Topics in Mathematical Finance

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A Dissertation submitted for the Degree of Doctor of Philosophy

To the memory of my mother

Contents

Preface

In the course of the past three years, I have benefited from the help of many people. In particular, I would like to thank my supervisor, Doug Kennedy, for his guidance and support, and my long-suffering office-mate, James Martin, for his patience. I also thank Martin Baxter for supplying the data for Section 3.4 and the Engineering and Physical Sciences Research Council for financial assistance.

This dissertation is dedicated to the memory of my mother, who died shortly before its completion.

The work in this dissertation is entirely my own, and includes nothing which is the outcome of work done in collaboration. It is not substantially the same as any that I have submitted for any degree or diploma or other qualification at any University.

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17th January, 1999

1

Introduction

In this thesis we consider two problems in the area of option pricing. The fundamental problem of option pricing is the following: suppose S_t denotes the price of some stock t years hence, and we are offered a contract which promises to pay £X in one years time, where X is specified as a function of $\{S_t : t \in [0,1]\}$, the path taken by the stock in the coming year. How much is this contract worth today? An elegant answer to this question is provided by the 'arbitrage-free' pricing methodology pioneered by Black & Scholes (1973), Merton (1973), Harrison & Kreps (1979) and Harrison & Pliska (1981). The premise is that any realistic financial market should not permit the manufacture of a risk-free profit from zero initial capital, the so-called *no-arbitrage* assumption. It turns out that for many classes of stock models, there is a common, unique price for X consistent with the no-arbitrage property. The most tractable of these, the Black-Scholes model, is still an area of active research; in Chapter 3 we will develop efficient techniques for calculating the arbitrage-free price of various types of option in this model. This model frequently arises when the underlying stochastic processes are assumed to be lognormal; in Chapter 2 we examine some theoretical properties of a class of interest-rate models based on a Gaussian random field. The problem of option pricing in one of these models often reduces to a calculation in a Black-Scholes model.

The remainder of this chapter is divided as follows: in Sections 1.1 and 1.2 we describe the arbitrage-free pricing methodology, first in discrete time and then continuous time. In Section 1.3 we present the Black-Scholes model; finally in Section 1.4 we give an introduction to interest-rate models.

1.1 Option pricing in discrete time

First consider a finite, discrete time model for the stock price: S_n, $n = 0, 1, \ldots, N$, where S_0 is deterministic, and the sample space Ω is finite. Let $\mathcal{F}_n$ denote the history of the stock price up to time n and suppose that we are trying to price a contract which makes a single payout of £X at time N, where X is a specified function of $\mathcal{F}_N$. We assume that an investor can buy and sell stock in arbitrary quantity without transaction costs; they can also 'short-sell' stock (hold a negative number of units) and can invest in a bank account paying interest at a constant interest-rate r (so £1 invested in the bank account at time 0 will be worth £$(1+r)^n$ at time n). A concept central to the arbitrage-free pricing methodology is the following:

Definition 1.1 A probability measure $\mathbb{Q}$ is said to be an *equivalent martingale measure* (EMM) if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ (that is, $\mathbb{P}$ and $\mathbb{Q}$ have the same null sets) and the discounted stock price $(1+r)^{-n}S_n$ is a $\mathbb{Q}$-martingale: for all $m > n$, we have

$$\mathbb{E}^{\mathbb{Q}}(S_m \mid \mathcal{F}_n) = (1+r)^{(m-n)}S_n.$$

Definition 1.2 A *self-financing trading strategy* is a pair (ξ, ζ) of processes such that ξ_n and ζ_n are $\mathcal{F}_n$-measurable and such that the trading strategy which holds ξ_n units of stock and £ζ_n over the time period $[n, n+1]$ requires no injections or removals of cash after time-0:

$$\xi_n S_{n+1} + (1+r)\zeta_n = \xi_{n+1}S_{n+1} + \zeta_{n+1} \qquad \text{for } n = 0, \ldots, (N-1).$$

An important property of self-financing trading strategies is the following:

Lemma 1.3 *If (ξ, ζ) is a self-financing trading strategy and $\mathbb{Q}$ is an EMM, the discounted value process $V_n = (1+r)^{-n}(\xi_n S_n + \zeta_n)$ is a $\mathbb{Q}$-martingale.*

Proof It is enough to consider the expected increment in V over a single step, so choose an arbitrary $n \in \{0, 1, \ldots, (N-1)\}$ and note that since (ξ, ζ) is self-financing and $(1+r)^{-n}S_n$ is a $\mathbb{Q}$-martingale we have:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(V_{n+1} - V_n \mid \mathcal{F}_n) &= (1+r)^{-n}\mathbb{E}^{\mathbb{Q}}[(1+r)^{-1}(\xi_{n+1}S_{n+1} + \zeta_{n+1}) - \xi_n S_n - \zeta_n \mid \mathcal{F}_n] \\
&= (1+r)^{-n}\mathbb{E}^{\mathbb{Q}}[(1+r)^{-1}(\xi_n S_{n+1} + (1+r)\zeta_n) - \xi_n S_n - \zeta_n \mid \mathcal{F}_n] \\
&= (1+r)^{-n}\xi_n\mathbb{E}^{\mathbb{Q}}[(1+r)^{-1}S_{n+1} - S_n \mid \mathcal{F}_n] \\
&= 0.
\end{aligned}$$

Thus V_n is a $\mathbb{Q}$-martingale. $\square$

Definition 1.4 A self-financing strategy is said to be an *arbitrage* if it satisfies (i) $\xi_0 S_0 + \zeta_0 = 0$, (ii) $\xi_N S_N + \zeta_N \geq 0$, and (iii) $\mathbb{E}(\xi_N S_N + \zeta_N) > 0$.

Definition 1.5 A stock model is *complete* if for each random variable X with $\mathbb{E}|X| < \infty$, there is a self-financing trading strategy (ξ, ζ) with $\xi_N S_N + \zeta_N \equiv X$. We say that such a trading strategy *hedges* X.

We can now state a version of the most important theorem in option pricing.

Theorem 1.6 (Harrison & Pliska) *A finite, discrete time stock model is arbitrage-free if and only if there is at least one EMM. The model is complete and arbitrage-free if and only there is exactly one EMM.*

Proof First suppose that $\mathbb{Q}$ is an EMM and there is also an arbitrage (ξ, ζ). Since (ξ, ζ) is self-financing, its discounted value process V_n is a $\mathbb{Q}$-martingale; thus $\mathbb{E}^{\mathbb{Q}} V_n = V_0$ for all n. But (ξ, ζ) is an arbitrage so $V_0 = 0$. Also from the properties of an arbitrage, we have $V_N \geq 0$ and $\mathbb{E} V_N > 0$; thus as $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, we have $\mathbb{E}^{\mathbb{Q}} V_N > 0$, giving a contradiction.

