Chapter 2

Extracting yield curves from bond prices

2.1 Introduction

As discussed in Chapter 1, the clearest picture of the term structure of interest rates is obtained by looking at the yields of zero-coupon bonds of different maturities. However, most traded bonds are coupon bonds, not zero-coupon bonds. This chapter discusses methods to extract or estimate a zero-coupon yield curve from the prices of coupon bonds at a given point in time.

Section 2.2 considers the so-called bootstrapping technique. It is sometimes possible to construct zero-coupon bonds by forming certain portfolios of coupon bonds. If so, we can deduce an arbitrage-free price of the zero-coupon bond and transform it into a zero-coupon yield. This is the basic idea in the bootstrapping approach. Only in bond markets with sufficiently many coupon bonds with regular payment dates and maturities can the bootstrapping approach deliver a decent estimate of the whole zero-coupon yield curve. In other markets, alternative methods are called for.

We study two alternatives to bootstrapping in Sections 2.3 and 2.4. Both are based on the assumption that the discount function is of a given functional form with some unknown parameters. The value of these parameters are then estimated to obtain the best possible agreement between observed bond prices and theoretical bond prices computed using the functional form. Typically, the assumed functional forms are either polynomials or exponential functions of maturity or some combination. This is consistent with the usual perception that discount functions and yield curves are continuous and smooth. If the yield for a given maturity was much higher than the yield for another maturity very close to the first, most bond owners would probably shift from bonds with the low-yield maturity to bonds with the high-yield maturity. Conversely, bond issuers (borrowers) would shift to the low-yield maturity. These changes in supply and demand will cause the gap between the yields for the two maturities to shrink. Hence, the equilibrium yield curve should be continuous and smooth. The unknown parameters can be estimated by least-squares methods.

We focus here on two of the most frequently applied parameterization techniques, namely cubic splines and the Nelson-Siegel parameterization. An overview of some of the many other approaches suggested in the literature can be seen in Anderson, Breedon, Deacon, Derry, and Murphy (1996, Ch. 2). For some recent procedures, see Jaschke (1998) and Linton, Mammen, Nielsen, and Tanggaard (2001).

2.2 Bootstrapping

In many bond markets only very few zero-coupon bonds are issued and traded. (All bonds issued as coupon bonds will eventually become a zero-coupon bond after their next-to-last payment date.) Usually, such zero-coupon bonds have a very short maturity. To obtain knowledge of the market zero-coupon yields for longer maturities, we have to extract information from the prices of traded coupon bonds. In some markets it is possible to construct some longer-term zero-coupon bonds by forming portfolios of traded coupon bonds. Market prices of these "synthetical" zero-coupon bonds and the associated zero-coupon yields can then be derived.

Example 2.1 Consider a market where two bullet bonds are traded, a 10% bond expiring in one year and a 5% bond expiring in two years. Both have annual payments and a face value of 100. The one-year bond has the payment structure of a zero-coupon bond: 110 dollars in one year and nothing at all other points in time. A share of 1/110 of this bond corresponds exactly to a zero-coupon bond paying one dollar in a year. If the price of the one-year bullet bond is 100, the one-year discount factor is given by

$$B_t^{t+1} = \frac{1}{110} \cdot 100 \approx 0.9091.$$

The two-year bond provides payments of 5 dollars in one year and 105 dollars in two years. Hence, it can be seen as a portfolio of five one-year zero-coupon bonds and 105 two-year zero-coupon bonds, all with a face value of one dollar. The price of the two-year bullet bond is therefore

$$B_{2,t} = 5B_t^{t+1} + 105B_t^{t+2},$$

cf. (1.4). Isolating B_t^{t+2}, we get

$$B_t^{t+2} = \frac{1}{105}B_{2,t} - \frac{5}{105}B_t^{t+1}. \tag{2.1}$$

If for example the price of the two-year bullet bond is 90, the two-year discount factor will be

$$B_t^{t+2} = \frac{1}{105} \cdot 90 - \frac{5}{105} \cdot 0.9091 \approx 0.8139.$$

From (2.1) we see that we can construct a two-year zero-coupon bond as a portfolio of 1/105 units of the two-year bullet bond and $-5/105$ units of the one-year zero-coupon bond. This is equivalent to a portfolio of 1/105 units of the two-year bullet bond and $-5/(105 \cdot 110)$ units of the one-year bullet bond. Given the discount factors, zero-coupon rates and forward rates can be calculated as shown in Section 1.2. □

