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ESTIMATION OF ZERO-COUPON YIELD CURVES BASED ON EXCHANGE FUND BILLS AND NOTES IN HONG KONG

Key Points :

- *Many central banks monitor on an on-going basis zero-coupon yield curves estimated on data on coupon-paying government bonds. Such yield curves have been shown to contain information about macroeconomic conditions, market expectations of future interest and inflation rates, and the pure time value of money of financial market participants.*

- *Empirical evidence shows that both the Svensson model and the Nelson and Siegel model, which are commonly used by central banks, fit the data for Exchange Fund Bills and Notes (EFBNs) very well. The Svensson model, with an additional hump, is slightly better than the Nelson and Siegel model.*

- *Both price and yield errors minimisation approaches give very good estimates of the zero-coupon yield curve. Although the yield errors minimisation method is computationally more difficult, it is better to use this approach in deriving the zero-coupon yield curve as the primary interest in the term structure analysis lies in the interest rates.*

- *Although the use of the term structure as an indicator of monetary policy and of expectations of future interest rates is limited in Hong Kong, a zero-coupon yield curve from EFBNs may be useful for further development of local bond market as a "benchmark" for pricing risk-bearing bonds.*

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I. INTRODUCTION

The yield curve, or the term structure of interest rates, of zero-coupon government bonds has been of great interest to most central banks in their daily monitoring of financial markets. Some empirical studies show that the shape of the yield curve may be useful in predicting the macroeconomic cycle (Monetary Authority of Singapore (1999)). Moreover, forward rates, or the slope of the yield curve, are informative of market expectations of future interest rates and inflation rates. They are used by several central banks including the Bank of England, the Federal Reserve and Sveriges Riksbank as monetary policy indicators. Furthermore, both forward and spot rates contain information about financial market participants' time value of money because they use these rates as benchmarks for their investment analysis and for pricing their bonds and other financial instruments.

There exist various methods to estimate zero-coupon yield curves. According to BIS (1999), most central banks have adopted either the Nelson and Siegel (1987) method or the extended version suggested by Svensson (1994), with the exceptions of US, Japan and United Kingdom which apply the "smoothing splines" method. This paper illustrates the application of the Nelson and Seigel model, and the Svensson model in deriving the zero-coupon yield curve for Hong Kong. The rest of the paper is organised as follows: Section II explains the usefulness of the zero-coupon yield curve, Section III discusses the two methods in details; Section IV describes the data and the estimation results. Conclusions are presented in the last section.

II. USEFULNESS OF THE ZERO-COUPON YIELD CURVE

The yield to maturity (YTM), the single discount rate on an investment that makes the sum of the present value of all cash flows equal to the current price of the investment, has been a common measure of the rate of return. However, using a single discount rate at different time periods is problematic because it assumes that all future cash flows from coupon payments will be reinvested at the derived YTM. This assumption neglects the reinvestment risk that creates investment uncertainty over the entire investment horizon. Another shortcoming of YTM is that the yields of bonds of the maturity depend on the patterns of their cash flows, which is often referred to as the coupon effect. As a result, the YTM of a coupon bond is not a good measure of the pure price of time and not the most appropriate yield measure in the term structure analysis.

On the other hand, zero-coupon securities eliminate the exposure to reinvestment risks as there is no cash flow to reinvest. The yields on the zero-coupon securities, called the *spot rate*, are not affected by the coupon effect since there are no coupon payments. Also, unlike the yield to maturity, securities having the same maturity have theoretically the same spot rates, which provide the *pure* price of time. As a result, it is preferable to work with zero-coupon yield curves rather than YTM when analysing the yield curve.

III. METHODOLOGY

One useful piece of information that can be derived from government bonds are the implied forward interest rates as they reflect the market's expectations about the future path of interest rates (Anderson and Sleath (1999)). The Svensson (1994) model (and the Nelson and Siegel (1987) model which is a restricted version of Svensson's model) has been preferred by many central banks because of its parsimonious nature to fit the implied forward rates from the raw data. Specifically, the Svensson model assumes that the instantaneous forward rate ($f(m,\beta)$) relates to the term to maturity as follows:

$$f(m,\beta) = \beta_0 + \beta_1 \exp\left(\frac{-m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(\frac{-m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(\frac{-m}{\tau_2}\right) \tag{1}$$

where m denotes time to maturity, $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$ is the parameter set to be estimated, and β_0, τ_1, τ_2 and $\beta_0+\beta_1$ are assumed to be greater than 0.[1]