Conversely, suppose that there are no arbitrage opportunities, and define the sets

$$\mathcal{X}^+ = \{X : X \geq 0, \mathbb{E}X \geq 1\} \tag{1.1}$$

$$\mathcal{X}^0 = \{X : X = V_N(\xi, \zeta) \text{ where } (\xi, \zeta) \text{ is self-financing and } V_0(\xi, \zeta) = 0\} \tag{1.2}$$

Since Ω is finite, the set of $\mathcal{F}_N$-measurable functions is a finite-dimensional vector space. Note that $\mathcal{X}^+$ is closed and convex and that $\mathcal{X}^0$ is a linear subspace. Since there are assumed to be no arbitrage opportunities, $\mathcal{X}^+$ and $\mathcal{X}^0$ are disjoint. By the separating hyperplane theorem, there exists a linear functional L with $L(X) = 0$ for $X \in \mathcal{X}^0$ and $L(X) > 0$ for $X \in \mathcal{X}^+$. Now define the probability measure $\mathbb{Q}$ by $\mathbb{Q}(A) = L(\mathrm{I}(A))/L(1)$ where $\mathrm{I}(A)$ denotes the indicator function of A. Since $L(X) > 0$ for any $X \geq 0$ with $\mathbb{E}(X) = 1$, $\mathbb{P}$ and $\mathbb{Q}$ are equivalent. We will show that $\mathbb{Q}$ is an EMM.

Let τ be a stopping time with values in $0, 1, \ldots, N$ and let (ξ, ζ) be the strategy

$$\begin{aligned} \xi_n &= \mathrm{I}(\tau > n) \\ \zeta_n &= \mathrm{I}(\tau \leq n)(1+r)^{(n-\tau)} S_\tau - (1+r)^n S_0. \end{aligned}$$

Observe that (ξ, ζ) is self-financing and $V_0(\xi, \zeta) = 0$, so we have $L(\xi_N S_N + \zeta_N) = 0$. By the definition of $\mathbb{Q}$, for any random variable Z, $\mathbb{E}^{\mathbb{Q}}(Z) = L(Z)/L(1)$, so we have $\mathbb{E}^{\mathbb{Q}}(\xi_N S_N + \zeta_N) = 0$, and hence

$$\begin{aligned} (1+r)^N S_0 &= \mathbb{E}^{\mathbb{Q}}\big(\mathrm{I}(\tau \leq N)(1+r)^{(N-\tau)} S_\tau\big) \\ &= (1+r)^N \mathbb{E}^{\mathbb{Q}}\big((1+r)^{-\tau} S_\tau\big). \end{aligned}$$

Thus $S_0 = \mathbb{E}^{\mathbb{Q}}\big((1+r)^{-\tau} S_\tau\big)$ for any stopping time τ with values in $0, 1, \ldots, N$ which implies that $(1+r)^{-n} S_n$, $n = 0, 1, \ldots, N$ is a $\mathbb{Q}$-martingale.

Turning to the other other equivalence, we will first show that if the market is complete, there can be only one EMM. Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be EMMs, let $A \in \mathcal{F}_N$ be an arbitrary event and let (ξ, ζ) be the hedging strategy for $\mathrm{I}(A)$. Since (ξ, ζ) is self-financing, its discounted payoff function is a $\mathbb{Q}_1$-martingale. Thus we have

$$\begin{aligned} V_0(\xi, \zeta) &= (1+r)^{-N} \mathbb{E}^{\mathbb{Q}_1}[\mathrm{I}(A)] \\ &= (1+r)^{-N} \mathbb{Q}_1(A). \end{aligned}$$

But $\mathbb{Q}_2$ is also an EMM, so using the same argument we deduce that $\mathbb{Q}_1(A) = \mathbb{Q}_2(A)$ and there can be at most one EMM.

Conversely, suppose that the market is arbitrage-free but not complete, so there exists an EMM $\mathbb{Q}_1$ and also a contract X which cannot be hedged. Let

$$\mathcal{S} = \{V_N(\xi, \zeta) : (\xi, \zeta) \text{ is self-financing}\},$$

a finite dimensional vector space containing 1, and write $X = A + U$ where $A \in \mathcal{S}$ and U is non-zero and orthogonal to $\mathcal{S}$ (in particular $\mathbb{E}(U) = 0$). Since $\mathbb{P}$ and $\mathbb{Q}_1$ are equivalent and Ω is finite, we can find $\epsilon > 0$ such that

$$\mathbb{Q}_2(A) = \mathbb{Q}_1(A) + \epsilon\, \mathbb{E}[U\ \mathrm{I}(A)]$$

defines a probability measure equivalent to $\mathbb{Q}_1$ and $\mathbb{P}$. An argument similar to that used in the second part of this proof shows that $(1 + r)^{-n} S_n$ is a $\mathbb{Q}_2$-martingale. Thus the EMM is not unique. □

If the model is complete and arbitrage-free, the time-0 value of an option must be the initial value of its hedging portfolio. From Lemma 1.3 and the fact that if (ξ, ζ) hedges X, we have $\xi_N S_N + \zeta_N = X$, we deduce the *pricing equation* for the initial value of X:

$$X_0 = (1 + r)^{-N} \mathbb{E}^{\mathbb{Q}} X \tag{1.3}$$

where $\mathbb{Q}$ is the EMM. In general, the time-n value of X is given by

$$X_n = (1 + r)^{(n-N)} \mathbb{E}^{\mathbb{Q}}(X \mid \mathcal{F}_n).$$

Observe that once we have identified the EMM $\mathbb{Q}$, the problem of pricing X is just an expectation calculation.

1.2 Option pricing in continuous time

We now turn to the continuous time setting. The stock price S_t, $t \in [0, T]$, is assumed to be a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the filtration $\mathcal{F}$ is the $\mathbb{P}$-augmentation of the filtration generated by a one-dimensional Brownian motion B. The stock price S_t then admits the decomposition $S_t = M_t + A_t$ where A_t is a continuous process of finite variation and $M_t = \int_0^t \sigma_u \, dB_u$ for some previsible process σ_u, (see Rogers & Williams (1987), Theorem IV.36.5, for example).