The example above can easily be generalized to more periods. Suppose we have M bonds with maturities of $1, 2, \ldots, M$ periods, respectively, one payment date each period and identical payment date. Then we can construct successively zero-coupon bonds for each of these maturities and hence compute the market discount factors $B_t^{t+1}, B_t^{t+2}, \ldots, B_t^{t+M}$. First, B_t^{t+1} is computed using the shortest bond. Then, B_t^{t+2} is computed using the next-to-shortest bond and the already computed value of B_t^{t+1}, etc. Given the discount factors $B_t^{t+1}, B_t^{t+2}, \ldots, B_t^{t+M}$, we can compute the zero-coupon interest rates and hence the zero-coupon yield curve up to time $t + M$ (for the M selected maturities). This approach is called **bootstrapping** or **yield curve stripping**.

Bootstrapping also applies to the case where the maturities of the M bonds are not all different and regularly increasing as above. As long as the M bonds together have at most M different payment dates and each bond has at most one payment date, where none of the bonds provide payments, then we can construct zero-coupon bonds for each of these payment dates and compute the associated discount factors and rates. Let us denote the payment of bond i ($i = 1, \ldots, M$) at time $t + j$ ($j = 1, \ldots, M$) by Y_{ij}. Some of these payments may well be zero, e.g. if the bond matures before time $t + M$. Let $B_{i,t}$ denote the price of bond i. From (1.4) we have that the discount factors $B_t^{t+1}, B_t^{t+2}, \ldots, B_t^{t+M}$ must satisfy the system of equations

$$\begin{pmatrix} B_{1,t} \\ B_{2,t} \\ \vdots \\ B_{M,t} \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} & \ldots & Y_{1M} \\ Y_{21} & Y_{22} & \ldots & Y_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{M1} & Y_{M2} & \ldots & Y_{MM} \end{pmatrix} \begin{pmatrix} B_t^{t+1} \\ B_t^{t+2} \\ \vdots \\ B_t^{t+M} \end{pmatrix}. \tag{2.2}$$

The conditions on the bonds ensure that the payment matrix of this equation system is non-singular so that a unique solution will exist.

For each of the payment dates $t + j$, we can construct a portfolio of the M bonds, which is equivalent to a zero-coupon bond with a payment of 1 at time $t + j$. Denote by $x_i(j)$ the number of units of bond i which enters the portfolio replicating the zero-coupon bond maturing at $t + j$. Then we must have that

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{21} & \ldots & \ldots & \ldots & Y_{M1} \\ Y_{12} & Y_{22} & \ldots & \ldots & \ldots & Y_{M2} \\ \vdots & \vdots & \ddots & & & \vdots \\ Y_{1j} & Y_{2j} & \ldots & \ldots & \ldots & Y_{Mj} \\ \vdots & \vdots & & & \ddots & \vdots \\ Y_{1M} & Y_{2M} & \ldots & \ldots & \ldots & Y_{MM} \end{pmatrix} \begin{pmatrix} x_1(j) \\ x_2(j) \\ \vdots \\ x_j(j) \\ \vdots \\ x_M(j) \end{pmatrix}, \tag{2.3}$$

where the 1 on the left-hand side of the equation is at the j'th entry of the vector. Of course, there will be the following relation between the solution $(B_t^{t+1}, \ldots, B_t^{t+M})$ to (2.2) and the solution $(x_1(j), \ldots, x_M(j))$ to (2.3):[1]

$$\sum_{i=1}^{M} x_i(j) B_{i,t} = B_t^{t+j}. \tag{2.4}$$

Thus, first the zero-coupon bonds can be constructed, i.e. (2.3) is solved for each $j = 1, \ldots, M$, and next (2.4) can be applied to compute the discount factors.

