on the Pure Expectations Theory where it is assumed that investors required increased rate premiums to invest outside of their preferred investment horizons. Since shorter term debt markets are more liquid, investors frequently require increased rate compensation for bearing liquidity risk as loan terms to maturity increase. Where LP_t increases as t increases, the Liquidity Premium Theory defines interest rates as follows:

(3) $$(1+y_{0,n})^n = \prod_{t=1}^{n} (1+y_{t-1,t})+LP_t$$

Equation (3), which is based on Equation (1) simply states that investors may require additional interest for certain maturities as compensation for illiquidity that may exist in the market.

An alternative explanation (and perhaps complementary explanation) of term structure, *The Market Segmentation Theory* states that various types of borrowing and lending institutions have strong debt maturity preferences. Such institutions will bid up or down interest rates for the various maturity dates based on their preferences. This theory has attracted more attention in the professional than in the academic literature. This theory is very much to the Preferred Habitat Theory which describes maturity matching of assets and liabilities as the basis for much segmentation.

The yield curve can be obtained empirically by examining the payoffs associated with a bond simultaneously with the bond's purchase price. Let D_t be the discount function for time t; that is, $D_t = 1/(1+y_{0,t})^t$. This means that a cash flow paid at time t will be discounted by multiplying it by the discount function D_t:

$$PV = CF_t \cdot D_t = CF_t /(1+y_{0,t})^t$$

A little algebra produces the following spot rate:

$$y_{0,t} = (1/D_t)^{1/t} - 1$$

Thus, one can obtain the spot rates $y_{0,t}$ from the bond's current purchase price P_0 and expected future cash flows from coupon payments and face value CF_t . Thus, consider a \$1000 face value bond making a single interest payment at an annual rate of 5%. Suppose this bond is currently selling for 102 (meaning 102% or 1020) and that it matures in one year when its coupon payment is made. The one year spot rate implied by this bond is determined as follows:

$$1020 = (50 + 1000) \cdot D_1 = (1050)/(1+y_{0,1})^1$$
$$D_1 = 1020/1050 = (.9714286)^{1/1} \;\; ; 1/.9714286 - 1 = y_{0,1} = .0294$$

Thus, the one year spot rate is 2.95%. However, a difficulty arises when the bond has more than one cash flow. As spot rates may vary over time, there may be a spot rate for each period, hence, a spot rate for each cash flow. Consider a \$1000 face value two year bond making interest payments at an annual rate of 5%. Suppose this bond is currently selling for 101.75 (meaning 101.75% or 1017.5) and that it matures in two years when its second coupon payment is made. The two spot rate implied by this bond is bootstrapped from the one year spot rate as follows:

$$1017.5 = 50 \cdot .9714286 + (50 + 1000) \cdot D_2$$
$$D_2 = [1017.5 - (50 \cdot .9714286)]/[50 + 1000] = .9227891$$
$$(1/.9227891)^{1/2} - 1 = y_{0,2} = .0410$$

More generally, this bootstrapping process is applied as follows:

$$PV = cF \cdot D_1 + cF \cdot D_2 + cF \cdot D_3 + \ldots + cF \cdot D_{n-1} + (cF + F) \cdot D_n = \Sigma cF/(1+y_{0,t})^t + (cF + F) /(1+y_{0,t})^n$$
$$PV = \Sigma CF_t \cdot D_t + (cF + F) /(1+y_{0,t})^n = \Sigma CF_t /(1+y_{0,t})^t + (cF + F) /(1+y_{0,t})^n$$
$$D_n = [P_0 - (\Sigma cF \cdot D_t)]/[cF + F]$$

Bootstrapping requires that there be one bond maturing in each year t so that its D_t can be used to determine (bootstrap) the D_t for the bond maturing in one year subsequent. Thus, one starts by determining D_1 , D_2 and so on until all D_t values have been determined. These expressions are used to bootstrap spot rates from bond prices, maturities and coupon rates in Table 2 and in Figure 1 mapping out the yield curve. Any i-year forward rate, $y_{t-i,t}$, from year t-i to year t is determined from $(D_t / D_t)^{1/i} - 1$.

Maturity	%Coupon	Ask Price	D_t	Spot Rate
1	5.00	102	0.9714286	2.94%
2	5.00	101 3/4	0.9227891	4.10%
3	5.00	101 1/2	0.8764658	4.49%
4	5.00	101 1/4	0.8323484	4.69%
5	5.00	101 1/4	0.7927128	4.76%
6	5.00	101 1/4	0.7549645	4.80%
7	5.00	101 1/4	0.7190138	4.83%
8	5.00	101 1/4	0.6847751	4.85%
9	5.25	102 1/4	0.64455	5.00%
10	5.25	102 1/4	0.612399	5.03%
11	5.25	102 1/4	0.5818518	5.05%
12	5.25	102 1/4	0.5528283	5.06%
13	5.50	104	0.5193962	5.17%
14	5.50	104	0.4923187	5.19%
15	5.50	104	0.4666528	5.21%
16	5.75	105 3/4	0.4331835	5.37%

Table 2: Bootstrapping Spot Rates

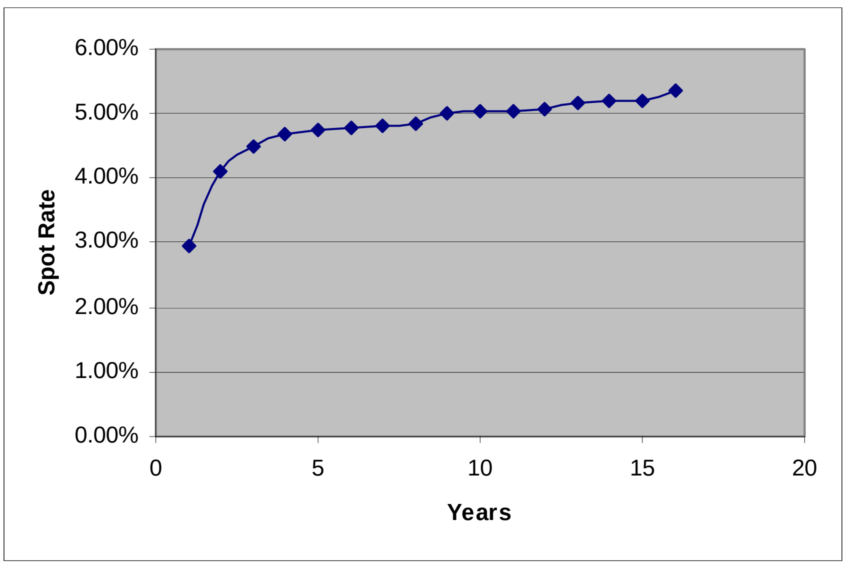

Figure 1: Mapping the Yield Curve

D. Term Structure Estimation with Coupon Bonds

The *spot rate* is the yield at present prevailing for zero coupon bonds of a given maturity. The t year spot rate is denoted here by $y_{0,t}$, which represents the interest rate on a loan to be made at time zero and repaid in its entirety at time t. Spot rates may be estimated from bonds with known future cash flows and their current prices. We are able to obtain spot rates from yields implied from series of bonds when we assume that the Law of One Price holds.

The yield curve represents yields or spot rates of bonds with varying terms to maturity. For example at a given point in time, the yield for one-year bonds may be 5% ($y_{0,1} = .05$), while the yield for five-year bonds may be 10% ($y_{0,5} = .10$). This section is concerned with how interest rates or yields vary with maturities of bonds. The simplest bonds to work with from an arithmetic perspective are *pure discount notes*, notes which make no interest payments. Such notes make only one payment at one point in time — on the maturity date of the note. Determining the relationship between yield and term to maturity for these bonds is quite trivial. The return one obtains from a pure discount note is strictly a function of capital gains; that is, the difference between the face value of the note and its purchase price. Short-term U.S. Treasury Bills are an example of pure discount (or zero coupon) notes. Coupon bonds are somewhat more difficult to work with from an arithmetic perspective because they make payments to bondholders at a variety of different periods.

Simultaneous Estimation of Discount Functions

A coupon bond may be treated as a portfolio of pure discount notes, with each coupon being treated as a separate note maturing on the date the coupon is paid. This slightly complicates the process for determining yields, but is necessary to avoid associating wrong yields with given time periods. Consider an example involving three bonds whose characteristics are given in Table 1. The three bonds are trading at known prices with a total of eight annual coupon payments among them (three for bonds A and B and 2 for bond C). Bond yields or spot rates must be determined simultaneously to avoid associating contradictory rates for the annual coupons on each of the three bills.