The zero-coupon yield or spot rate $s(m,\beta)$, which gives the time value of money, is obtained by integrating the instantaneous forward rates:

[1] The values of τ_1 and τ_2 must be positive as these determine the positions of the "humps" of the yield curve. When the time to settlement approaches infinity, the forward rate approaches the constant β_0, which must be positive. When the time to settlement approaches zero, the forward rate approaches the constant $\beta_0 + \beta_1$, which must also be positive and should be equal to the overnight interbank rate.

$$s(m,\beta)=\beta_0+\beta_1*\left[\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}}\right]+\beta_2*\left[\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}}-\exp\left(-\frac{m}{\tau_1}\right)\right]$$

$$+\beta_3*\left[\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}}-\exp\left(-\frac{m}{\tau_2}\right)\right] \qquad (2)$$

Equation (2) implies that the zero-coupon yield $s(m,\beta)$ depends on the time to maturity (m) of the bond and the set of parameters (β). Given any spot rate $s(m,\beta)$, the discount factor $d(m,\beta)$ used to obtain the present value of future cash flows is given as:

$$d(m,\beta) = \exp\left(-\frac{s(m,\beta)}{100}m\right) \qquad (3)$$

For any coupon-bearing bond with maturity of T years, its price $P^e(\beta)$ can be approximated by the sum of the discounted semi-annual coupon payments (c_k) and the final principal (V) as follows:

$$P^e(\beta)=c_1*n*d(n,\beta)+\sum_{k=2}^{[2T]}c_k*d(n+0.5*(k-1))+V*d(T,\beta) \qquad (4)$$

where [2T] = 2T if 2T is an integer,
[2T] = (integral part of 2T) +1 if [2T] is a non-integer,
n is the number of years from trading date to the first coupon payment,
c_k is the k[th] coupon payment.

To obtain the parameters $\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2$, the approximated prices P^e (in terms of these parameters) are compared with the observed prices of all outstanding EFBNs. Precisely, these parameters are estimated by minimising the sum of squared bond-price errors weighted by $(1/\Phi)^2$:

2 BIS (1999) notes that using bond prices in the estimation irrespective to their durations will lead to over-fitting of the long-term bond prices at the expense of the short-term prices. To correct this problem, one approach is to weight the price error of each bond by the inverse of its duration.

$$\underset{\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2}{Min} \quad \sum_{j=1}^{n} \left\{ \left[P_j - P_j^{e} \left(\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2 \right) \right] / \Phi_j \right\}^2 \qquad (5)$$

where P_j is the observed price of bond *j*, *n* is the total number of bonds outstanding and Φ*j* equals the duration of bond *j*.

Instead of comparing the bond prices, an alternative method to obtain the parameters is to minimise the deviation between estimated and observed yields.[3] In this case, the estimation procedure involves two stages. First, a discount function similar to equation (3) is used to compute estimated prices, such that

$$P_j^{e}(\beta) = c_1 * n * d(n,\beta) + \sum_{k=2}^{[2T]} c_k * d(n + 0.5*(k-1)) + V * d(T,\beta) \qquad (6)$$

In the second stage, the estimated yield to maturity for the bond, denoted $Y_j^{e}(\beta)$, is estimated from equation (6) by solving the following equation

$$P_j^{e}(\beta) = \sum_{k=1} coupon * \exp(-Y_j^{e}(\beta)k/2) + V * \exp(-Y_j^{e}(\beta)T) \qquad (7)$$

The parameter set β is obtained by minimising the sum of squared yield errors between the observed yield to maturity Y_j and the corresponding estimated yield to maturity $Y_j^{e}(\beta)$

$$\underset{\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2}{Min} \quad \sum_{j=1}^{n} \left[Y_j - Y_j^{e}(\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2) \right]^2 \qquad (8)$$

Optimisations are performed by applying a non-linear minimisation procedure separately to the sum of squared (weighted) price errors in equation (5), and the sum of squared yield errors in equation (8), with different set of initial values, subject to the constraints on the parameter values. Once the estimated parameters are obtained, the implied forward rate and the zero-coupon yield (spot rate) curve can be computed by substituting these parameters into equations (1) and (2).