Example 2.2 In Example 2.1 we considered a two-year 5% bullet bond. Assume now that a two-year 8% serial bond with the same payment dates is traded. The payments from this bond are 58 dollars in one year and 54 dollars in two years. Assume that the price of the serial bond

[1] In matrix notation, Equation (2.2) can be written as $\boldsymbol{B}_{\text{cpn}} = \mathbf{Y}\boldsymbol{B}_{\text{zero}}$ and Equation (2.3) can be written as $\boldsymbol{e}_j = \mathbf{Y}^\top \boldsymbol{x}(j)$, where $\boldsymbol{e}_j$ is the vector on the left hand side of (2.3), and the other symbols are self-explanatory (the symbol $^\top$ indicates transposition). Hence,

$$\boldsymbol{x}(j)^\top \boldsymbol{B}_{\text{cpn}} = \boldsymbol{x}(j)^\top \mathbf{Y}\boldsymbol{B}_{\text{zero}} = \boldsymbol{e}_j^\top \boldsymbol{B}_{\text{zero}} = B_t^{t+j},$$

which is equivalent to (2.4).

is 98 dollars. From these two bonds we can set up the following equation system to solve for the discount factors B_t^{t+1} and B_t^{t+2}:

$$\begin{pmatrix} 90 \\ 98 \end{pmatrix} = \begin{pmatrix} 5 & 105 \\ 58 & 54 \end{pmatrix} \begin{pmatrix} B_t^{t+1} \\ B_t^{t+2} \end{pmatrix}.$$

The solution is $B_t^{t+1} \approx 0.9330$ and $B_t^{t+2} \approx 0.8127$. □

More generally, if there are M traded bonds having in total N different payment dates, the system (2.2) becomes one of M equations in N unknowns. If $M > N$, the system may not have any solution, since it may be impossible to find discount factors consistent with the prices of all M bonds. If no such solution can be found, there will be an arbitrage opportunity.

Example 2.3 In the Examples 2.1 and 2.2 we have considered three bonds: a one-year bullet bond, a two-year bullet bond, and a two-year serial bond. In total, these three bonds have two different payment dates. According to the prices and payments of these three bonds, the discount factors B_t^{t+1} and B_t^{t+2} must satisfy the following three equations:

$$\begin{aligned} 100 &= 110 B_t^{t+1}, \\ 90 &= 5 B_t^{t+1} + 105 B_t^{t+2}, \\ 98 &= 58 B_t^{t+1} + 54 B_t^{t+2}. \end{aligned}$$

No solution exists. In Example 2.1 we found that the solution to the first two equations is

$$B_t^{t+1} \approx 0.9091 \qquad \text{and} \qquad B_t^{t+2} \approx 0.8139.$$

In contrast, we found in Example 2.2 that the solution to the last two equations is

$$B_t^{t+1} \approx 0.9330 \qquad \text{and} \qquad B_t^{t+2} \approx 0.8127.$$

If the first solution is correct, the price on the serial bond should be

$$58 \cdot 0.9091 + 54 \cdot 0.8139 \approx 96.68, \tag{2.5}$$

but it is not. The serial bond is mispriced relative to the two bullet bonds. More precisely, the serial bond is too expensive. We can exploit this by selling the serial bond and buying a portfolio of the two bullet bonds that *replicates* the serial bond, i.e. provides the same cash flow. We know that the serial bond is equivalent to a portfolio of 58 one-year zero-coupon bonds and 54 two-year zero-coupon bonds, all with a face value of 1 dollar. In Example 2.1 we found that the one-year zero-coupon bond is equivalent to 1/110 units of the one-year bullet bond, and that the two-year zero-coupon bond is equivalent to a portfolio of $-5/(105 \cdot 110)$ units of the one-year bullet bond and 1/105 units of the two-year bullet bond. It follows that the serial bond is equivalent to a portfolio consisting of

$$58 \cdot \frac{1}{110} - 54 \cdot \frac{5}{105 \cdot 110} \approx 0.5039$$

units of the one-year bullet bond and

$$54 \cdot \frac{1}{105} \approx 0.5143$$

units of the two-year bullet bond. This portfolio will give exactly the same cash flow as the serial bond, i.e. 58 dollars in one year and 54 dollars in two years. The price of the portfolio is

$$0.5039 \cdot 100 + 0.5143 \cdot 90 \approx 96.68,$$

which is exactly the price found in (2.5). □

In some markets, the government bonds are issued with many different payment dates. The system (2.2) will then typically have fewer equations than unknowns. In that case there are many solutions to the equation system, i.e. many sets of discount factors can be consistent both with observed prices and the no-arbitrage pricing principle.