Table 1
Coupon Bonds A, B and C

BOND	CURRENT PRICE	FACE VALUE	COUPON RATE	YEARS TO MATURITY
A	962	1000	.10	3
B	1010.4	1000	.12	3
C	970	1000	.10	2

Let D_t be the discount function for time t; that is, $D_t = 1/(1+y_{0,t})^t$. Since $y_{0,t}$ is the spot rate or discount rate that equates the present value of a bond with its current price, the following equations may be solved for discount functions then spot rates:

$$962 = 100D_1 + 100D_2 + 1100D_3$$
$$1010.4 = 120D_1 + 120D_2 + 1120D_3$$
$$970 = 100D_1 + 1100D_2$$

This system of equations may be represented by the following system of matrices:

$$\begin{bmatrix} 100 & 100 & 1100 \\ 120 & 120 & 1120 \\ 100 & 1100 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 962 \\ 1010.4 \\ 970 \end{bmatrix}$$

$$\boldsymbol{CF} \cdot \boldsymbol{d} = \boldsymbol{P_0}$$

To solve this system we first invert Matrix **CF**, then use this inverse to premultiply Vector $\mathbf{P_0}$ to obtain Vector **d**:

$$\begin{bmatrix} -.0616 & .0605 & -.001 \\ .0056 & -.0055 & -.001 \\ .0060 & -.0050 & 0 \end{bmatrix} \begin{bmatrix} 962 \\ 1010.4 \\ 970 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} .90 \\ .80 \\ .72 \end{bmatrix}$$

$$\boldsymbol{CF}^{-1} \cdot \boldsymbol{P_0} = \boldsymbol{d}$$

Thus, we find from solving this system for Vector **d** that $D_1 = .9$, $D_2 = .8$ and $D_3 = .72$. Since $D_t = 1/(1+y_{0,t})^t$, $1/D_t = (1+y_{0,t})^t$, and $y_{0,t} = 1/D^{1/t} - 1$. Thus, spot rates are determined as follows:

$$\frac{1}{D_1} - 1 = .1111$$

$$\frac{1}{D_2^{\frac{1}{2}}} - 1 = .1180$$

$$\frac{1}{D_3^{\frac{1}{3}}} - 1 = .1157$$

Note that there exists a different spot rate (or discount rate) for each term to maturity; however, the spot rates for all cash flows generated by all bonds at a given period in time are the same. Thus, $y_{0,t}$ will vary over terms to maturity, but will be the same for all of the bonds in a given time period. The original system of three equations is solved in a step-by-step process in Appendix A at the end of the chapter.

Regression Estimation

In the section above, we were concerned with the yield curve describing the relationships among spot and forward rates over different intervals of time. We defined the following discount function D_t:

$$D_t = \frac{1}{(1 + y_t)^t}$$

where y_t is the spot rate which varied over time. Our solution technique for the different discount functions D_t and yields y_t required that we analyze a series of bonds maturing and making coupon payments on specific dates. In particular, our solution technique required that we have at least one bond for each yield we wished to estimate and that bonds make payments on identical dates. In reality, we may have difficulty finding bonds which make payments on common dates; furthermore, the bonds which we select may not be priced consistently. Our solution technique would not imply a spot rate for any date that would not be consistent with at least one bond payment.

Here, we will consider an alternative technique for mapping out a yield curve. Suppose that a fixed income manager believes that the following equation describes the relationship between bond discount functions and time (t):

$$D_t = a + b_1 t + b_2 t^2 + \varepsilon_t$$

where a, b_1 and b_2 are multiple OLS regression coefficients. We can use the multiple regression technique to determine spot rates from the data in Table 10 derived from zero coupon bonds. Based on a two-independent-variable OLS model, what would this fund manager predict the for the yield for a 2.5 year bond? Note that none of the bonds mature or make a coupon payment in exactly 2.5 years, so that we cannot compute $D_{2.5}$ using the solution technique from above. Thus, to estimate the 2.5 year yield, we shall perform an OLS regression of D_t on t and t^2.

The first step in our computations is to calculate each value for D_t from y_t. We find that D_1 = .917431, D_2 = .82946, D_3 = .741162, D_4 = .6635 and D_5 = .590785. We regress D_t against t and t^2 to obtain the following regression equation and t-statistics:

$$D_t = \underset{(552.74)}{1.014818} - \underset{(-33.1)}{.09956t} - \underset{(5.99)}{.002939t^2}$$

Table 10
Bond Yields and Maturity Data

Bond	Yield	t	t^2
A	.060	1	1
B	.082	2	4
C	.100	3	9
D	.114	4	16
E	.125	5	25

Inserting t = 2.5 into this equation, we find that $D_{2.5}$ = .784287. This leads to a yield $y_{2.5}$ solution of .102072. Our standard error estimates for a, b_1 and b_2 are, respectively, .001836, .003 and .000491. Thus, based on resulting t-statistics, our estimates for a, b_1 and b_2 are statistically significant at the .01 level.

E. Arbitrage With Riskless Bonds

The example provided above consists of three priced riskless bonds defining spot rates for all three relevant years. The cash flow structure of any three-year bond (for example, Bond D) added to the market can be replicated with some portfolio of bonds A, B, and C as long as its cash payments to investors are on the same dates as those made by at least one (in this example, two) of the three bonds A, B, and C. For example, assume that there now exists Bond D, a three-year, 11.5% coupon bond selling in this market for $990. This bond will make payments of $115 in years 1 and 2 in addition to a $1115 payment in year 3. A portfolio of bonds A, B and C can be comprised to generate the exact cash flow series. Thus, Bond D can be replicated by a portfolio of our first three bonds with the following weights: $w_A = .25$, $w_B = .75$ and $w_C = 0$, which are determined by the following system of equations or matrices:

$$115 = 100w_A + 120w_B + 100w_C$$
$$115 = 100w_A + 120w_B + 1100w_C$$
$$1115 = 1100w_A + 1120w_B$$

$$\begin{bmatrix} 100 & 120 & 100 \\ 100 & 120 & 1100 \\ 1100 & 1120 & 0 \end{bmatrix} \begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix} = \begin{bmatrix} 115 \\ 115 \\ 1115 \end{bmatrix}$$

$$\mathbf{CF} \cdot \mathbf{w} = \mathbf{cf_D}$$

To solve this system we first invert Matrix **CF**, then use it to premultiply Vector $\mathbf{cf_D}$ to obtain vector **w**:

$$\begin{bmatrix} -.0616 & .0056 & .0060 \\ .0605 & -.0055 & -.0050 \\ -.0010 & .0010 & 0 \end{bmatrix} \begin{bmatrix} 115 \\ 115 \\ 1115 \end{bmatrix} = \begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix} = \begin{bmatrix} .25 \\ .75 \\ 0 \end{bmatrix}$$

$$\mathbf{CF}^{-1} \cdot \mathbf{cf_D} = \mathbf{w}$$

Thus, we find from this system that $w_A = .25$, $w_B = .75$ and $w_C = 0$. We determine the value of the portfolio replicating Bond D by weighting their current market prices: (.25 × $962) + (.75 × $1010.4) = $998.3. Based on the portfolio's price, the value of Bond D is $998.3, although its current market price is $990. Thus, one gains an arbitrage profit from the purchase of this bond for $990 financed by the sale of the portfolio of Bonds A and B at a price of $998.3. Here, we simply swap a portfolio comprised of Bonds A and B for Bond D. Our cash flows in years 1,2 and 3 will be zero, although we receive a positive cash flow now of $8.3. This is a clear arbitrage profit. This arbitrage opportunity will persist until the value of the portfolio equals the value of Bond D. Thus, spot rates must be consistent for all bonds of the same risk class and maturity.