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[3] The choice of whether the sum of squared yield or price errors should be minimised depends on the purpose of the study. If the purpose of the analysis is to estimate interest rates, then it may be preferable to conduct the minimisation of yield errors.

Similar procedures can also be applied in the estimation of Nelson and Siegel's model. The only difference is that the third term of both the forward rate in equation (1) and spot rate in equation (2) under Sveenson's model does not exist in Nelson and Siegel's framework. In other words, there are only 4 parameters ($\beta_0, \beta_1, \beta_2, \tau_1$) to be determined under Nelson-Siegel's model.

IV. DATA AND RESULTS

a. DATA

The estimations are based on the price information of Exchange Fund Bills and Notes (EFBNs) on 11 March and 7 May 2002 obtained from Bloomberg. There are about 60 observations on each day. About half of them are quoted at Bloomberg's market consensus prices and the remainings are quoted with the theoretical prices calculated by Bloomberg, based on prices of similar bond of the same credit quality and rating.[4] The data used for deriving the yield curve in this study are entirely from market quotes to ensure greater accuracy. For each day, the maturity of the EFBNs with market quotes ranges from 190 to 3,555 days.

EFBNs are Hong Kong dollar debt securities issued by the Hong Kong Monetary Authority. They constitute direct, unsecured, unconditional and general obligations of the Hong Kong SAR Government for the account of the Exchange Fund and rank pari passu with all other unsecured indebtedness of the Hong Kong SAR Government for the account. The Bills are discount securities with maturities of 3 months, 6 months and 1 year. The Notes are coupon-bearing securities with maturities of 2, 3, 5, 7 and 10 years. All coupons are paid semi-annually. Both the Bills and the Notes are exempt from profits tax and stamp duty.

b. ESTIMATION RESULTS

Table 1 reports the estimation parameters for the zero-coupon yield curves by the Svensson and the Nelson-Siegel models on 11 March and 7 May 2002 based on the market-quoted EFBNs. The spot and implied forward rates estimated from these parameters are also presented in Charts 1 to 4.

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4 The estimation by Bloomberg is subject to error because of the liquidity premium on EFBN as the only eligible securities for discount window purpose.

From the table and charts, the following observations are noted:

- As shown from the statistics of R^2, both Svensson and Nelson-Seigel models fit the data very well, with the Svensson's model giving a better fit when yield errors are minimised.

- Both price and yield errors minimisation methods produce good fit to the data, as evidenced by the high R^2. However, the parameter estimates are sometimes sensitive to the method being used. For example, the estimate of β_0 is generally smaller in the price errors minimisation method than the yield errors minimisation method, with the difference being greater under the Svensson model. The difference implies that the estimated forward rate at long maturity will be larger in the yield errors minimisation method than the price one.[5]

- The forward rates estimated for 7 May 2002 are generally smaller than those for 11 March 2002. This may reflect that the interest rate outlook of the market has turned to more bearish recently. Such downward shift of the forward rate curve may essentially reflect the sentiment change in the US interest rate outlook under the linked exchange rate system.

V. CONCLUSION

The above analysis shows that the zero-coupon yield curves for EFBNs can be satisfactorily fitted by both Svensson and Nelson-Siegel models using either price or yield errors minimisation approach. Based on the R^2, the Svensson model, with an additional hump, fits the data slightly better than the Nelson-Siegel model irrespective of which minimisation method is used.

As discussed, both price and yield errors minimisation methods are seen to fit the data well. The yield errors minimisation approach is however computationally more difficult than the price one as it involves an extra iteration stage in the estimation process. As a result, convergence problems occur more frequently and the optimisation process is more sensitive to the choice of initial values. Nevertheless, since the primary interest for the yield curve analysis lies in the interest rates, it is better to use the yield errors minimisation approach.

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5 At the long end of the yield curve, the Svensson model is assumed to converge to a constant level given by β_0. The rationale for this constraint is from the assumption that as forward rates reflect expectations about future short interest rates, it seems implausible that agents will perceive a different path for the future short rate in 30 years time compared with 25 years.