2.3 Cubic splines

Bootstrapping can only provide knowledge of the discount factors for (some of) the payment dates of the traded bonds. In many situations information about market discount factors for other future dates will be valuable. In this section and the next, we will consider methods to estimate the entire discount function $T \mapsto B_t^T$ (at least up to some large T). To simplify the notation in what follows, let $\bar{B}(\tau)$ denote the discount factor for the next τ periods, i.e. $\bar{B}(\tau) = B_t^{t+\tau}$. Hence, the function $\bar{B}(\tau)$ for $\tau \in [0, \infty)$ represents the time t market discount function. In particular, $\bar{B}(0) = 1$. We will use a similar notation for zero-coupon rates and forward rates: $\bar{y}(\tau) = y_t^{t+\tau}$ and $\bar{f}(\tau) = f_t^{t+\tau}$. The methods studied in this and the following sections are both based on the assumption that the discount function $\tau \mapsto \bar{B}(\tau)$ can be described by some functional form involving some unknown parameters. The parameter values are chosen to get a close match between the observed bond prices and the theoretical bond prices computed using the assumed discount function.

The approach studied in this section is a version of the cubic splines approach introduced by McCulloch (1971) and later modified by McCulloch (1975) and Litzenberger and Rolfo (1984). The word spline indicates that the maturity axis is divided into subintervals and that the separate functions (of the same type) are used to describe the discount function in the different subintervals. The reasoning for doing this is that it can be quite hard to fit a relatively simple functional form to prices of a large number of bonds with very different maturities. To ensure a continuous and smooth term structure of interest rates, one must impose certain conditions for the maturities separating the subintervals.

Given prices for M bonds with time-to-maturities of $T_1 \leq T_2 \leq \cdots \leq T_M$. Divide the maturity axis into subintervals defined by the "knot points" $0 = \tau_0 < \tau_1 < \cdots < \tau_k = T_M$. A spline approximation of the discount function $\bar{B}(\tau)$ is based on an expression like

$$\bar{B}(\tau) = \sum_{j=0}^{k-1} G_j(\tau) I_j(\tau),$$

where the G_j's are basis functions, and the I_j's are the step functions

$$I_j(\tau) = \begin{cases} 1, & \text{if } \tau \geq \tau_j, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\bar{B}(\tau) = G_0(\tau)$ for $\tau \in [\tau_0, \tau_1)$, $\bar{B}(\tau) = G_0(\tau) + G_1(\tau)$ for $\tau \in [\tau_1, \tau_2)$, etc. We demand that the G_j's are continuous and differentiable and ensure a smooth transition in the knot points τ_j. A polynomial spline is a spline where the basis functions are polynomials. Let us consider a cubic spline, where

$$G_j(\tau) = \alpha_j + \beta_j(\tau - \tau_j) + \gamma_j(\tau - \tau_j)^2 + \delta_j(\tau - \tau_j)^3,$$

and $\alpha_j, \beta_j, \gamma_j$, and δ_j are constants.

For $\tau \in [0, \tau_1)$, we have

$$\bar{B}(\tau) = \alpha_0 + \beta_0\tau + \gamma_0\tau^2 + \delta_0\tau^3. \tag{2.6}$$

Since $\bar{B}(0) = 1$, we must have $\alpha_0 = 1$. For $\tau \in [\tau_1, \tau_2)$, we have

$$\bar{B}(\tau) = \left(1 + \beta_0\tau + \gamma_0\tau^2 + \delta_0\tau^3\right) + \left(\alpha_1 + \beta_1(\tau - \tau_1) + \gamma_1(\tau - \tau_1)^2 + \delta_1(\tau - \tau_1)^3\right). \tag{2.7}$$

To get a smooth transition between (2.6) and (2.7) in the point $\tau = \tau_1$, we demand that

$$\bar{B}(\tau_1-) = \bar{B}(\tau_1+), \tag{2.8}$$

$$\bar{B}'(\tau_1-) = \bar{B}'(\tau_1+), \tag{2.9}$$

$$\bar{B}''(\tau_1-) = \bar{B}''(\tau_1+), \tag{2.10}$$

where $\bar{B}(\tau_1-) = \lim_{\tau\to\tau_1,\tau<\tau_1} \bar{B}(\tau)$, $\bar{B}(\tau_1+) = \lim_{\tau\to\tau_1,\tau>\tau_1} \bar{B}(\tau)$, etc. The condition (2.8) ensures that the discount function is continuous in the knot point τ_1. The condition (2.9) ensures that the graph of the discount function has no kink at τ_1 by restricting the first-order derivative to approach the same value whether t approaches τ_1 from below or from above. The condition (2.10) requires the same to be true for the second-order derivative, which ensures an even smoother behavior of the graph around the knot point τ_1.