Again we assume that a risk-free bank account is available, now paying interest at the continuous rate r, so £1 invested at time 0 grows into £$\exp(rt)$ by time t.

Remark 1.7 The restriction to a constant interest-rate is mainly for notational convenience. If r_t is a strictly positive deterministic process (so £1 invested in the bank account at time 0 is worth $\exp(\int_0^t r(u)\,du)$ by time t) the natural generalisations of the results in this section can be proved in a very similar way.

We now replace Definitions 1.1–1.4 with their continuous time versions.

Definition 1.8 A probability measure $\mathbb{Q}$ is said to be an *equivalent martingale measure* (EMM) if $\mathbb{Q}$ is equivalent to $\mathbb{P}$, and the discounted stock price $\exp(-rt)S_t$ is a local $\mathbb{Q}$-martingale.

Definition 1.9 A *self-financing trading strategy* is a pair (ξ, ζ) of previsible processes such that the process $\xi_t S_t + \zeta_t$ is a semimartingale (for example, $\mathbb{P}(\int_0^T (\xi_t^2 \sigma_t^2 + |\zeta_t|)\,dt < \infty) = 1$ is sufficient for this) and such that

$$\xi_t S_t + \zeta_t = \xi_0 S_0 + \zeta_0 + \int_0^t \xi_u\,dS_u + \int_0^t r\zeta_u\,du \quad \text{for all } t \in [0,T]. \tag{1.4}$$

Before we define what we mean by a continuous-time arbitrage, it is necessary to impose some restrictions on the set of trading strategies to exclude examples such as the following: suppose $A_t = 0$ and $\sigma_t = 1$ for all t and consider the strategy which buys 1 unit of stock at time 0, and at time $1 - 2^{-n}$ for $n = 1, 2, \ldots$ either sells all its stock, if its gain by this time is at least £1, and otherwise doubles its stock holding. By time 1, this strategy shows a profit of at least £1 with probability one, and also makes a finite number of trades with probability one. To exclude this type of strategy, we introduce the concept of an admissible trading strategy.

Definition 1.10 A self-financing trading strategy (ξ, ζ) is said to be *admissible* if for some EMM $\mathbb{Q}$, the discounted value process $V_t := \exp(-rt)(\xi_t\,S_t + \zeta_t)$ is a $\mathbb{Q}$-martingale.

Lemma 1.11 *If (ξ, ζ) is a self-financing trading strategy and $\mathbb{Q}$ is an EMM then the discounted value process V_t is a local $\mathbb{Q}$-martingale.*

Proof Applying Itô's Lemma to the semimartingale V_t gives

$$dV_t = -rV_t\,dt + e^{-rt}(\xi_t\,dS_t + r\zeta_t\,dt),$$

and applying it to $\exp(-rt)S_t$ we have

$$d(\exp(-rt)S_t) = -r\exp(-rt)S_t\,dt + \exp(-rt)\,dS_t.$$

Combining these gives

$$\begin{aligned} dV_t &= -rV_t\,dt + \xi_t[d(e^{-rt}S_t) + re^{-rt}S_t\,dt] + re^{-rt}\zeta_t\,dt \\ &= \xi_t\,d(e^{-rt}S_t). \end{aligned}$$

Thus V_t is a local $\mathbb{Q}$-martingale. □

Definition 1.12 An *arbitrage* is a self-financing trading strategy (ξ, ζ) for which (i) the strategy (ξ, ζ) is admissible, (ii) we have $\xi_0 S_0 + \zeta_0 = 0$, $\xi_T S_T + \zeta_T \geq 0$ and $\mathbb{E}(\xi_T S_T + \zeta_T) > 0$ (the generation of a riskless profit from zero capital).

Lemma 1.13 *There are no arbitrages.*

Proof If (ξ, ζ) is an arbitrage, there is an EMM $\mathbb{Q}$ under which its discounted value process is a $\mathbb{Q}$-martingale, so $V_0 = \mathbb{E}^{\mathbb{Q}} V_T$. Since $V_T \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(V_T > 0) > 0$, from the equivalence of $\mathbb{P}$ and $\mathbb{Q}$, we have $\mathbb{P}^{\mathbb{Q}}(V_T) > 0$. But $V_0 = 0$, giving a contradiction. □

Remark 1.14 The existence of an EMM also excludes another type of arbitrage. If (ξ, ζ) is a self-financing trading strategy which satisfies condition (ii) in Definition 1.12, and has a discounted value process which is bounded below, $V_t \geq K$ for some $K \in \mathbb{R}$, then there can be no EMMs. To see this, suppose $\mathbb{Q}$ is an EMM; from Lemma 1.11, V_t is a local $\mathbb{Q}$-martingale. A local-martingale bounded below is a supermartingale, from Fatou's Lemma, so we have $\mathbb{E}^{\mathbb{Q}} V_T \leq \mathbb{E}^{\mathbb{Q}} V_0$. Thus (ii) cannot hold, by an argument similar to the proof of Lemma 1.13.

Definition 1.15 A arbitrage-free stock price model is said to be *complete* if for each random variable X and EMM $\mathbb{Q}$ with $\mathbb{E}^{\mathbb{Q}}|X| < \infty$, there is an admissible self-financing trading strategy (ξ, ζ) with $\xi_T S_T + \zeta_T \equiv X$. We say that (ξ, ζ) *hedges* X.

Clearly if (ξ, ζ) hedges X and our model is to remain arbitrage-free, the value of X at time 0 must be $\xi_0 S_0 + \zeta_0$. Since its discounted value process is a martingale under some EMM $\mathbb{Q}$, the time-t value of the claim, X_t satisfies the *pricing equation*

$$X_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}(X \mid \mathcal{F}_t). \tag{1.5}$$

Remark 1.16 In a way similar to the proof of Theorem 1.6 we can show that if the stock model is complete, then there can be at most one EMM. The only complication is that we need V_t, the discounted value process of the hedging strategy used in the proof of Theorem 1.6 to be a martingale under an arbitrary EMM. Since V_t is bounded (as it is the conditional expectation of an indicator function, and is a martingale under the EMM admitting (ξ, ζ)), and a bounded local-martingale is a martingale, this condition holds.

Theorem 1.17 *Suppose $\sigma > 0$ and $|\sigma_t^{-1}(rS_t - A'_t)| < K$ for some constant K, then there is a unique EMM $\mathbb{Q}$, whose Radon-Nikodým derivative is given by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^T \gamma_t \, dB_t - \tfrac{1}{2} \int_0^T |\gamma_t|^2 \, dt \right) \tag{1.6}$$

where $\gamma_t = \sigma_t^{-1}(rS_t - A'_t)$. In addition, the stock model is complete.