F. Fixed Income Portfolio Dedication

A fixed income fund is concerned with ensuring the provision of a relatively stable income over a given period of time. Typically, a fixed income fund must provide payments to its creditors, clients or owners for a given period. For example, a pension fund is often expected to make a series of fixed payments to pension fund participants. Such funds must invest their assets to ensure that their liabilities are paid. In many cases, fixed income funds will purchase assets such that their cash flows exactly match the liability payments that they are required to make. This exact matching strategy is referred to as dedication and is intended to minimize the risk of the fund. The process of dedication is much the same as the arbitrage swaps discussed above; the fund manager merely determines the cash flows associated with his liability structure and replicates them with a series of default risk free bonds. For example, assume that a pension manager needs to make payments to pension plan participants of $1,500,000 in one year; $2,500,000 in two years; and $4,000,000 in three years. He wishes to match these cash flows with a portfolio of bonds E, F and G whose characteristics are given in Table 3. These three bonds must be used to match the cash flows associated with the fund's liability structure. For example, in year 1, Bond E will pay $1100 (1000+100), F will pay $120 and G will pay $100. These payments must be combined to total $1,500,000. Cash flows must be matched in years 2 and 3 as well.

Table 3
Coupon Bonds E, F and G

BOND	CURRENT PRICE	FACE VALUE	COUPON RATE	YEARS TO MATURITY
E	1010	1000	.10	1
F	1100	1000	.12	2
G	950	1000	.10	3

Only one matching strategy exists for this scenario. The following system may be solved for **b** to determine exactly how many of each of the bonds are required to satisfy the fund's cash flow requirements:

$$\begin{bmatrix} 1100 & 120 & 100 \\ 0 & 1120 & 100 \\ 0 & 0 & 1100 \end{bmatrix} \begin{bmatrix} b_E \\ b_F \\ b_G \end{bmatrix} = \begin{bmatrix} 1{,}500{,}000 \\ 2{,}500{,}000 \\ 4{,}000{,}000 \end{bmatrix}$$

$$\mathbf{CF} \cdot \mathbf{b} = \mathbf{L}$$

Inverting Matrix **CF** and multiplying by Vector **L**, we find that the purchase of 824.9704 Bonds E, 1907.467 Bonds F and 3636.363 Bonds G satisfy the manager's exact matching requirements. The fund's time zero payment for these bonds totals $6,385,979.9292.

G. **Bond Duration**

Bonds and other debt instruments issued by the United States Treasury are generally regarded to be free of default risk and of relatively low liquidity risk. However, these bonds, particularly those with longer terms to maturity are subject to market value fluctuations after they are issued, primarily due to changes in interest rates offered on new issues. Generally, interest rate increases on new bond issues decrease values of bonds which are already outstanding; interest rate decreases on new bond issues increase values of bonds which are already outstanding. The duration model is intended to describe the proportional change in the value of a bond that is induced by a change in interest rates or yields of new issues.

Many analysts use present value models to value treasury issues, frequently using yields to maturity of new treasury issues to value existing issues with comparable terms. It is important for analysts to know how changes in new-issue interest rates will affect values of bonds with which they are concerned. Bond *duration* measures the proportional sensitivity of a bond to changes in the market rate of interest. Consider a two-year 10% coupon treasury issue which is currently selling for $986.48. The yield to maturity y of this bond is 12%. Default risk and liquidity risk are assumed to be zero; interest rate risk will be of primary importance. Assume that this bond's yield or discount rate is the same as the market yields of comparable treasury issues (which might be expected in an efficient market) and that bonds of all terms to maturity have the same yield. Further assume that investors have valued the bond such that its market price equals its present value; that is, the discount rate k for the bond equals its yield to maturity y. If market interest rates and yields were rise for new treasury issues, then the yield of this bond would rise accordingly. However, since the contractual terms of the bond will not change, its market price must drop to accommodate a yield consistent with the market. Assume that the value of an n-year bond paying interest at a rate of c on face value F is determined by a present value model with the yield y of comparable issues serving as the discount rate k:

(1) $$PV = \sum_{t=1}^{n} \frac{cF}{(1+y)^t} + \frac{F}{(1+y)^n}$$

Assume that the terms of the bond contract, n, F and c are constant. Just what is the proportional change in the price of a bond induced by a proportional change in market interest rates (technically, a proportional change in [1+y])? This may be approximated by the bond's Macaulay Simple Duration Formula as follows:

(2) $$\frac{\Delta PV}{PV} \div \frac{\Delta(1+y)}{(1+y)} \approx Dur = \frac{dPV}{PV} \div \frac{d(1+y)}{(1+y)} = \frac{dPV}{d(1+y)} \times \frac{(1+y)}{PV}$$

Equation (2) provides a good approximation of the proportional change in the price of a bond in a market meeting the assumptions described above induced by an infinitesimal proportional change in (1 + y). To compute the bond's sensitivity, we first rewrite Equation (1) in polynomial form (to take derivatives later) and substitute y for k (since they are assumed to be equal):

(3) $$PV = \sum_{t=1}^{n} \frac{cF_t}{(1+y)^t} + \frac{F}{(1+y)^n} = \sum_{t=1}^{n} cF(1+y)^{-t} + F(1+y)^{-n}$$

First, find the derivative of PV with respect to (1+y):

(4) $$\frac{dPV}{d(1+y)} = \sum_{t=1}^{n} -tcF(1+y)^{-t-1} - nF(1+y)^{-n-1}$$

Equation (4) is rewritten:

(5) $$\frac{dPV}{d(1+y)} = \frac{\sum_{t=1}^{n} -tcF(1+y)^{-t} - nF(1+y)^{-n}}{(1+y)}$$

Since the market rate of interest is assumed to equal the bond yield to maturity, the bond's price will equal its present value. Next, multiply both sides of Equation (5) by $(1+y) \div P_0$ to maintain consistency with Equation (2):

(6) $$Dur = \frac{dPV}{d(1+y)} \times \frac{(1+y)}{P_0} = \frac{\sum_{t=1}^{n} -tcF(1+y)^{-t} - nF(1+y)^{-n}}{P_0}$$

Thus, duration is defined as the proportional price change of a bond induced by a infinitesimal proportional change in (1+y) or 1 plus the market rate of interest:

(7) $$Dur = \frac{dPV}{d(1+y)} \times \frac{(1+y)}{P_0} = \frac{\sum_{t=1}^{n} \frac{-tcF}{(1+y)^t} + \frac{-nF}{(1+y)^n}}{P_0}$$

Since the market rate of interest will likely determine the yield to maturity of any bond, the duration of the bond described above is determined as follows from Equation (7):

(8) $$Dur = \frac{\frac{-1 \cdot .1 \cdot 1000}{(1+.12)} + \frac{-2 \cdot .1 \cdot 1000}{(1+.12)^2} + \frac{-2 \cdot 1000}{(1+.12)^2}}{986.48} = -1.87$$

This duration level of -1.87 suggests that the proportional decrease in the value of this bond will equal 1.87 times the proportional increase in market interest rates. This duration level also implies that this bond has exactly the same interest rate sensitivity as a *pure discount bond* (a bond making no coupon payments) which matures in 1.87 years.

Application of the Simple Macaulay Duration model does require several important assumptions. First, it is assumed that yields are invariant with respect to maturities of bonds; that is, the yield curve is flat. Furthermore, it is assumed that investors' projected reinvestment rates are identical to the bond yields to maturity. Any change in interest rates will be infinitesimal and will also be invariant with respect to time. The accuracy of this model will depend on the extent to which these assumptions hold.

H. Fixed Income Immunization

Earlier, we discussed bond portfolio dedication, which is concerned with matching terminal cash flows or values of bond portfolios with required payouts associated with liabilities. This process assumes that no transactions will take place within the portfolio and that cash flows associated with liabilities will remain as originally anticipated. Clearly, these assumptions will not hold for many institutions. Alternatively, one may hedge fixed income portfolio risk by using *immunization* strategies, which are concerned with matching the present values of asset portfolios with the present values of cash flows associated with future liabilities. More specifically, immunization strategies are primarily concerned with matching asset durations with liability durations. If asset and liability durations are matched, it is expected that the net fund value (equity or surplus) will not be affected by a shift in interest rates; asset and liability changes offset each other. Again, this simple immunization strategy is dependent on the following:

1. Changes in $(1 + y)$ are infinitesimal.
2. The yield curve is flat (yields do not vary over terms to maturity).
3. Yield curve shifts are parallel.
4. Only interest rate risk is significant.