The estimated spot and forward rates for the yield curves on 11 March 2002 and 7 May 2002 indicate a downward shift of the curves on the latter date. This may imply market anticipation of a delay in the rise of future interest rates. However, such shift essentially reflects the sentiment regarding US interest rate as Hong Kong does not have an independent monetary policy under the linked exchange rate system. As such, the use of yield curve as an indicator of monetary policy and expectations of future interest rates in Hong Kong is limited as the term structure may only reflect market conditions in the US.[6] Nevertheless, with further development of the local debt market, the derivation of a zero-coupon yield curve for government bonds is important as this can serve as a "benchmark" for pricing other risk-bearing bonds.

[6] A study of the term structure of short-term interbank rates in Hong Kong (Gerlach (2001)) found that term spreads – the slope of the yield curve at the short end – contain little information about future short-term rates. Instead, time-varying liquidity premia are found in the data.

Table 1. Estimation Results based on Market Consensus Prices of EFBNs

Estimation Method	**Yield Errors Minimisation**			
Estimation Parameters	**Svensson**	**Nelson & Siegel**	**Svensson**	**Nelson & Siegel**
	11 March 2002		**7 May 2002**	
β_0	7.69	7.14	7.42	6.86
β_1	-5.69	-5.14	-5.83	-5.27
β_2	-5.49	-4.79	-5.02	-4.67
β_3	-3.58	-	-2.57	-
τ_1	0.55	0.83	0.79	0.96
τ_2	1.87	-	2.63	-
R^2	0.9989	0.9968	0.9993	0.9986
# of observations	32	32	32	32

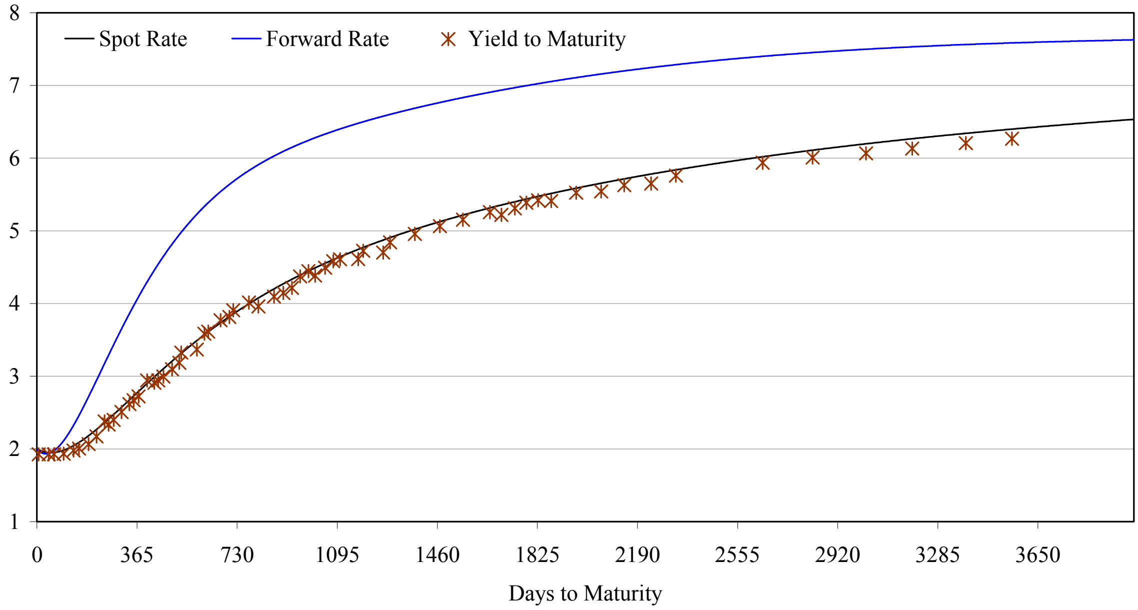
Estimation Method	**Price Errors Minimisation**			
Estimation Parameters	**Svensson**	**Nelson & Siegel**	**Svensson**	**Nelson & Siegel**
	11 March 2002		**7 May 2002**	
β_0	7.41	7.05	7.01	6.78
β_1	-5.41	-5.05	-5.43	-5.18
β_2	-5.03	-4.55	-3.78	-4.44
β_3	-4.43	-	-5.03	-
τ_1	0.44	0.84	0.52	0.98
τ_2	1.38	-	1.31	-
R^2	0.9999	0.9999	0.9999	0.9999
# of observations	32	32	32	32