Proof The stock price satisfies the SDE

$$dS_t = \sigma_t \, dB_t + A'_t \, dt,$$

and with γ and $\mathbb{Q}$ as defined in the statement of the Lemma, the Cameron-Martin-Girsanov Theorem (see Rogers & Williams (1987), Theorem IV.38.5 for example) shows that the process $\tilde{B}_t = B_t - \int_0^t \gamma_u \, du$ is a $\mathbb{Q}$-Brownian motion, provided that the expectation of the right-hand side of (1.6) equals one. The condition $|\gamma_t| < K$ together with Novikov's criterion (see Revuz & Yor (1994), Chapter VIII, Proposition 1.15) are sufficient for this purpose. Noting that the discounted stock price $\exp(-rt)S_t$ has SDE

$$\begin{aligned} d(e^{-rt}S_t) &= e^{-rt}[\sigma_t \, dB_t + (A'_t - rS_t)\, dt] \\ &= e^{-rt}\sigma_t \, d\tilde{B}_t \end{aligned} \tag{1.7}$$

we see that $\mathbb{Q}$ is an EMM. In terms of the $\mathbb{Q}$-Brownian motion $\tilde{B}_t$, the stock price follows the SDE

$$dS_t = \sigma_t \, d\tilde{B}_t + rS_t \, dt.$$

Uniqueness of the EMM will follow from completeness, which we now prove.

Let X be a random variable with $\mathbb{E}^{\mathbb{Q}}|X| < \infty$ for some EMM $\mathbb{Q}$. Defining $v_t = \mathbb{E}^{\mathbb{Q}}(X \mid \mathcal{F}_t)$, which is a continuous $\mathbb{Q}$-martingale with $v_T = X$, we will look for a trading strategy whose discounted value process is v_t. Since $\tilde{B}$ has the predictable representation property with respect to $\mathcal{F}$, there is a previsible process h_t with $\int_0^T h_t^2 \, dt < \infty$ and $v_t = \mathbb{E}^{\mathbb{Q}}X + \int_0^t h_u \, d\tilde{B}_u$. Suppose the strategy (ξ, ζ) is self-financing and has discounted value process v_t. Writing $E_t = \exp(rt)v_t$ for its undiscounted value process, we have

$$\begin{aligned} dE_t &= e^{rt}(rv_t \, dt + h_t \, d\tilde{B}_t) \\ &= \xi_t \, dS_t + r\zeta_t \, dt \end{aligned}$$

if the strategy is to be self-financing. Substituting $dS_t = \sigma_t \, d\tilde{B}_t + rS_t \, dt$ and equating finite-variation and non-finite-variation terms gives

$$\begin{aligned} \xi_t &= e^{rt}h_t/\sigma_t \\ \xi_t rS_t + r\zeta_t &= re^{rt}v_t. \end{aligned}$$

The first of these we can use to define ξ_t, while the second determines ζ_t. The strategy (ξ, ζ) is the required hedge for X. $\square$

Remark 1.18 If S_t is Markov and X depends only on S_T, then the time-t value of the option has the form $V(S_t, t)$ for some function V. If V is sufficiently smooth, we can apply Itô's Lemma, giving $\xi_t = \partial V(s,t)/\partial s$. Thus with a way of calculating the time-t value of the option, we can obtain the hedging strategy by differentiating.

1.3 The Black-Scholes model

The Black-Scholes model assumes that the stock price follows a diffusion with SDE

$$dS_t = S_t(\sigma\, dB_t + \mu\, dt)$$

where σ and μ are constants and σ, the *volatility* is strictly positive. This fits into the framework of the previous section through the choices $\sigma_t = \sigma S_t$, $A_t = \mu \int_0^t S_u\, du$. These choices for σ_t and A_t reflect the idea that it is more likely that the percentage increments in the stock price will be stationary over time than the increments themselves.

Proposition 1.19 *(i) The Black-Scholes model is complete. (ii) The time-*0 *value of a contract paying the $\mathcal{F}_T$-measurable amount X at time T is* $\exp(-rT)\mathbb{E}^{\mathbb{Q}}X$ *where* $\log S_t$ *is a Brownian motion with drift under $\mathbb{Q}$, specifically,* $\log S_t = (r - \frac{1}{2}\sigma^2)t + \sigma \tilde{B}_t$ *for a $\mathbb{Q}$-Brownian motion $\tilde{B}_t$. In particular, the law of* $\log S_T$ *under $\mathbb{Q}$ is normal with mean $\big(r - \frac{1}{2}\sigma^2\big)T$ and variance $\sigma^2 T$.*

Proof (i) Since $|\sigma_t^{-1}(A_t' - rS_t)| = \sigma^{-1}|\mu - r|$ is constant, the conditions of Theorem 1.17 hold for the Black-Scholes model; thus the model is complete.

Parts (ii) and (iii) follow from the SDE for $\exp(rt)S_t$ under $\mathbb{Q}$ given in (1.7). □

The popularity of the Black-Scholes model is due to its tractability, which arises from the simple law of $\{S_t\}$ under the martingale measure. This law is unchanged if μ is replaced with a fairly general stochastic process (e.g. a bounded previsible process) so the assumption of constant μ is not very significant. With a deterministic but time-dependent volatility and interest-rate, S_T is still lognormal, so pricing a contract of the form $X(S_T)$ is generally no harder, but $\log S_t$ is now a Brownian motion with time-dependent drift, and the problem of pricing a fully path-dependent contract can be significantly harder.

Example 1.20 Convention dictates that our first example of a option which can be priced explicitly in the Black-Scholes model is the European Call option. This grants the holder the right, but not the obligation, to buy one unit of stock at a predetermined *strike price* K, at the *exercise time* T. At time T, we see that the option effectively has the value $(S_T - K)^+ = \max(S_T - K, 0)$ since the option will be exercised and the stock immediately re-sold if the stock price has risen above the strike price (giving an immediate profit of $S_T - K$) and be allowed to lapse otherwise. We can calculate the time-0 value of this option using (1.5),

$$\begin{aligned} X_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T - K)^+ \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\big[S_0 \exp\big((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}N\big) - K\big]^+ \end{aligned}$$

for some random variable N which is standard normal under $\mathbb{Q}$. This has a closed form solution known as the Black-Scholes formula:

$$X_0 = S_0 \Phi\left(\frac{\log(S_0/K) + \big(r + \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log(S_0/K) + \big(r - \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}}\right)$$

where Φ denotes the normal distribution function.