F. Convexity

Earlier, we used duration to determine the approximate change in a bond's value induced by a change in interest rates (1 + y). However, the accuracy of the duration model is reduced by finite changes in interest rates, as we might reasonably expect. Duration may be regarded as a first order approximation (it only uses the first derivative) of the change in the value of a bond induced by a change in interest rates. *Convexity* is determined by the second derivative of the bond's value with respect to (1 + y). The first derivative of the bond's price with respect to (1 + y) is given:

$$\frac{\partial P_0}{\partial(1+y)} = \sum_{t=1}^{n} -tcF(1+y)^{-t-1} - nF(1+y)^{-n-1} \tag{1}$$

We find the second derivative by determining the derivative of the first derivative as follows:

$$\begin{aligned}\frac{\partial^2 P_0}{\partial(1+y)^2} &= \left[\sum_{t=1}^{n} -t(-t-1)cF(1+y)^{-t-2}\right] - \left[\, n(-n-1)F(1+y)^{-n-2}\,\right] \\ &= \left[\sum_{t=1}^{n} \frac{(t^2+t)cF}{(1+y)^{t+2}}\right] + \left[\frac{(n^2+n)F}{(1+y)^{n+2}}\right]\end{aligned} \tag{2}$$

Convexity is merely the second derivative of P_0 with respect to (1 + y) divided by P_0. The first two derivatives may be used in a Taylor series to approximate new bond prices induced by changes in interest rates as follows:

$$\begin{aligned}P_1 &\approx P_0 + f'(1+y_0)\cdot[\Delta(1+y)] + \frac{1}{2!}\cdot f''(1+y_0)\cdot[\Delta(1+y)]^2 \\ P_1 &\approx P_0 + \left[\sum_{t=1}^{n} \frac{-tcF_t}{(1+y_0)^{t+1}} - \frac{nF}{(1+y_0)^{n+1}}\right]\cdot[\Delta y] \\ &\quad + \frac{1}{2}\left[\sum_{t=1}^{n} \frac{(t^2+t)\cdot cF_t}{(1+y_0)^{t+2}} + \frac{(n^2+n)\cdot F}{(1+y_0)^{n+2}}\right]\cdot[\Delta y]^2\end{aligned} \tag{3}$$

Consider a five-year ten percent \$1000-face-value coupon bond currently selling at par (face value). We may compute the present yield to maturity of this bond as $y_0 = .10$. The first derivative of the bond's value with respect to (1+y) at $y_0 = .10$ is found from Equation (1) to be 3790.786769 (duration is 3790.786769 · 1.1 ÷ 1000 = 4.169865446); the second derivative is found from Equation (2) to be 19,368.34238 (convexity is 19,368.34238 ÷ 1000 = 19.36834238). If bond yields were to drop from .10 to .08, the actual value of this bond would increase to 1079.8542, as determined from a standard present value model. If we were to use the duration model (first-order approximation from the Taylor expansion, based only on the first derivative), we estimate that the value of the bond increases to 1075.815735. If we use the convexity model second-order approximation from Equation (2), we estimate that the value of the bond increases to 1079.689403. The second-order approximation may also be written as:

$$\begin{aligned}P_1 &= P_0 + Dur\cdot[\Delta(1+y)\cdot P_0/(1+y)] \\ &\quad + \left(\frac{1}{2}\right)\cdot P_0\cdot Con\cdot(\Delta(1+y))^2\end{aligned} \tag{4}$$

Note that this second estimate with the second-order approximation generates a revised bond value which is significantly closer to the bond's actual value as measured by the present value model. Therefore, the duration and immunization models are substantially improved by the second order approximations of bond prices (the convexity model). The fund manager wishing to hedge portfolio risk should not simply match durations (first derivatives) of assets and liabilities, he should also match their convexities (second derivatives).

EXERCISES

1. Consider an example where we can borrow money today for one year at 5%; $y_{0,1}$ = .05. Suppose that we are able to obtain a commitment to obtain a one year loan one year from now at an interest rate of 8%. Thus, the one year forward rate on a loan originated in year equals 8%. According to the Pure Expectations Theory, what is the two year spot rate of interest $y_{0,2}$?

2. Suppose that the one year spot rate $y_{0,1}$ of interest is 5%. Investors are expecting that the one year spot rate one year from now will increase to 6%; thus, the one year forward rate $y_{1,2}$ on a loan originated in one year is 6%. Furthermore, assume that investors are expecting that the one year spot rate two years from now will increase to 7%; thus, the one year forward rate $y_{2,3}$ on a loan originated in two years is 7%. Based on the pure expectations hypothesis, what is the three year spot rate?

3. Suppose that the one year spot rate $y_{0,1}$ of interest is 5%. Investors are expecting that the one year spot rate one year from now will increase to 7%; thus, the one year forward rate $y_{1,2}$ on a loan originated in one year is 7%. Furthermore, assume that the three-year spot rate equals 7% as well. What is the anticipated one year forward rate $y_{2,3}$ on a loan originated in two years based on the pure expectations hypothesis?

4.** Bond A, a three year 7% issue, currently sells for 964.3227. Bond B is a two year 8% issue currently selling for 1010.031. Bond C is a three year 6% issue currently selling for 938.4063. Based on this information, answer the following:

c. What are the one, two and three year spot rates of interest?
d. What are the one- and two-year forward rates on loans originating one year from now?
e. What is the one-year forward rate on a loan originated in two years?

5. Assume that there are two three-year bonds with face values equaling $1000. The coupon rate of bond A is .05 and .08 for bond B. A third bond C also exists, with a maturity of two years. Bond C also has a face value of $1000; it has a coupon rate of 11%. The prices of the three bonds are $878.9172, $955.4787 and $1055.419, respectively.

a.** What are the spot rates implied by these bonds?
b. Find a portfolio of bonds A, B and C which would replicate the cash flow structure of bond D, which has a face value of $1000, a maturity of three years and a coupon rate of 3%.

6. A life insurance company expects to make payments of $30,000,000 in one year, $15,000,000 in two years $25,000,000 in three years and $35,000,000 in four years to satisfy claims of policyholders. These anticipated cash flows are to be matched with a portfolio of the following $1000 face value bonds:

BOND	CURRENT PRICE	COUPON RATE	YEARS TO MATURITY
1	1000	.10	1
2	980	.10	2
3	1000	.11	3
4	1000	.12	4

How many of each of the four bonds should the company purchase to exactly match its anticipated payments to policyholders?

7. Find the duration of the following pure discount bonds:

a. A $1000 face value bond maturing in one year currently selling for $900.

b. A $1000 face value bond maturing in two years currently selling for $800.
c. A $2000 face value bond maturing in three years currently selling for $1400.
d. A portfolio consisting of one of each of the three bonds listed in parts a, b and c of this problem.

8. What is the relationship between the maturity of a pure discount bond and its duration?

9. Find the duration of each of the following $1000 face value coupon bonds assuming coupon payments are made annually:

a. 3 year 10% bond currently selling for $900
b. 3 year 12% bond currently selling for $900
c. 4 year 10% bond currently selling for $900
d. 3 year 10% bond currently selling for $800

10. Based on duration computations, what would happen to the prices of each of the bonds in Question 9 if market interest rates (1+r) were to decrease by 10%?

11. What is the duration of a portfolio consisting of one of each of the bonds listed in problem 9?

12. Find durations and convexities for each of the following bonds:

a. A 10% five year bond selling for $1079.8542 yielding 8%
b. A 12% five year bond selling for $1000 yielding 12%

13.a. Use the duration (first order) approximation models to estimate bond value increases induced by changes in interest rates (yields) to 10% for each of the bonds in Problem 12 above.
b. Use the convexity (second order) approximation models to estimate bond value increases induced by changes in interest rates (yields) to 10% for each of the bonds in Problem 12 above.
c. Find the present values of each of the bonds in Problem 12 above after yields (discount rates) change to 10%.

14.**The following table lists five pure discount bonds along with their yields and terms to maturity.

BOND	YIELD	t
A	.060	1
B	.082	2
C	.100	3
D	.114	4
E	.125	5

Based on a multiple regression model, with t and t^2 as independent variables, what would you predict for the yield of a 4.5-year bond?