The condition (2.8) implies $\alpha_1 = 0$. Differentiating (2.6) and (2.7), we find

$$\bar{B}'(\tau) = \beta_0 + 2\gamma_0\tau + 3\delta_0\tau^2, \quad 0 \le \tau < \tau_1,$$

and

$$\bar{B}'(\tau) = \beta_0 + 2\gamma_0\tau + 3\delta_0\tau^2 + \beta_1 + 2\gamma_1(\tau - \tau_1) + 3\delta_1(\tau - \tau_1)^2, \quad \tau_1 \le \tau < \tau_2.$$

The condition (2.9) now implies $\beta_1 = 0$. Differentiating again, we get

$$\bar{B}''(\tau) = 2\gamma_0 + 6\delta_0\tau, \quad 0 \le \tau < \tau_1,$$

and

$$\bar{B}''(\tau) = 2\gamma_0 + 6\delta_0\tau + 2\gamma_1 + 6\delta_1(\tau - \tau_1), \quad \tau_1 \le \tau < \tau_2.$$

Consequently, the condition (2.10) implies $\gamma_1 = 0$. Similarly, it can be shown that $\alpha_j = \beta_j = \gamma_j = 0$ for all $j = 1, \ldots, k-1$. The cubic spline is therefore reduced to

$$\bar{B}(\tau) = 1 + \beta_0\tau + \gamma_0\tau^2 + \delta_0\tau^3 + \sum_{j=1}^{k-1} \delta_j(\tau - \tau_j)^3 I_j(\tau). \tag{2.11}$$

Let $t_1, t_2, \ldots, t_N$ denote the time distance from today (date t) to the each of the payment dates in the set of all payment dates of the bonds in the data set. Let Y_{in} denote the payment of bond i in t_n periods. From the no-arbitrage pricing relation (1.4), we should have that

$$B_i = \sum_{n=1}^{N} Y_{in}\bar{B}(t_n),$$

where B_i is the current market price of bond i. Since not all the zero-coupon bonds involved in this equation are traded, we will allow for a deviation ε_i so that

$$B_i = \sum_{n=1}^{N} Y_{in} \bar{B}(t_n) + \varepsilon_i. \tag{2.12}$$

We assume that ε_i is normally distributed with mean zero and variance σ^2 (assumed to be the same for all bonds) and that the deviations for different bonds are mutually independent. We want to pick parameter values that minimize the sum of squared deviations $\sum_{i=1}^{M} \varepsilon_i^2$.

Substituting (2.11) into (2.12) yields

$$B_i = \sum_{n=1}^{N} Y_{in} \left\{ 1 + \beta_0 t_n + \gamma_0 t_n^2 + \delta_0 t_n^3 + \sum_{j=1}^{k-1} \delta_j (t_n - \tau_j)^3 I_j(\tau) \right\} + \varepsilon_i,$$

which implies that

$$B_i - \sum_{n=1}^{N} Y_{in} = \beta_0 \sum_{n=1}^{N} Y_{in} t_n + \gamma_0 \sum_{n=1}^{N} Y_{in} t_n^2 + \delta_0 \sum_{n=1}^{N} Y_{in} t_n^3 + \sum_{j=1}^{k-1} \delta_j \sum_{n=1}^{N} Y_{in} (t_n - \tau_j)^3 I_j(t_n) + \varepsilon_i.$$

Given the prices and payment schemes of the M bonds, the $k+2$ parameters $\beta_0, \gamma_0, \delta_0, \delta_1, \ldots, \delta_{k-1}$ can now be estimated using ordinary least squares.[2] Substituting the estimated parameters into (2.11), we get an estimated discount function, from which estimated zero-coupon yield curves and forward rate curves can be derived as explained earlier in the chapter.