Other examples, where X depends on more than just S_T, and for which explicit pricing formulae still exist are *lookback options*, such as $X = S_T - \inf\{S_t : t \in [0,T]\}$, and *barrier options*, like $X = (S_T - K)\,\mathrm{I}\big(\sup_{t\in[0,T]} S_t < H\big)$ for a constant H. Derivations of pricing formulae for these examples can be found in Musiela & Rutkowski (1997). With a general deterministic time-dependent volatility process, neither of these types of options have simple closed-form expressions.

An important example of an option which does not have a simple pricing formula even in the Black-Scholes model with constant volatility is the Asian option: $X = \big(\int_0^T S_u\,du - K\big)^+$. An explicit formulae exists in the form an inverse Laplace transform (see Geman & Yor (1993) and Geman & Eydeland (1995)) but it appears to be difficult to evaluate numerically.

1.4 Interest-rate models

The underlying assets in an interest-rate model are the *(zero-coupon) bonds*, for each time t and future time $T \geq t$, we can buy and sell a T*-bond*, an asset making a single cash payout of £1 at time T. We denote the time-t price of a T-bond $P_{t,T}$ and refer to the curve $\{P_{t,\cdot}\}$ as the *term structure* at time t. Note that $P_{T,T} = 1$, and in general we would expect $P_{t,T} < 1$ for $t < T$. We will assume that a bank account is available, or equivalently assume the existence of a *cash bond* whose value at time t is

$$Y_t = \exp\bigg(\int_0^t r_u\,du\bigg)$$

where r_t is the instantaneous riskless interest-rate, or *spot-rate*, at time t. Note that unlike in the discussion of stock-models we must allow the spot-rate to be non-constant; indeed, unless r_t is non-deterministic, the no-arbitrage condition implies that $P_{t,T}$ takes the (non-random) value $\exp(-\int_t^T r_u\,du)$. For simplicity, we will consider trading strategies which only ever invest in the savings account and a finite number bonds.

Remark 1.21 Consider a contract paying out the random amount X at time T, which will become known by the earlier time $T' < T$; a simple argument using the no-arbitrage condition implies that the time-T' value of X is $XP_{T',T}$. Similarly, the time-t value of a contract worth $\sum_i a_i P_{T,T_i}$ at time T, (for constants a_i, T_i, $i = 1,\ldots,n$) is $\sum_i a_i P_{t,T_i}$.

In what follows we will use two types of interest-rate. Normally we will work with *continuously compounded* rates—a loan of £1 at rate ρ for a period of length t requires a final payment of £$\exp(\rho t)$, but occasionally we will refer to *simple compounding*, in this case a payment of £$(1+\rho t)$ is indicated.

Definition 1.22 The δ-*LIBOR rate* $L_t^\delta := \delta^{-1}(P_{t,t+\delta}^{-1} - 1)$, is the simple rate of interest available at time t on a loan from the current moment to the later time $t + \delta$. The *instantaneous forward rate* $F_{t,T}$ at time t for time $T > t$ is defined by $F_{t,T} = -(\partial/\partial T)\log P_{t,T}$ (assuming the derivative exists).

Remark 1.23 Consider an agreement fixed at time $t < T$ to loan £1 over $[T, T+\delta]$ at (continuously compounded) rate κ. At time T the borrower will receive £1 and subsequently return £$\exp(\kappa\delta)$ at time $T+\delta$. The borrower's position is thus equivalent to holding a T-bond and shorting $\exp(\kappa\delta)$ $(T+\delta)$-bonds, a portfolio whose time t value is $[P_{t,T} - \exp(\kappa\delta)P_{t,T+\delta}]$. If we consider the rate $\tilde{\kappa}$, determined by the term-structure at time t, at which we would be willing to both borrow and lend over the period $[T, T+\delta]$, the no-arbitrage assumption implies that $\tilde{\kappa} = \delta^{-1}\log(P_{t,T}/P_{t,T+\delta})$. From this observation, we see that we can interpret the instantaneous forward rate $F_{t,T}$ as the rate implied by the time-t term structure for the period $[T, T+dT]$. We assume that $r_t = F_{t,t}$.

1.4.1 Interest-Rate derivatives

We now introduce the major interest-rate derivatives.

Definition 1.24 A *cap* at rate κ for the period $[T, T+\delta]$ gives the holder the option, at time T, of borrowing £1 at rate κ over $[T, T+\delta]$.

If $L_T^\delta > \kappa$ the owner of the cap will borrow £1 from the seller of the cap and invest the money in the market at the higher rate. When the interest payments are exchanged at time $T+\delta$, the holder of the cap will show a profit of

$$P_{T,T+\delta}^{-1} - (1 + \kappa\delta).$$

Since the profit available by exercising the option is known at time T, it follows from Remark 1.21 that the time-T value of the cap is

$$[1 - (1+\kappa\delta)P_{T,T+\delta}]^+.$$

Thus a cap is an option on a linear function of a bond price. In a complete model with EMM $\mathbb{Q}$, the value of the cap is given by

$$Y_t\,\mathbb{E}^{\mathbb{Q}}\big(Y_T^{-1}[1 - (1+\kappa\delta)P_{T,T+\delta}]^+ \,\big|\, \mathcal{F}_t\big). \tag{1.8}$$

Letting $\tilde{\kappa}_T$ be the solution of $1 - (1+\tilde{\kappa}_T\delta)P_{T,T+\delta} = 0$, this may be written as

$$Y_t\,\mathbb{E}^{\mathbb{Q}}\big[Y_T^{-1}P_{T,T+\delta}\,\delta(\tilde{\kappa}_T - \kappa)^+\big]$$

so the cap effectively pays $\delta(\tilde{\kappa}_T - \kappa)^+$ at time $T+\delta$.

Definition 1.25 A *swap* at rate κ is an agreement between two parties, the *payer* and the *receiver*, denoted P and R respectively, in which P pays R interest on a loan of £1 at rate κ over each of the time intervals $[T_0, T_1], \ldots, [T_{n-1}, T_n]$, while R pays P interest on a loan of £1 at the LIBOR rate $L^{(T_j - T_{j-1})}_{T_{j-1}}$ over $[T_{j-1}, T_j]$, $j = 1, \ldots, n$. We consider the case where the payments are made in arrears, so the interest payments corresponding to the time interval $[T_{j-1}, T_j]$ are exchanged at time T_j.