EXERCISE SOLUTIONS

1. According to the Pure Expectations Theory, we compute the two year spot rate as follows:

$$(1+y_{0,2})^2 = \prod_{t=1}^{2}(1+y_{t-1,t}) = (1+.05)(1+.08) = 1.134$$

$$y_{0,2} = [(1+.05)(1+.08)]^{1/2} - 1 = \sqrt{1.134} - 1 = .0648944$$

2. The three year rate is based on a geometric mean of the short term spot rates as follows:

$$(1+y_{0,3})^3 = \prod_{t=1}^{3}(1+y_{t-1,t}) = (1+.05)(1+.06)(1+.07) = 1.19091$$

$$y_{0,3} = [(1+.05)(1+.06)(1+.07)]^{1/3} - 1 = \sqrt[3]{1.19091} - 1 = .0599686$$

3. The three-year rate is based on a geometric mean of the short term spot rates as follows:

$$(1+y_{0,3})^3 = (1.07)^3 = 1.22504 = \prod_{t=1}^{3}(1+y_{t-1,t}) = (1+.05)(1+.07)(1+y_{2,3})$$

We solve for $y_{2,3}$ as follows:

$$1.22504 \div [(1+.05)(1+.07)] - 1 = y_{2,3} = .07$$

4.a. First, assume that bonds are priced with the following present value functions:

Equation Set A

$$964.3227 = 70D_1 + 70D_2 + 1070D_3$$
$$1010.031 = 80D_1 + 1080D_2 + 0D_3$$
$$938.4063 = 60D_1 + 60D_2 + 1060D_3$$

We will call this first set of equations Equation set A. This set of equations has three equations with three unknown values for which we need solutions. Note that the coefficient for D_3 in Equation 2A equals zero. If we can produce another equation with a zero coefficient for D_3, we will be able to solve for two unknowns in a two equation system. We will call this next set of equations Equation Set B. If we multiply Equation 1A by (1060/1070), the coefficient for D_3 Equation 1B will match the coefficient for D_3 in Equation 3A (which will be re-written as Equation 2B):

Equation Set B

69.34579D1 + 69.34579D2 + 1060 D3 = 955.3103 *1060/1070 * Equation 1A*
60D1 + 60D2 + 1060D3 = 938.4063 *Equation 3A = Equation 2B*

Next, subtract Equation 2B from Equation 1B to eliminate D3 and its coefficient 1060. The result of this subtraction is the first equation in Equation Set C. The second equation in Equation Set C is the second equation in Equation Set A multiplied by (19.25234/80). This multiplication will enable us to subtract later to eliminate unknown D1:

Equation Set C

9.345794D1 + 9.345794D2 + 0D3 = 16.90401 *Equation 1B - Equation 2B*
9.345794D1 + 126.1682D2 + 0D3 = 117.9943 *Equation 2A * (9.345794/80)*

When we subtract the second equation in Equation Set C from the first, Variable D1 will be eliminated:

0D1 + -116.822D2 + 0D3 = -101.09 *Equation 1C - Equation 2C*

Now, we can easily solve for D2 = -101.09/-116.822 = .865333. Now, using Equation 1C, we can solve for D1 = (16.90401 - 9.345794 * D2)/9.345794 = 0.943396. Finally, we use Equation 1A to solve for D3 = (964.3227 - 70*D1 - 70*D2)/1070) = .782908. Now that we have solved for our three discount functions D_1, D_2 and D_3, we can obtain spot rates:

$$\frac{1}{D_1} - 1 = .06 = y_{0,1}$$

$$\frac{1}{D_2^{\frac{1}{2}}} - 1 = .075 = y_{0,2}$$

$$\frac{1}{D_3^{\frac{1}{3}}} - 1 = .085 = y_{0,3}$$

b. Obtain the one and two year forward rates on loans originated in year one as follows:

$$y_{1,2} = \frac{(1+.075)^2}{(1+.06)} - 1 = 1.0902123 - 1 = .0902123$$

$$y_{1,3} = \sqrt{\frac{(1+.085)^3}{(1+.06)}} - 1 = \sqrt{1.2049897} - 1 = .0977202$$

c. The one year forward rate on a loan originated in two years is computed as follows:

$$y_{2,3} = \frac{(1+.085)^3}{(1.075)^2} - 1 = 1.1052799 - 1 = .1052799$$

5. a. First, solve the following system for the discount functions **d**:

$$\underbrace{\begin{bmatrix} 50 & 50 & 1050 \\ 80 & 80 & 1080 \\ 110 & 1110 & 0 \end{bmatrix}}_{\boldsymbol{CF}} \cdot \underbrace{\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}}_{\boldsymbol{d}} = \underbrace{\begin{bmatrix} 878.9172 \\ 955.4787 \\ 1055.4190 \end{bmatrix}}_{\boldsymbol{P_0}}$$

We find that $D_1 = .943396$, $D_2 = .857338$ and $D_3 = .751314$. The spot rates are obtained as follows:

$$\frac{1}{D_1} - 1 = \frac{1}{.943396} - 1 = .06$$

$$\frac{1}{D_2^{\frac{1}{2}}} - 1 = \frac{1}{.857338^{\frac{1}{2}}} - 1 = .08$$

$$\frac{1}{D_3^{\frac{1}{3}}} - 1 = \frac{1}{.751314^{\frac{1}{3}}} - 1 = .10$$

b. The weights are found by solving for **w** as follows:

$$\underbrace{\begin{bmatrix} 50 & 80 & 110 \\ 50 & 80 & 1110 \\ 1050 & 1080 & 0 \end{bmatrix}}_{\boldsymbol{CF}} \cdot \underbrace{\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix}}_{\boldsymbol{w}} = \underbrace{\begin{bmatrix} 30 \\ 30 \\ 1030 \end{bmatrix}}_{\boldsymbol{P_0}}$$

$$\underbrace{\begin{bmatrix} -.03996 & .00396 & .002666 \\ .03885 & -.00385 & -.001660 \\ -.00100 & .00100 & 0 \end{bmatrix}}_{\boldsymbol{CF}^{-1}} \cdot \underbrace{\begin{bmatrix} 30 \\ 30 \\ 1030 \end{bmatrix}}_{\boldsymbol{P_0}} = \underbrace{\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix}}_{\boldsymbol{w}}$$

We find that $w_A = 1.666666$, $w_B = -.666666$ and that $w_C = 0$. means that bond D is replicated by a portfolio consisting of 1.666666 of bond A and -.666666 of bond B.

6. The following system may be solved for **b** to determine exactly how many of each of the bonds are

required to satisfy the fund's cash flow requirements:

$$\underbrace{\begin{bmatrix} 1100 & 100 & 110 & 120 \\ 0 & 1100 & 110 & 120 \\ 0 & 0 & 1110 & 120 \\ 0 & 0 & 0 & 1120 \end{bmatrix}}_{\boldsymbol{CF}} \cdot \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{\boldsymbol{b}} = \underbrace{\begin{bmatrix} 30{,}000{,}000 \\ 15{,}000{,}000 \\ 25{,}000{,}000 \\ 35{,}000{,}000 \end{bmatrix}}_{\boldsymbol{P_0}}$$

First, we invert Matrix **CF** to obtain $\mathbf{CF^{-1}}$:

$$\underbrace{\begin{bmatrix} .000909 & -.000083 & -.00008 & -.000079 \\ 0 & .000909 & -.00009 & -.000087 \\ 0 & 0 & .00090 & -.000096 \\ 0 & 0 & 0 & .000892 \end{bmatrix}}_{\boldsymbol{CF^{-1}}}$$

Thus by inverting matrix **CF** to obtain $\mathbf{CF^{-1}}$, and pre-multiplying vector $\mathbf{P_0}$ by $\mathbf{CF^{-1}}$ to obtain solutions vector **b**, we find that the purchase of 21,193.5 Bonds 1, 8,312.858 Bonds 2, 19,144.14 Bonds 3 and 31,250 Bonds 4 satisfy the insurance company's exact matching requirements.