Source: HKMA

Chart 1. Estimation Results of EFBN Zero-Coupon Yield Curve based on Market Consensus Prices of EFBNs

Estimation Method: Yield Errors Minimisation
11 March 2002

Svensson's Model



Nelson & Siegel's Model

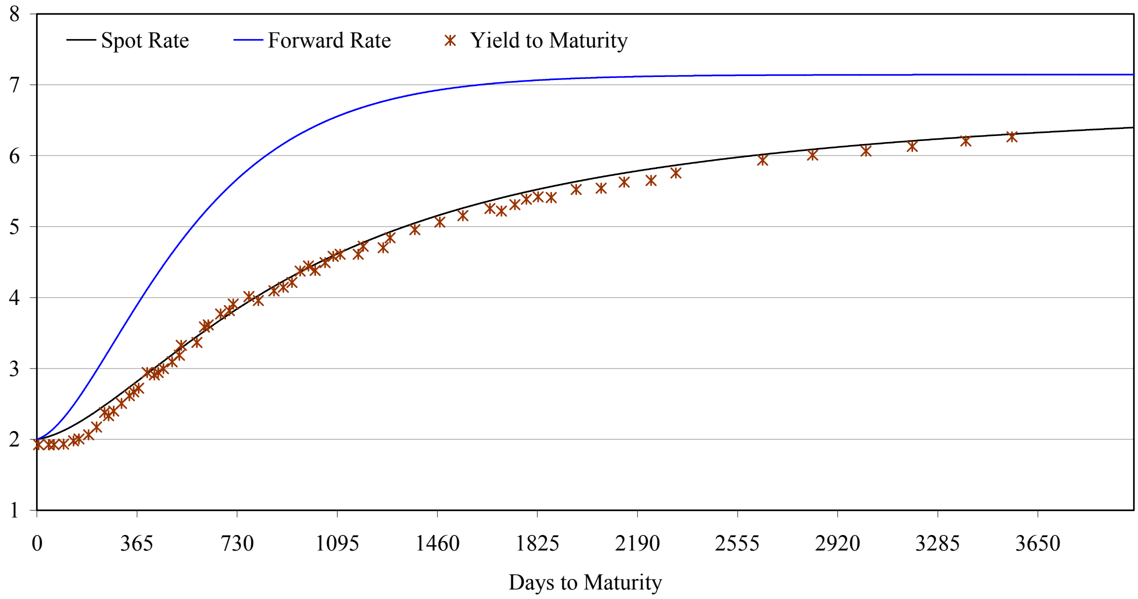
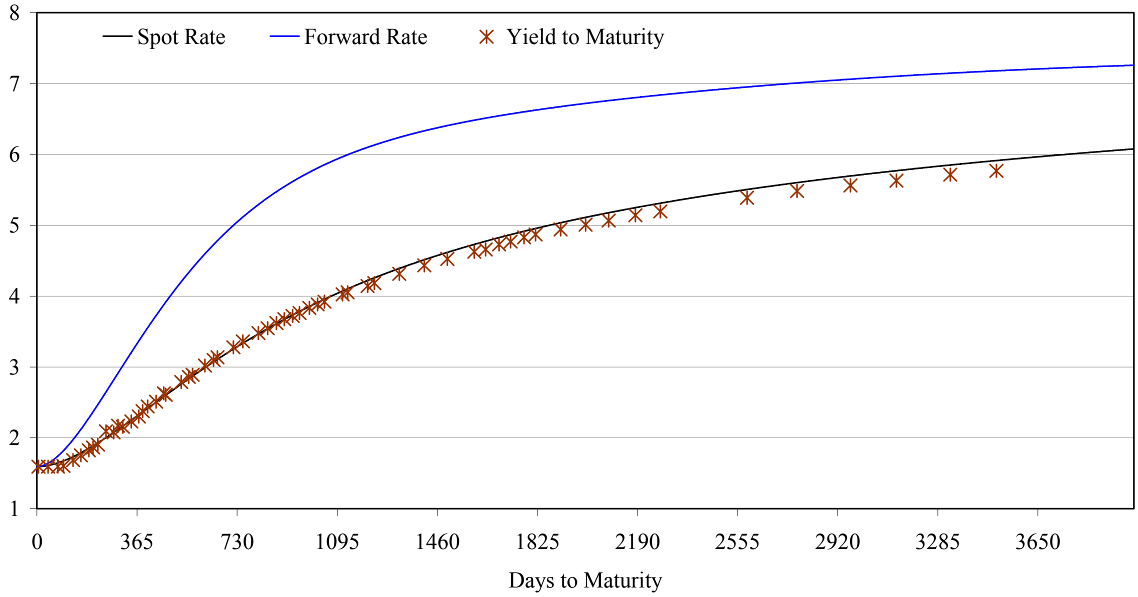