It remains to describe how the number of subintervals k and the knot points τ_j are to be chosen. McCulloch suggested to let k be the nearest integer to $\sqrt{M}$ and to define the knot points by

$$\tau_j = T_{h_j} + \theta_j (T_{h_j+1} - T_{h_j}),$$

where $h_j = [j \cdot M/k]$ (here the square brackets mean the integer part) and $\theta_j = j \cdot M/k - h_j$. In particular, $\tau_k = T_M$. Alternatively, the knot points can be placed at for example 1 year, 5 years, and 10 years, so that the intervals broadly correspond to the short-term, intermediate-term, and long-term segments of the market, cf. the preferred habitats hypothesis discussed in Section 5.7.

Figure 2.1 shows the discount function on the Danish government bond markets on February 14, 2000 estimated using cubic splines and data from 14 bonds with maturities up to 25 years. Figure 2.2 shows the associated zero-coupon yield curve and the term structure of forward rates.

Discount functions estimated using cubic splines will usually have a credible form for maturities less than the longest maturity in the data set. Although there is nothing in the approach that ensures that the resulting discount function is positive and decreasing, as it should be according to (1.1), this will almost always be the case. As the maturity approaches infinity, the cubic spline discount function will approach either plus or minus infinity depending on the sign of the coefficient of the third order term. Of course, both properties are unacceptable, and the method cannot be expected to provide reasonable values beyond the longest maturity T_M, since none of the bonds are affected by that very long end of the term structure.

Two other properties of the cubic splines approach are more disturbing. First, the derived zero-coupon rates will often increase or decrease significantly for maturities approaching T_M, cf.

[2] See, for example, Johnston (1984).

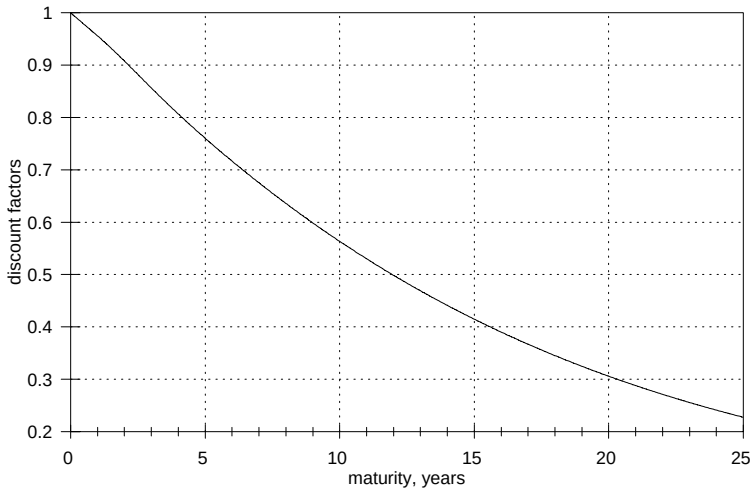

Figure 2.1: The discount function, $\tau \mapsto \bar{B}(\tau)$, estimated using cubic splines and prices of Danish government bonds February 14, 2000.

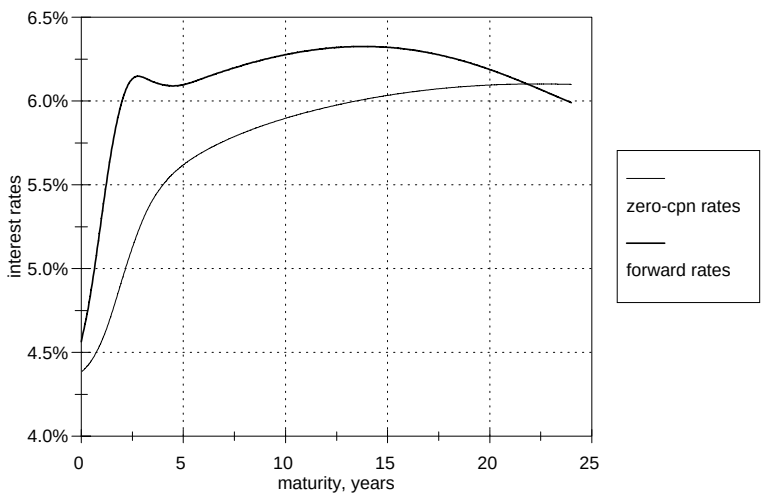

Figure 2.2: The zero-coupon yield curve, $\tau \mapsto \bar{y}(\tau)$, and the term structure of forward rates, $\tau \mapsto \bar{f}(\tau)$, estimated using cubic splines and prices of Danish government bonds February 14, 2000.