7. a. i. First, find the yield to maturity (ytm) of the bond:

$$0 = NPV = \sum_{t=1}^{n} \frac{CF_t}{(1+ytm)^t} - P_0 \;\; ; \text{ yield to maturity = ytm}$$

$$0 = NPV = \frac{1000}{(1+ytm)^1} - 900 \;\; ; \;\; Solve\ for\ ytm$$

ytm = .111

ii. Use ytm from Part i in the duration formula:

$$Dur = \frac{\sum_{t=1}^{n} \boldsymbol{t} \cdot \frac{CF_t}{(1+ytm)^t}}{P_0}$$

Note:negative signs are omitted

$$Dur = \frac{1 \cdot \frac{1{,}000}{1.111}}{900} = Dur = 1\ year$$

b. i. $$0 = NPV = \sum_{t=1}^{n} t \frac{CF_t}{(1+ytm)^t} - P_0 = \frac{1,000}{(1 = ytm)^t} - 800$$

Ytm = .118

ii. $$Dur = \frac{\sum_{t=1}^{n} t \times \frac{CF_t}{(1+ytm)^t}}{P_0} = \frac{2 \times \frac{1,000}{1.118^2}}{800} = 2$$

c. Ytm = .126 ; $$Dur = \frac{3 \times \frac{2,000}{(1.126)^3}}{1,400} = 3$$

d. There are several ways to work this problem. First, consider the cash flows of the portfolio:

P_0 = 900 + 800 + 1400 = 3100

CF_1 = 1000 ; CF_2 = 1000 ; CF_3 = 2000

$$0 = NPV = \frac{1000}{(1+ytm)^1} + \frac{1000}{(1+ytm)^2} + \frac{2000}{(1+ytm)^3} - 3100 \quad ; \text{Ytm} = .122$$

$$Dur = \frac{\sum_{t=1}^{n} t \times \frac{CF_t}{(1+ytm)^t}}{P_0} = \frac{\frac{1 \times 1000}{1.122} + \frac{2 \times 1000}{(1.122)^2} + \frac{3 \times 2000}{(1.122)^3}}{3100}$$

Dur = 2.161 years

Second, notice that the portfolio duration is a weighted average of the bond durations:

(900/3100) × 1 + (800/3100) × 2 + (1400/3100) × 3 = 2.161

8. The duration of a pure discount bond equals its maturity.

9. a. i. First find the bond's Ytm:

$$0 = NPV = \frac{100}{(1+ytm)^1} + \frac{100}{(1+ytm)^2} + \frac{100+1000}{(1+ytm)^3} - 900 \quad ; \text{Ytm} = .143$$

ii. Now, use Ytm to find Duration:

$$Dur = \frac{\frac{1 \times 100}{1.143} + \frac{2 \times 100}{(1.143)^2} + \frac{3 \times 1100}{(1.143)^3}}{900} = 2.722$$

b. $$0 = NPV = \frac{120}{(1+ytm)^1} + \frac{120}{(1+ytm)^2} + \frac{1120}{(1+ytm)^3} - 900$$

Ytm = .165

$$Dur = \frac{\frac{1 \cdot 120}{1.165} + \frac{2 \cdot 120}{(1.165)^2} + \frac{3 \cdot 1120}{(1.165)^3}}{900} = 2.672$$

c. $$0 = NPV = \frac{100}{(1+ytm)^1} + \frac{100}{(1+ytm)^2} + \frac{100}{(1+ytm)^3} + \frac{1100}{(1+ytm)^4} - 900$$

Ytm = .134

$$Dur = \frac{\frac{1 \cdot 100}{1.134} + \frac{2 \cdot 100}{(1.134)^2} + \frac{3 \cdot 100}{(1.134)^3} + \frac{4 \cdot 1100}{(1.134)^4}}{900} = 3.456$$

d. $$0 = NPV = \frac{100}{(1+ytm)^1} + \frac{100}{(1+ytm)^2} + \frac{1120}{(1+ytm)^3} - 800$$

ytm = .194

$$Dur = \frac{\frac{1 \cdot 100}{1.194} + \frac{2 \cdot 100}{(1.194)^2} + \frac{3 \cdot 1100}{(1.194)^3}}{800} = 2.703$$

10. a. $\%\Delta P_0 = Dur \times \%\Delta(1+r)$; $\%\Delta P_0 = 2.722 \times .10 = .2722$

$\%\Delta P_0 = .2722$; $\Delta P_0 = .2722 \times 900 = 244.98$;

The new price is 900 + 244.98 = 1144.98

b. $\%\Delta P_0 = 2.672 \times .10 = .2622$; $\Delta P_0 = 235.98$; Price = 1135.98

c. $\%\Delta P_0 = 3.456 \times .10 = .3456$; $\Delta P_0 = 311.04$; Price = 1211.04

d. $\%\Delta P_0 = .2703$; $\Delta P_0 = 216.24$; new price = 1016.24

11. $$Dur = 2.722 \cdot \frac{900}{3500} + 2.672 \cdot \frac{900}{3500} + 3.456 \cdot \frac{900}{3500} + 2.703 \cdot \frac{800}{3500} = 2.893$$

12. Durations and convexities are as follows:

a. Dur = 4.203743015 ; Con = 20.31015, calculated as follows:

DUR A	CONV. A
92.59259258	158.7664481
171.467764	441.0179114
238.1496722	816.6998358
294.0119409	1260.339253
3743.207581	19255.18302

Add then divide by P0 = 1079.8542 to obtain:

-4.203743015   20.31015527

b. Dur = 4.0373493 ; Con = 17.86, calculated as follows:

DUR B	CONV. B
107.1428571	170.8272595
191.3265306	457.5730164
256.2408892	817.0946722
305.0486776	1215.914691
3177.590392	15198.93363

Add then divide by P0 = 1000 to obtain:

-4.037349347   17.86034327

13.a. P_{1A} = 995.7906904 = 1079.8542 - 4.203743·(.1 - .08)/(1.08)·1079.8542

P_{1B} = 1072.095524 = 1000 - 4.037349·(.1-.12)/(1.12)·1000

b. P_{1A} = 1000.1770910 = 1079.8542 - 4.203743·(.1 - .08)/(1.08)·1079.8542 + .5·1079.8542·20.3101·$(.1-.08)^2$

P_{1B} = 1075.667592 =1000 - 4.037349·(.1-.12)/(1.12)·1000 +.5·1000·17.86·$(.1-.12)^2$

c. P_{1A} = 1000.00 = (100/.1)·(1-1/$(1.1)^5$) + 1000/$(1.1)^5$

P_{1B} = 1075.815735 = (120/.1)·(1-1/$(1.1)^5$) + 1000/$(1.1)^5$

The new bond values given in part c are precise. Note how much better the bond convexity model in part b estimates revised bond prices than the duration model in part a.

14. To estimate the 4.5 year yield, we shall perform an OLS regression of D_t on t and t^2. The first step in our computations is to calculate each value for D_t from y_t. We find that D_1 = .943396, D_2 = .854172, D_3 = .751315, D_4 = .649321 and D_5 = .554929. We regress D_t against t and t^2 to obtain the following regression equation:

$$D_t = \underset{(217.0)}{1.040426} - \underset{(12.0)}{.09412t} - \underset{(.53)}{.00068t^2}$$

Inserting t=4.5 into this equation, we find that $D_{4.5}$ = .60319. This leads to a yield $y_{4.5}$ solution of .118892. Also, note that our estimates for a and b_1 are statistically significant at the .01 level; b_2 is not. Although our computations for $y_{4.5}$ are correct based on the relationship defined by the fixed income analyst, one should question use of the t^2 independent variable since it was not statistically significant. One might obtain more accurate forecasts for y by omitting this t^2 term from his regressions, assuming that his data and measurements are reliable.

APPENDIX A: SOLVING SYSTEMS OF EQUATIONS

Systems of linear equations with multiple variables are often solved algebraically using the addition method. This method uses addition and multiplication principles. In order to use this method to solve the system completely, we normally must have the same number of equations as variables; ie., for two variables, we need two equations, and for three variables, we need three equations.

Two Variable Systems: Solving for x and y.

For a two variable system, we simply add or multiply the equations to cancel out one variable. We then solve for the second variable, and plug it into either equation to solve for the first variable which we originally canceled. For example:

(A) $.05 = .05x + .12y$

(B) $.08 = .10x + .30y$

Notice that if we multiply the first equation by -2, then add the two equations, we can eliminate the x term.

$$-2 \cdot .05 = -2 \cdot .05x + -2 \cdot .12y \ ,$$

which gives us the following equation:

$$-.10 = -.10x + -.24y$$

We then add this equation to our original equation (B):

$$\begin{array}{rl} & -.10 = -.10x + -.24y \\ + & .08 = \ .10x + \ .30y \\ \hline & -.02 = \ 0 \ + \ .06y \end{array}$$

Now we solve for y:

$$\begin{aligned} .06y &= -.02 \\ y &= -.02/.06 \\ y &= -.333 \end{aligned}$$

Now that we have found y, we must find x. This can be done by substituting -.333 for y into either of our original equations:

$$\begin{aligned} .05 &= .05x + -.333 \cdot .12 \\ .05 &= .05x + -.04 \\ .05x &= .09 \\ x &= .09/.05 \\ x &= 1.8 \end{aligned}$$

To check our solution, we can also substitute -.333 for y into the second equation, to ensure we get a value of 1.8 for x.

$$\begin{aligned} .08 &= .10x + .30 \cdot (-.333) \\ .10x &= .18 \\ x &= 1.8 \end{aligned}$$

Three Variable Systems: Solving for x, y, and z.