Chart 2. Estimation Results of EFBN Zero-Coupon Yield Curve based on Market Consensus Prices of EFBNs

Estimation Method: Yield Errors Minimisation
7 May 2002

Svensson's Model



Nelson & Siegel's Model

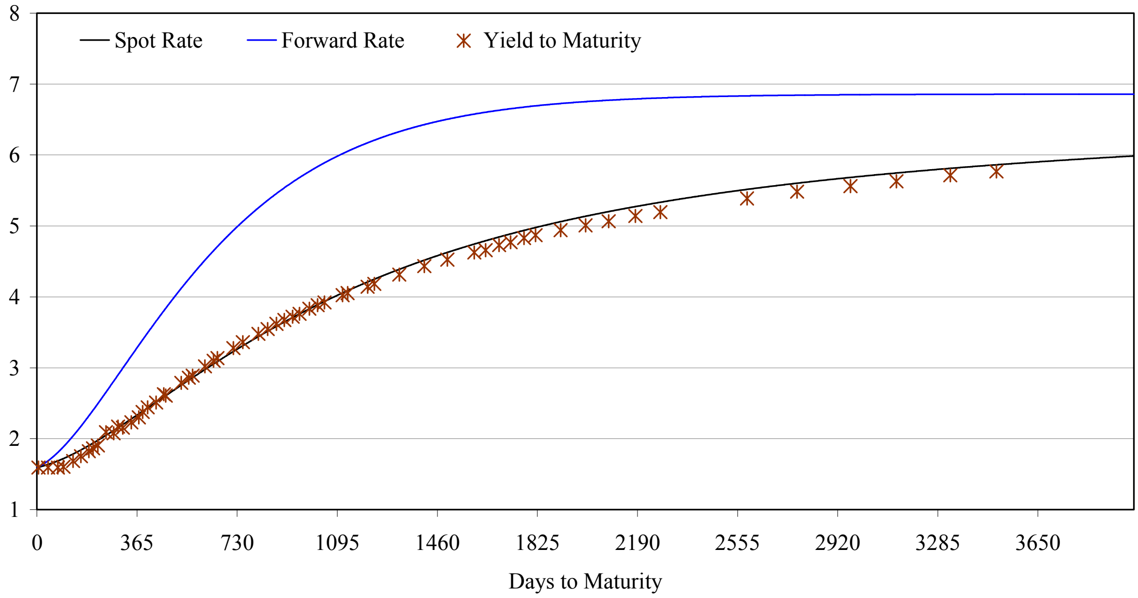
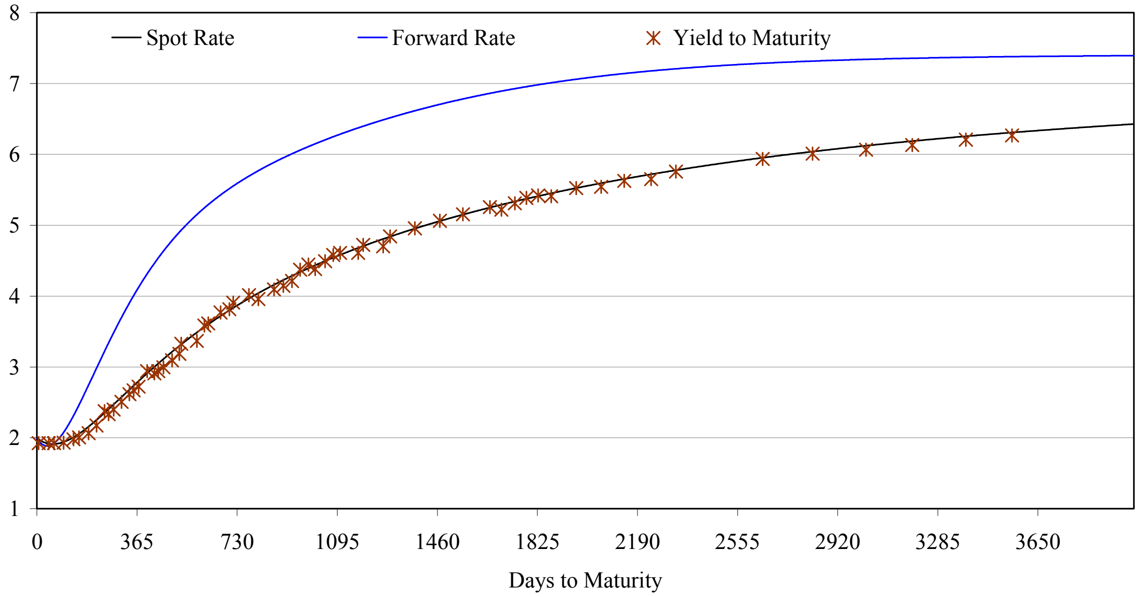