Normally $T_j - T_{j-1} = \delta$ is the same for all j. At time T_{j-1}, the LIBOR rate $L^{\delta}_{T_{j-1}}$ becomes known and P subsequently shows a profit of $P^{-1}_{T_{j-1},T_j} - (1 + \kappa\delta)$ when the interest payments are exchanged at time T_j. Thus the time-T_{j-1} value of this 'leg' of the swap is $1 - (1 + \kappa\delta) P_{T_{j-1},T_j}$ and the time-t value, for $t < T_{j-1}$, is

$$P_{t,T_{j-1}} - (1 + \kappa\delta) P_{t,T_j}.$$

We conclude that from the payer's point of view, for $t < T_0$, the time-t value of the whole swap is given by

$$P_{t,T_0} - P_{t,T_n} - \kappa\delta \sum_{j=1}^{n} P_{t,T_j}. \tag{1.9}$$

Definition 1.26 A *swaption* with rate κ is an option giving the holder the right to assume the rôle of the payer in a swap contract at a specified time $T < T_0$.

From (1.9), we see that a swaption is effectively an option on a linear combination of bond-prices with time t value

$$Y_t \, \mathbb{E}^{\mathbb{Q}} \Big[Y_T^{-1} \big(P_{T,T_0} - P_{T,T_n} - \kappa\delta \textstyle\sum_{j=1}^{n} P_{T,T_j} \big)^+ \,\Big|\, \mathcal{F}_t \Big]. \tag{1.10}$$

For the case where $\{Y_T, P_{T,T_0}, \ldots, P_{T,T_n}\}$ are joint lognormal under $\mathbb{Q}$, we will present an efficient way of estimating (1.10) in Chapter 3.

1.4.2 Models for the spot-rate

Early attempts at modelling interest-rates concentrated on modelling the spot-rate directly. The model proposed by Vasicek (1977) assumes that r_t satisfies the SDE

$$dr_t = \alpha(\beta - r_t)\, dt + \sigma\, dB_t$$

where B_t is a Brownian motion under an EMM, while Cox, Ingersoll & Ross (1985) (CIR) suggest the model

$$dr_t = \alpha(\beta - r_t)\, dt + \sigma \sqrt{r_t}\, dB_t,$$

which has the advantage of guaranteeing a non-negative spot-rate.

In order to calibrate a spot-rate model using the current bond prices, it is useful to have a formula for the price of zero coupon bonds in terms of the model's parameters and the current spot-rate. In the Vasicek model, bond prices are given by the formula

$$P_{t,T} = a(t,T)e^{-b(t,T)r_t} \tag{1.11}$$

where, if $\alpha \neq 0$,

$$\begin{aligned} b(t,T) &= \alpha^{-1}\big(1 - e^{-\alpha(T-t)}\big) \\ a(t,T) &= \exp\left(\frac{[b(t,T) - T + t](\alpha^2\beta - \frac{1}{2}\sigma^2)}{\alpha^2} - \frac{\sigma^2 b(t,T)^2}{4\alpha}\right) \end{aligned}$$

while if $\alpha = 0$, $b_{t,T} = T - t$ and $a_{t,T} = \exp\big(\sigma^2(T-t)^3/6\big)$. In the CIR model, bond prices are again given by (1.11) but now $a(t,T)$ and $b(t,T)$ are given by

$$\begin{aligned} b(t,T) &= \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma+\alpha)(e^{\gamma(T-t)} - 1) + 2\gamma} \\ a(t,T) &= \left(\frac{2\gamma e^{(\alpha+\gamma)(T-t)/2}}{(\gamma+\alpha)(e^{\gamma(T-t)} - 1) + 2\gamma}\right)^{2\alpha\beta/\sigma^2} \end{aligned}$$

where $\gamma^2 = \alpha^2 + 2\sigma^2$.

Since both of these models have a finite number of parameters but there are potentially an infinite number of bond prices, we cannot hope to fit either of these models perfectly to a general initial term structure. The models of Ho & Lee (1986) and Hull & White (1990) address this problem by introducing time-dependent coefficients; in the continuous time version of Ho & Lee's model, the spot-rate has SDE

$$dr_t = a_t\,dt + \sigma\,dB_t$$

while Hull & White suggest

$$dr_t = (a_t - br_t)\,dt + \sigma\,dB_t.$$

By suitable choice of the function a_t, both models can be made to fit any initial term structure. Unfortunately a_t must be re-calibrated as time evolves since we cannot capture all possible movements in the term structure with a single driving Brownian motion. An important advance in interest-rate modelling which solves this problem is described in the next section.

Both the Vasicek and CIR models are examples of *affine term structure models* as identified by Duffie & Kan (1994). If the term structure is described by a n-dimensional state variable X, which satisfies an SDE of the form

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\cdot dW_t$$

where μ and $\sigma\sigma^\top$ are affine functions, $r_t = R(X_t)$ for R affine and W is an n-dimensional Brownian motion under an EMM, the zero-coupon bond prices have the form $P_{t,T} = a_{t,T}\exp(-b_{t,T}\cdot X_t)$ where the functions a and b can be obtained by solving a pair of differential equations. In the Vasicek and CIR models, the spot rate itself serves as a state variable.

1.4.3 Forward-rate models

Instead of modelling the spot-rate and deriving a pricing formula for the zero-coupon bonds, an alternative approach is to model the collection of instantaneous forward-rates, $\{F_{t,T}\}$. Since the bond prices are given by $P_{t,T} = \exp(-\int_t^T F_{t,u}\,du)$, this method will automatically fit the initial term structure.

The HJM framework

Heath, Jarrow & Morton (1992) consider modelling the evolution of the instantaneous forward-rates with a family of semimartingales adapted to the natural filtration of a finite-dimensional Brownian motion; they assume that for each $t < T$, the forward-rate $F_{t,T}$ satisfies the SDE

$$dF_{t,T} = \alpha_{t,T}\,dt + \sigma_{t,T}\cdot dB_t$$

where B is a an n-dimensional $\mathbb{P}$-Brownian motion, and α and σ are predictable processes satisfying $\int_0^T(|\alpha_{t,T}| + |\sigma_{t,T}|^2)\,dt < \infty$, for all T. Defining the vector processes $a_{t,T}$ and $b_{t,T}$ by

$$\begin{aligned} a_{t,T} &= -\int_t^T \sigma_{t,u}\,du, \\ b_{t,T} &= -\int_t^T \alpha_{t,u}\,du + \tfrac{1}{2}|a_{t,T}|^2, \end{aligned}$$

the dynamics of the zero-coupon bond prices $P_{t,T}$ can be shown to be

$$dP_{t,T} = P_{t,T}[(r_t + b_{t,T})\,dt + a_{t,T}\cdot dB_t].$$

To ensure the non-existence of arbitrage opportunities, we assume that there is a vector process γ_t with

$$b_{t,T} + a_{t,T}\gamma_t = 0 \tag{1.12}$$

and

$$\mathbb{E}\exp\left(\int_0^T \gamma_t\cdot dB_t - \tfrac{1}{2}\int_0^T |\gamma_t|^2\,dt\right) = 1.$$