Solving a system of three variables requires three equations. This is done algebraically in a system similar to the two variable/two equation example.

Consider the following example:

(A) $.05 = .04x + .09y + .15z$
(B) $.15 = .08x + .12y + .10z$
(C) $.30 = .12x + .06y + .25z$

To begin, we solve one equation for x. We could have started with another variable, but x in this example falls out rather nicely. Solving the first equation for x gives us the following:

$$.04x = .05 - .09y - .15z$$
$$x = .05/.04 - .09y/.04 - .15z/.04$$
$$x = 1.25 - 2.25y - 3.75z$$

Now we plug x (our revised version of equation A)into the other two equations:

(B1) $.15 = .08(1.25 - 2.25y - 3.75z) + .12y + .10z$
(C1) $.30 = .12(1.25 - 2.25y - 3.75z) + .06y + .25z$

Simplifying these two equations we get the following:

(B2) $.05 = -.06y - .20z$
(C2) $.15 = -.21y - .20z$

Now we have two equations, with two variables. This is solved exactly as our first example. Multiply the first equation by -1 and then add the two equations:

(B3) $-.05 = .06y + .20z$
(C3) $.15 = -.21y - .20z$

$$.10 = -.15y$$
$$y = -.66667$$

Now plug y into either equation B3 (or C3) and solve for z:

$$.05 = -.06 \cdot (-.66667) -.20z$$
$$z = -.05$$

Finally, we have y and z, and we can plug these into any of our three original equations to solve for x:

$$.15 = .08x + .12 \cdot (-.66667) + .10 \cdot (-.05)$$
$$x = 3$$

Using the values found for x, y, and z you can plug them into either or the other two original equations to check..

For a second application, consider the example involving estimation of spot rates from Section D. Here, we have three bonds whose characteristics are given in Table 1 from Section D. The example involves solving simultaneously the following syatem of equations for D1, D2 and D3:

$$962 = 100D_1 + 100D_2 + 1100D_3$$
$$1010.4 = 120D_1 + 120D_2 + 1120D_3$$
$$970 = 100D_1 + 1100D_2$$

First, we solve for D3 in the first equation:
1100D3 = 962 - 100D1 - 100D2
D3 = 0.874545 - 0.090909D1 - 0.090909D2 Divide both sides by 1100

Now, substitute this equation for D3 in the original second equation:
1010.4 = 120D1 + 120D2 + 1120(.874545 - .0909091D1 - .0909091D2)

Simplify by combining similar terms:
30.9096 = 18.18181D1 - 18.18181D2

Now, we have an equation with 2 unknowns. Combine with our original third equation:
970 = 100D1 + 1100D2
30.9096 = 18.18181D1 + 18.18181D2

Now, solve this 2X2 system for D1 and D2. First, solve the first equation for D2:
1100D2 = 970 - 100D1
D2 = 0.881818 - 0.090909D1 Divide both sides by 1100

Now, substitute this equation for D2 in the second equation:
30.9096 = 18.18181D1 + 18.18181(.881818 - .090909D1)

Simplify by combining similar terms:
14.87655 = 16.52892D1

Now, we know D1:
D1 = 0.900032

We substitute D2 into our original third equation to obtain D2:
(970 - 100*.900032)/1100 = D2
D2 = 0.799997

Finally, we substitute D1 and D2 into our original second equation to obtain D3:
(1010.4 - 120*.900032 - 120*.799997)/1120
D3 = 0.719997

APPENDIX B: SOLVING SYSTEMS OF EQUATIONS WITH MATRICES

Multiplication of Matrices

Two matrices **A** and **B** may be multiplied to obtain the product **AB** = **C** if the number of columns in the first Matrix **A** equal the number of rows **B** in the second.[1] If Matrix **A** is of dimension m × n and Matrix **B** is of dimension n × q, the dimensions of the product Matrix **C** will be m × q. Each element $c_{i,k}$ of Matrix **C** is determined by the following sum:

(1)
$$c_{i,k} = \sum_{j=1}^{n} a_{i,j} b_{j,k}$$

For example, consider the following product:

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 150 & 154 \\ 170 & 184 \\ 184 & 224 \end{bmatrix}$$
$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{C}$$

Matrix **C** in the above is found as follows:

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} (7\cdot7)+(4\cdot5)+(9\cdot9) & (7\cdot6)+(4\cdot1)+(9\cdot12) \\ (6\cdot7)+(4\cdot5)+(12\cdot9) & (6\cdot6)+(4\cdot1)+(12\cdot12) \\ (3\cdot7)+(2\cdot5)+(17\cdot9) & (3\cdot6)+(2\cdot1)+(17\cdot12) \end{bmatrix}$$
$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{C}$$

Notice that the number of columns (3) in Matrix **A** equals the number of rows in Matrix **B**. Also note that the number of rows in Matrix **C** equals the number of rows in Matrix **A**; the number of columns in **C** equals the number of columns in Matrix **B**.

Inversion of Matrices

An *inverse* Matrix $\mathbf{A}^{-1}$ exists for the square Matrix **A** if the product $\mathbf{A}^{-1}\mathbf{A}$ or $\mathbf{A}\mathbf{A}^{-1}$ equals the identity Matrix **I**. Consider the following product:

$$\begin{bmatrix} 2 & 4 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{30} & \frac{2}{15} \\ \frac{4}{15} & \frac{-1}{15} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\boldsymbol{A} \quad \boldsymbol{A}^{-1} = \boldsymbol{I}$$

One means for finding the inverse Matrix $\mathbf{A}^{-1}$ for Matrix **A** is through the use of a process called the *Gauss-Jordan Method*. This method will be performed on Matrix **A** by first augmenting it with the identity matrix as follows:

(A)
$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 8 & 1 & 0 & 1 \end{array}\right]$$

For the sake of convenience, call the above augmented Matrix **A** temporarily. Now, a series of row operations (addition, subtraction or multiplication of each element in a row) will be performed such that the identity matrix replaces the original Matrix **A** (on the left side). The right-side elements will comprise the inverse Matrix $\mathbf{A}^{-1}$. Thus, in our final augmented matrix, we will have ones along the principal diagonal on the left side and zeros elsewhere; the right side of the matrix will comprise the inverse of **A**. Allowable row operations include the following:

1. Multiply a given row by any constant. Each element in the row must be multiplied by the same constant.
2. Add a given row to any other row in the matrix. Each element in a row is added to the corresponding element in the same column of another row.
3. Subtract a given row from any other row in the matrix. Each element in a row is subtracted from the corresponding element in the same column of another row.
4. Any combination of the above. For example, a row may be multiplied by a constant before it is subtracted from another row.

Our first row operation will serve to replace the upper left corner value with a one. We multiply Row 1 in **A** (Row 1**A**) by .5 to obtain the following:

$$\begin{bmatrix} 1 & 2 & | & .5 & 0 \\ 8 & 1 & | & 0 & 1 \end{bmatrix} \qquad 1\boldsymbol{A} \times .5 = 1\boldsymbol{B}$$

where Row 1**B** replaces Row 1**A**. Now we obtain a zero in the lower left corner by multiplying Row 2 in **A** by 1/8 and subtracting the result from our new Row 1 to obtain Matrix **B** as follows:

(B)

$$\begin{bmatrix} 1 & 2 & | & .5 & 0 \\ 0 & \frac{15}{8} & | & .5 & \frac{-1}{8} \end{bmatrix} \qquad \begin{matrix} 1\boldsymbol{A} \times 0.5 = 1\boldsymbol{B} \\ 1\boldsymbol{B} - \left(2\boldsymbol{A} \times \frac{1}{8} \right) = 2\boldsymbol{B} \end{matrix}$$

Next, we obtain a 1 in the lower right corner of the left side of the matrix by multiplying Row 2**B** by 8/15:

$$\begin{bmatrix} 1 & 2 & | & .5 & 0 \\ 0 & 1 & | & \frac{4}{15} & \frac{-1}{15} \end{bmatrix} \qquad 2\boldsymbol{B} \times \frac{8}{15} = 2\boldsymbol{C}$$

We obtain a zero in the upper right corner of the left side matrix by multiplying Row 2 above by 2 and subtracting from Row 1 in **B**:

(C)

$$\begin{bmatrix} 1 & 0 & | & \frac{-1}{30} & \frac{2}{15} \\ 0 & 1 & | & \frac{4}{15} & \frac{-1}{15} \end{bmatrix} \qquad \begin{matrix} 1\boldsymbol{B} - (2\boldsymbol{C} \times 2) = 1\boldsymbol{C} \\ 2\boldsymbol{B} \times \frac{8}{15} = 2\boldsymbol{C} \end{matrix}$$

The left side of augmented Matrix **C** is the identity matrix; the right side of **C** is $\mathbf{A}^{-1}$.