Chart 3. Estimation Results of EFBN Zero-Coupon Yield Curve based on Market Consensus Prices of EFBNs

Estimation Method: Price Errors Minimisation
11 March 2002

Svensson's Model



Nelson & Siegel's Model

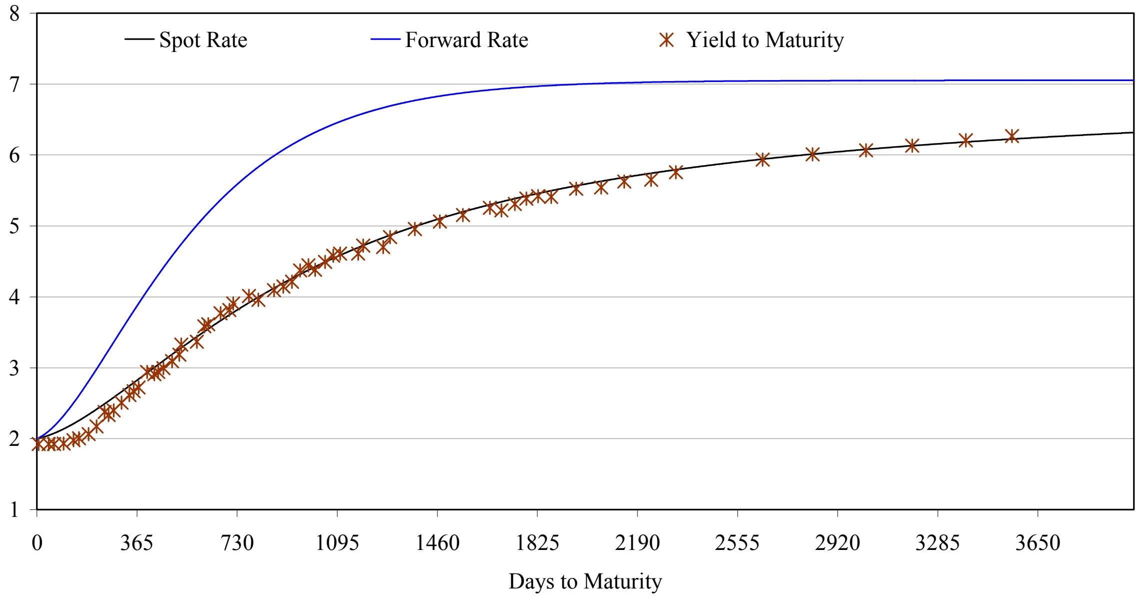
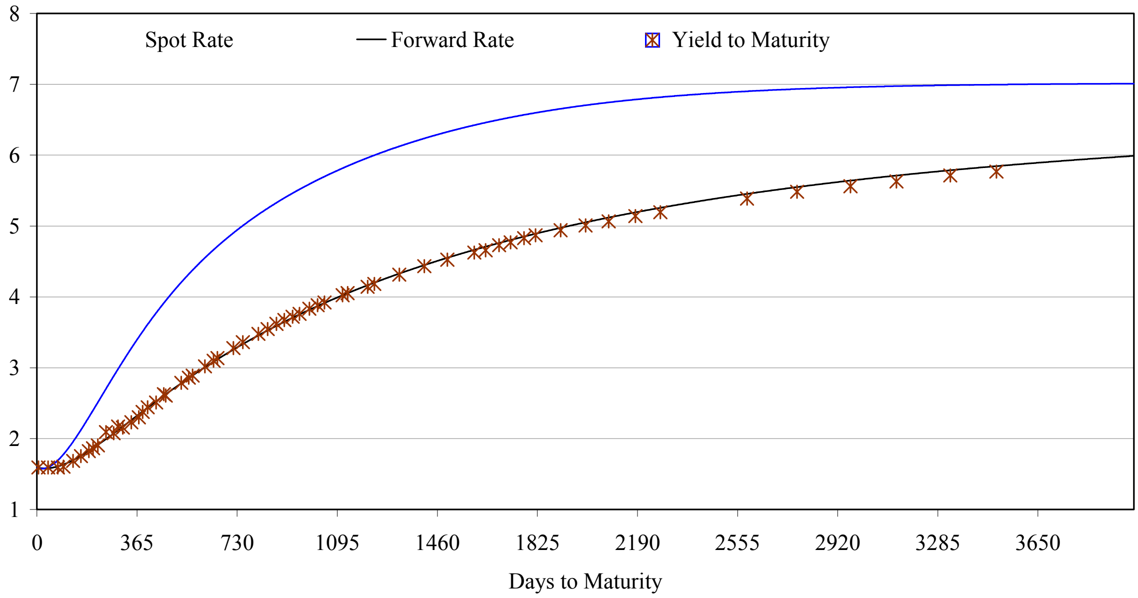