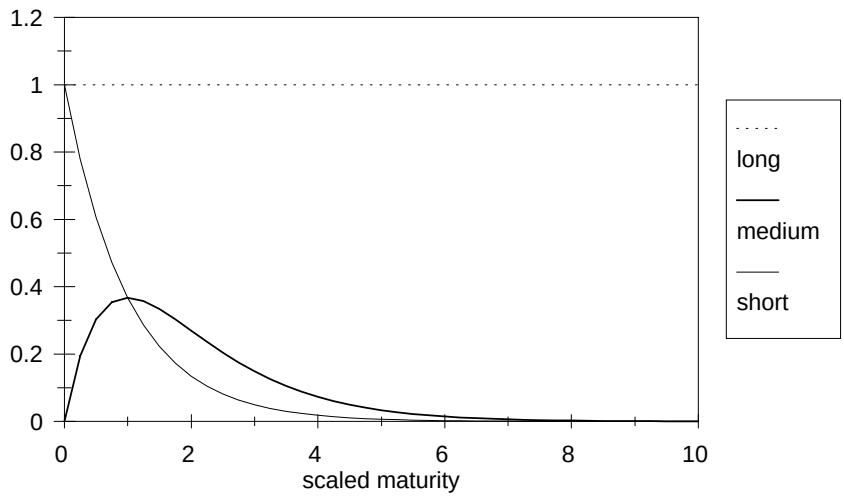

Figure 2.3: The three curves which the Nelson-Siegel parameterization combines.

Shea (1984, 1985). Second, the derived forward rate curve will typically be quite rugged especially near the knot points, and the curve tends to be very sensitive to the bond prices and the precise location of the knot points. Therefore, forward rate curves estimated using cubic splines should only be applied with great caution.

2.4 The Nelson-Siegel parameterization

Nelson and Siegel (1987) proposed a simple parameterization of the term structure of interest rates, which has become quite popular. The approach is based on the following parameterization of the forward rates:

$$\bar{f}(\tau) = \beta_0 + \beta_1 e^{-\tau/\theta} + \beta_2 \frac{\tau}{\theta} e^{-\tau/\theta}, \tag{2.13}$$

where β_0, β_1, β_2, and θ are constants to be estimated. The same constants are assumed to apply for all maturities, so no splines are involved. The simple functional form ensures a smooth and yet quite flexible curve. Figure 2.3 shows the graphs of the three functions that constitutes (2.13). The flat curve (corresponding to the constant term β_0) will by itself determine the long-term forward rates, the term $\beta_1 e^{-\tau/\theta}$ is mostly affecting the short-term forward rates, while the term $\beta_2 \tau/\theta e^{-\tau/\theta}$ is important for medium-term forward rates. The value of the parameter θ determines how large a maturity interval the non-constant terms will affect. The value of the parameters β_0, β_1, and β_2 determine the relative weighting of the three curves.

According to (1.20) on page 8, the term structure of zero-coupon rates is given by

$$\bar{y}(\tau) = \frac{1}{\tau} \int_0^\tau \bar{f}(u)\, du = \beta_0 + (\beta_1 + \beta_2) \frac{1 - e^{-\tau/\theta}}{\tau/\theta} - \beta_2 e^{-\tau/\theta},$$

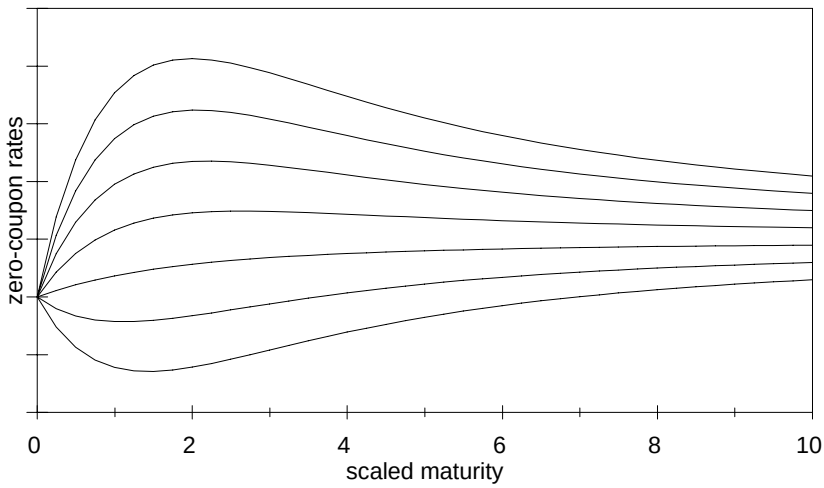

Figure 2.4: Possible forms of the zero-coupon yield curve using the Nelson-Siegel parameterization.