Because matrices cannot be divided as numbers are in arithmetic, one performs an analogous operation by inverting the matrix intended to be the "divisor" and postmultiplying this inverse by the first matrix to obtain a quotient. Thus, instead of dividing **A** by **B** to obtain **D**, one inverts **B** and obtains **D** by the product $\mathbf{AB}^{-1} = \mathbf{D}$. This concept is extremely useful for many types of algebraic manipulations.

Solving Systems of Equations

Matrices can be very useful in arranging systems of equations. Consider for example the following system of equations:

$$.05x_1 + .12x_2 = .05$$
$$.10x_1 + .30x_2 = .08$$

This system of equations may be represented as follows:

$$\begin{bmatrix} .05 & .12 \\ .10 & .30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .05 \\ .08 \end{bmatrix}$$
$$\boldsymbol{C} \quad \boldsymbol{x} = \boldsymbol{s}$$

We are not able to divide **C** by **s** to obtain **x**; instead, we invert **C** to obtain $\mathbf{C}^{-1}$ and multiply it by **s** to obtain **x**:

$$\mathbf{C}^{-1}\mathbf{s} = \mathbf{x}$$

Therefore, to solve for Vector **x**, we first invert **C** by augmenting it with the Identity Matrix:

(A) $$\left[\begin{array}{cc|cc} .05 & .12 & 1 & 0 \\ .10 & .30 & 0 & 1 \end{array}\right]$$

(B) $$\left[\begin{array}{cc|cc} 1 & 2.4 & 20 & 0 \\ 0 & .6 & -20 & 10 \end{array}\right] \qquad \begin{array}{l} Row\ \boldsymbol{B}1 = \boldsymbol{A}1 \times 20 \\ Row\ \boldsymbol{B}2 = (10 \times \boldsymbol{A}2) - \boldsymbol{B}1 \end{array}$$

(C) $$\left[\begin{array}{cc|cc} 1 & 0 & 100 & -40 \\ 0 & 1 & \frac{-100}{3} & \frac{50}{3} \end{array}\right] \qquad \begin{array}{l} Row\ \boldsymbol{C}1 = \boldsymbol{B}1 - (2.4 \times \boldsymbol{B}2) \\ Row\ \boldsymbol{C}2 = \boldsymbol{B}2 \times \frac{5}{3} \end{array}$$
$$\boldsymbol{I} \qquad \boldsymbol{C}^{-1}$$

Thus, we obtain Vector **x** with the following product:

(D) $$\begin{bmatrix} 100 & -40 \\ \frac{-100}{3} & \frac{50}{3} \end{bmatrix} \begin{bmatrix} .05 \\ .08 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.8 \\ \frac{-1}{3} \end{bmatrix}$$
$$\boldsymbol{C}^{-1} \quad \boldsymbol{s} = \boldsymbol{x} = \boldsymbol{x}$$

Thus, we find that $x_1 = 1.8$ and $x_2 = -1/3$.

Arbitrage is defined as the simultaneous purchase and sale of assets or portfolios yielding identical cash flows. Assets generating identical cash flows (certain or risky cash flows) should be worth the same amount. This is known as the Law of One Price. If assets generating identical cash flows sell at different prices, opportunities exist to create a profit by buying the cheaper asset and selling the more expensive asset. The ability to realize a profit from this type of transaction is known as an arbitrage opportunity. Solutions for multiple variables in systems of equations are most useful in the application of the Law of One Price and seeking arbitrage opportunity.

APPENDIX C: DERIVATIVES OF POLYNOMIALS

The derivative from calculus can be used to determine rates of change or slopes. For those functions whose slopes are constantly changing, the derivative is to find an instantaneous rate of change; that is, the change in y induced by the "tiniest" change in x. Assume that y is given as a function of variable x. If x were to increase by a small (infinitesimal — that is, approaching, though not quite equal to zero) amount h, by how much would y change? This rate of change is given by the derivative of y with respect to x, which is defined as follows:

(1) $$\frac{d^2y}{dx^2} = f''(x) = \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h}$$

One type of function which appears regularly in finance is the polynomial function. This type of function defines variable y in terms of a coefficient c (or series of coefficients c_j), variable x (or series of variables x_j) and an exponent n (or series of exponents n_j). Strictly speaking, the exponents in a polynomial equation must be non-negative integers; however, the rules that we discuss here still apply when the exponents assume negative or non-integer values. Where there exists one coefficient, one variable and one exponent, the polynomial function is written as follows:

(2) $$y = c \cdot x^n$$

For example, let c = 7 and n = 4. Thus, our polynomial is written as follows: $y = 7x^4$. The derivative of y with respect to x is given by the following function:

(3) $$\frac{dy}{dx} = c \cdot n \cdot x^{n-1}$$

Taking the derivative of y with respect to x in our example, we obtain: $dy/dx = 7 \cdot 4 \cdot x^3 = 28x^3$. Note that this derivative is always positive when x > 0; thus the slope of this curve is always positive when x > 0. Consider a second polynomial with more than one term (m terms total). In this second case, there will be one variable x, m coefficients (c_j) and m exponents (n_j):

(4) $$y = \sum_{j=1}^{m} c_j \cdot x^{n_j}$$

The derivative of such a function y with respect to x is given by:

(5) $$\frac{dy}{dx} = \sum_{j=1}^{m} c_j \cdot n_j \cdot x^{n_j - 1}$$

That is, simply take the derivative of each term in y with respect to x and sum these derivatives. Consider a second example, a second order (the largest exponent is 2) polynomial function given by: $y = 5x^2 - 3x + 2$. The derivative of this function with respect to x is: $dy/dx = 10x - 3$. This derivative is positive when x > .3, negative when x < .3 and zero when x = .3. Thus, when dy/dx > 0, y increases as x increases; when dy/dx < 0, y decreases as x increases, and when dy/dx = 0, y may be either minimized or maximized. Also notice that y is minimized when x = .3; at this point, dy/dx = 0.

As suggested above, derivatives can often be used to find minimum and maximum values of functions. To find the minimum value of y in function $y = 5x^2 - 3x + 2$, we set the first derivative of y with respect to x equal to zero and then solve for x. For our example, the minimum is found as follows:

$$10x - 3 = 0$$
$$10x = 3$$
$$x = \frac{3}{10}$$

In order to ensure that we have found a minimum (rather than a maximum), we check the second derivative. The second derivative is found by taking the derivative of the first derivative. If the second derivative is greater than zero, we have a minimum value for y (the function is concave up). When the second derivative is less than zero, we have a maximum (the function is concave down). If the second derivative is zero, we have neither a minimum nor a maximum. The second derivative in the above example is given by: $d^2y/dx^2 = 10$, also written $f''(x) = 10$. Since the second derivative 10 is greater than zero, we have found a minimum value for y. In many cases, more than one "local" minimum or maximum value will exist.

Consider a third example where our second order polynomial is given: $y = -7x^2 + 4x + 5$. The first derivative is: $dy/dx = -14x + 4$. Setting the first derivative equal to zero, we find our maximum as follows:

$$-14x + 4 = 0$$
$$-14x = -4$$
$$x = \frac{4}{14}$$

We check second order conditions (the second derivative) to ensure that this is a maximum. The second derivative is: $d^2y/dx^2 = -14$. Since -14 is less than zero, we have a maximum at 4/14.

1.