Chart 4. Estimation Results of EFBN Zero-Coupon Yield Curve based on Market Consensus Prices of EFBNs

Estimation Method: Price Errors Minimisation
7 May 2002

Svensson's Model



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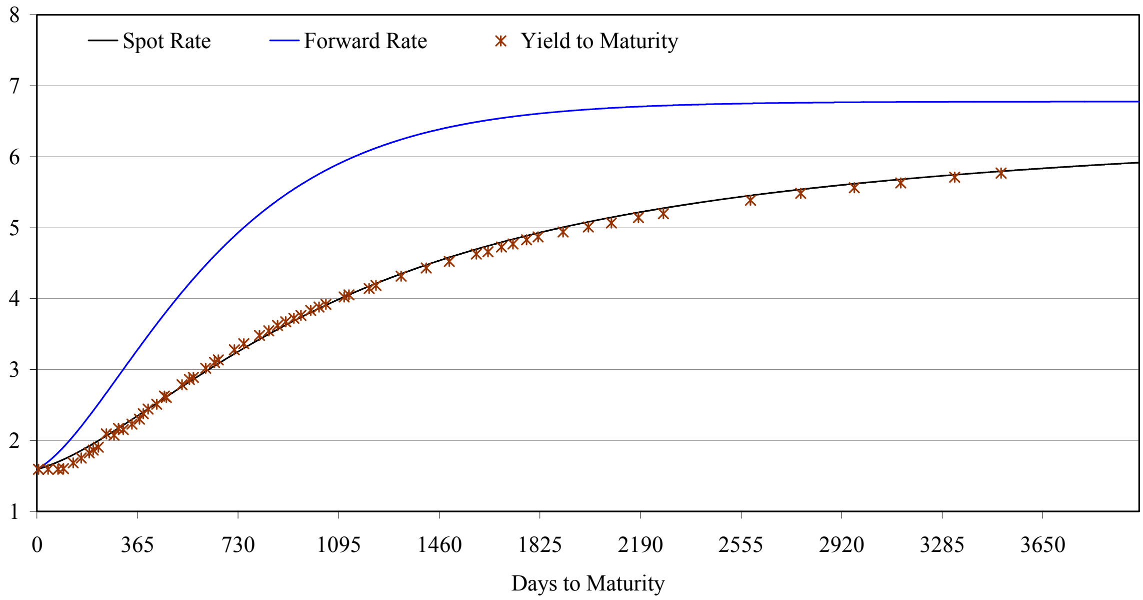

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