The probability measure $\mathbb{Q}$, defined by $d\mathbb{Q}/d\mathbb{P} = \exp\bigl(\int_0^T \gamma_t \cdot dB_t - \frac{1}{2}\int_0^T |\gamma_t|^2\, dt\bigr)$ is then an EMM, and the process $\tilde{B}_t = B_t - \int_0^t \gamma_u\, du$ a Brownian motion under this measure. Differentiating (1.12) with respect to T, noting that γ_t does not depend on T, gives

$$-\alpha_{t,T} + \sigma_{t,T} \cdot \int_t^T \sigma_{t,u}\, du - \sigma_{t,T} \cdot \gamma_t = 0,$$

thus the forward-rate dynamics under $\mathbb{Q}$ are

$$dF_{t,T} = \left(\sigma_{t,T} \cdot \int_t^T \sigma_{t,u}\, du\right) dt + \sigma_{t,T} \cdot d\tilde{B}_t. \tag{1.13}$$

The case where $\sigma_{t,T}$ is deterministic, the 'Gaussian HJM model', gives rise to lognormal bond prices, and is particularly tractable. Kennedy (1994) considers a more general form of Gaussian forward-rate model, based on a Gaussian random field, and derives the dynamics of the forward-rates under an EMM. We will examine some probabilistic properties of this type of interest rate model in Chapter 2.

1.4.4 Other models

Many other models have been proposed for interest-rates; we will mention a few here. Flesaker & Hughston (1996) propose a model in which the bond prices are modelled directly as $P_{t,T} = A_t^{-1}\mathbb{E}(A_T \mid \mathcal{F}_t)$ where A_t is a strictly positive supermartingale. They show that if $A_t = f_t + g_t W_t$ where $W_t = \exp\bigl(N_t - \frac{1}{2}\sigma^2 t\bigr)$ for a $\mathbb{P}$-Brownian motion N_t, then all caps and swaptions can be priced with a Black-Scholes-type formula. In addition, if f_t and g_t are strictly positive and strictly decreasing, we guarantee positive rates. A related approach was suggested in Rogers (1997). Another idea is to model the LIBOR rate of some fixed maturity as lognormal, see Goldys, Musiela & Sondermann (1994), Sandermann, Sondermann & Miltersen (1994), and Brace, Gątarek & Musiela (1997). This again gives a Black-Scholes-type formula for the price of caps, but does not posses an exact swaption valuation formula.

2

Gaussian Forward Rate Models

2.1 Introduction

In this chapter, we consider interest-rate models which use a continuous Gaussian random field to model forward-rates; this includes HJM-type models where the forward-rates are driven by a finite-dimensional Brownian motion, and also certain 'infinite factor' generalisations.

Let H denote the diagonal upper half plane,

$$H := \{(s,t) : t \geq s\}, \tag{2.1}$$

and suppose that $F_{s,t}$, $(s,t) \in H$, is the surface of instantaneous forward-rates, related to the zero-coupon bond prices, $P_{s,t}$, via

$$P_{s,t} = \exp\Big(-\int_s^t F_{s,u}\,du\Big). \tag{2.2}$$

We will assume that F is continuous, and to ensure the no-arbitrage condition, we assume that there exists a measure $\mathbb{Q}$ under which the discounted bond prices,

$$Z_{s,t} := \exp\Big(-\int_0^s F_{u,u}\,du\Big)P_{s,t} \tag{2.3}$$

are $\mathbb{Q}$-martingales with respect to the filtration

$$\tilde{\mathcal{F}}_s := \sigma(F_{u,v} : u \leq s, v \geq u). \tag{2.4}$$

Let us start by investigating some implications of the martingale measure assumption, following Kennedy (1997). From (2.2) and (2.3), we have

$$Z_{s,t} = \exp\Big(-\int_0^s F_{u,u}\,du - \int_s^t F_{s,u}\,du\Big),$$

so the martingale measure condition, $\mathbb{E}(Z_{s_2,t} \,|\, \tilde{\mathcal{F}}_{s_1}) = Z_{s_1,t}$ for $s_1 \leq s_2 \leq t$, amounts to

$$\mathbb{E}\big(\exp(A) \,\big|\, \tilde{\mathcal{F}}_{s_1}\big) = 1, \tag{2.5}$$

where

$$A = -\int_{s_1}^{s_2} (F_{u,u} - F_{s_1,u})\,du - \int_{s_2}^{t} (F_{s_2,u} - F_{s_1,u})\,du.$$

Since A is a Gaussian random variable, we have

$$\mathbb{E}\big(\exp(A) \,\big|\, \tilde{\mathcal{F}}_{s_1}\big) = \exp\big(\mathbb{E}(A \,|\, \tilde{\mathcal{F}}_{s_1}) + \tfrac{1}{2}\,\mathrm{var}(A \,|\, \tilde{\mathcal{F}}_{s_1})\big). \tag{2.6}$$

Also from the Gaussian property, $\mathrm{var}(A \,|\, \tilde{\mathcal{F}}_{s_1})$ is constant with probability one; combining this with (2.5) and (2.6), it follows that $\mathbb{E}(A \,|\, \tilde{\mathcal{F}}_{s_1})$ is constant with probability one; thus A is independent of $\tilde{\mathcal{F}}_{s_1}$. Now let $s \le s_1$ and $v \ge s$. Since Gaussian random variables are independent if and only if they are uncorrelated, A is uncorrelated with $F_{s,v}$, and we have

$$-\int_{s_1}^{s_2} \mathrm{cov}(F_{s,v}, F_{u,u} - F_{s_1,u})\,du - \int_{s_2}^{t} \mathrm{cov}(F_{s,v}, F_{s_2,u} - F_{s_1,u})\,du = 0,$$

which we can differentiate with respect to t to give

$$\mathrm{cov}(F_{s,v}, F_{s_2,t} - F_{s_1,t}) = 0 \qquad \text{for } s \le s_1 \le s_2 \le t, \text{ and } v \ge s; \tag{2.7}$$