which we will rewrite as

$$\bar{y}(\tau) = a + b\frac{1 - e^{-\tau/\theta}}{\tau/\theta} + ce^{-\tau/\theta}. \tag{2.14}$$

Figure 2.4 depicts the possible forms of the zero-coupon yield curve for different values of a, b, and c. By varying the parameter θ, the curves can be stretched or compressed in the horizontal dimension.

If we could directly observe zero-coupon rates $\bar{y}(T_i)$ for different maturities T_i, $i = 1, \ldots, M$, we could, given θ, estimate the parameters a, b, and c using simple linear regression on the model

$$\bar{y}(\tau) = a + b\frac{1 - e^{-\tau/\theta}}{\tau/\theta} + ce^{-\tau/\theta} + \varepsilon_i,$$

where $\varepsilon_i \sim N(0, \sigma^2)$, $i = 1, \ldots, M$, are independent error terms. Doing this for various choices of θ, we could pick the θ and the corresponding regression estimates of a, b and c that result in the highest R^2, i.e. that best explain the data. This is exactly the procedure used by Nelson and Siegel on data on short-term zero-coupon bonds in the U.S. market.

When the data set involves coupon bonds, the estimation procedure is slightly more complicated. The discount function associated with the forward rate structure in (2.14) is given by

$$\bar{B}(\tau) = \exp\left\{-a\tau - b\theta\left(1 - e^{-\tau/\theta}\right) - c\tau e^{-\tau/\theta}\right\}.$$

Substituting this into (2.12), we get

$$B_i = \sum_{n=1}^{N} Y_{in} \exp\left\{-at_n - b\theta\left(1 - e^{-t_n/\theta}\right) - ct_n e^{-t_n/\theta}\right\} + \varepsilon_i. \tag{2.15}$$

Since this is a non-linear expression in the unknown parameters, the estimation must be based on generalized least squares, i.e. non-linear regression techniques. See e.g. Gallant (1987).

2.5 Additional remarks on yield curve estimation

Above we looked at two of the many estimation procedures based on a given parameterized form of either the discount function, the zero-coupon yield curve, or the forward rate curve. A clear disadvantage of both methods is that the estimated discount function is not necessarily consistent with those (probably few) discount factors that can be derived from market prices assuming only no-arbitrage. The procedures do no punish deviations from no-arbitrage values.

A more essential disadvantage of all such estimation procedures is that they only consider the term structure of interest rates at one particular point in time. Estimations at two different dates are completely independent and do not take into account the possible dynamics of the term structure over time. As we shall see in Chapters 7 and 8, there are many dynamic term structure models which also provide a parameterized form for the term structure at any given date. Applying such models, the estimation can (and should) be based on bond price observations at different dates. Typically, the possible forms of the term structure in such models resemble those of the Nelson-Siegel approach. We will return to this discussion in Chapter 7.

Finally, we will emphasize that the estimated term structure of interest rates should be used with caution. An obvious use of the estimated yield curve is to value fixed income securities. In particular, the coupon bonds in the data set used in the estimation can be priced using the estimated discount function. For some of the bonds the price according to the estimated curve will be lower (higher) than the market price. Therefore, one might think such bonds are overvalued (undervalued) by the market. (In an estimation like (2.12) this can be seen directly from the residual ε_i.) It would seem a good strategy to sell the overvalued and buy the undervalued bonds. However, such a strategy is not a riskless arbitrage, but a risky strategy, since the applied discount function is not derived from the no-arbitrage principle only, but depends on the assumed parametric form and the other bonds in the data set. With another parameterized form or a different set of bonds the estimated discount function and, hence, the assessment of over- and undervaluation can be different.

2.6 Exercises

EXERCISE 2.1 Find a list of current price quotes on government bonds at an exchange in your country. Derive as many discount factors and zero-coupon yields as possible using only the no-arbitrage pricing principle, i.e. use the bootstrapping approach.