thus $\tilde{\mathcal{F}}_{s_1}$ and $\sigma(F_{s_2,t} - F_{s_1,t} : s_1 \le s_2 \le t)$ are independent. As F is Gaussian, its distribution is determined by its mean and covariance structure; accordingly, for $\alpha, \beta \in H$, define

$$\mu_\alpha := \mathbb{E}(F_\alpha) \tag{2.8}$$

$$\Gamma(\alpha, \beta) := \mathrm{cov}(F_\alpha, F_\beta). \tag{2.9}$$

Condition (2.7) implies that we can write

$$\Gamma(s_1, t_1, s_2, t_2) = c(s_1 \wedge s_2, t_1, t_2) \tag{2.10}$$

for some function c, symmetric in t_1 and t_2, where $s_1 \wedge s_2$ denotes $\min(s_1, s_2)$. The martingale measure assumption also gives rise to an HJM-style drift condition:

$$\mu_{s,t} - \mu_{0,t} = \int_0^t [c(s \wedge v, v, t) - c(0, v, t)]\,dv. \tag{2.11}$$

A proper proof of this can be found in Kennedy (1994). To give an indication of why (2.11) is plausible, we will use the HJM drift constraint (1.13) and the fact that the covariance structure of the infinitesimal increment in the forward-rate curve over $[s, s + ds]$ is given by $\sigma(s, t_1) \cdot \sigma(s, t_2)\,ds$ for an HJM model, and by

$$c'(s, t_1, t_2)\,ds = (\partial/\partial s)c(s, t_1, t_2)\,ds$$

in the current framework. Integrating the HJM drift term over $[0, s]$ gives

$$\begin{aligned}
\mu_{s,t} - \mu_{0,t} &= \int_0^s \sigma(u,t) \cdot \int_u^t \sigma(u,v)\, dv\, du \\
&= \int_0^s \int_u^t c'(u,v,t)\, dv\, du \\
&= \int_0^t \int_0^{s\wedge v} c'(u,v,t)\, du\, dv \\
&= \int_0^t [c(s\wedge v, v, t) - c(0,v,t)]\, dv.
\end{aligned}$$

Thus μ is completely determined once c and $\{\mu_{0,t} : t \geq 0\}$ have been specified. Also, c and $\{\mu_{0,t} : t \geq 0\}$ can be chosen arbitrarily, provided (2.10) and (2.11) hold for some symmetric non-negative definite function $\Gamma : H^2 \times H^2 \to \mathbb{R}$. All the conditions we discuss in this chapter will be conditions on Γ—we will only ever be interested in the 'stochastic' part of the field F. To simplify the notation, henceforth we will ignore the deterministic part of F, and assume $\mu_\alpha \equiv 0$, with the understanding that the actual forward-rates are the sum of F and a deterministic function. It is also convenient to assume that F is defined on all of $\mathbb{R}^2$ rather than just H; note that setting $F_{s,t} = F_{t,t}$ for $s \geq t$ preserves property (2.7) (trivially); indeed F now satisfies the condition that $\mathcal{I}_s$ is independent of $\mathcal{F}_s$, denoted

$$\mathcal{I}_s \quad \perp \quad \mathcal{F}_s \tag{2.12}$$

where

$$\mathcal{I}_s := \sigma(F_{s',t} - F_{s,t} : s' \geq s, t \in \mathbb{R}) \tag{2.13}$$

$$\mathcal{F}_s := \sigma(F_{s',t} : s' \leq s, t \in \mathbb{R}) \tag{2.14}$$

which we will refer to as the *independent increments property*. A continuous Gaussian field F satisfying (2.12) will be called a *random field model*. Another property we shall frequently consider is *stationarity*

$$\left(F_{\alpha_1}, \ldots, F_{\alpha_n}\right) =_{\mathcal{D}} \left(F_{S(x)\alpha_1}, \ldots, F_{S(x)\alpha_n}\right), \quad \text{for } x \in \mathbb{R},\ n \in \mathbb{N},\ \alpha_i \in \mathbb{R}^2 \tag{2.15}$$

where $=_{\mathcal{D}}$ is equality in law and $S(x)$ denotes translation parallel to the line $s = t$:

$$S(x)(s,t) = (s + x, t + x).$$

One benefit of assuming Gaussian forward-rates is that it becomes easy to investigate 'global' properties such as Markov conditions. In Section 2.2 we will follow a similar course to Kennedy (1997) by investigating different versions of the Markov property, and seeing how they restrict the form of the covariance structure. Kennedy (1994) shows how transformations of the Brownian Sheet, the zero-mean, continuous Gaussian field on $[0, \infty)^2$ with covariance structure

$$\text{cov}(X_{s_1,t_1}, X_{s_2,t_2}) = (s_1 \wedge s_2)(t_1 \wedge t_2)$$

can be used to construct random field models. In Section 2.2.1 we show that if such a field is stationary, its covariance structure is determined by just three parameters and has the form

$$\operatorname{cov}(F_{s_1,t_1}, F_{s_2,t_2}) = \sigma^2 \exp\bigl(-\lambda(t_1 \wedge t_2 - s_1 \wedge s_2) - \mu|t_1 \vee t_2 - s_1 \wedge s_2|\bigr), \tag{2.16}$$

for $0 \leq \lambda \leq 2\mu$, where $s_1 \vee s_2$ denotes $\max(s_1, s_2)$, which is also the form of the covariance structure under the strongest formulation of Kennedy (1997). Kennedy (1997) also shows that this covariance structure arises from the transformation

$$F_{s,t} = \sigma e^{-\mu t} X_{e^{\lambda s}, e^{(2\mu-\lambda)t}},$$

where X is a Brownian Sheet.

In the remainder of this section we will introduce some more notation and prove a few elementary results about stationary random field models.

Remark 2.1 If F is stationary, the distribution of F is determined by the stationary distribution of the forward-rate curve $\{F_{0,t} : t \in \mathbb{R}\}$. To see this, recall that the independent increments property implies that $\Gamma(s,t,s',t') = c(s \wedge s', t, t')$ for some function c; by stationarity we have

$$\begin{aligned} c(s \wedge s', t, t') &= \operatorname{cov}\bigl(F_{s \wedge s', t}, F_{s \wedge s', t'}\bigr) \\ &= \operatorname{cov}\bigl(F_{0, t - s \wedge s'}, F_{0, t' - s \wedge s'}\bigr) \\ &= f(t - s \wedge s',\, t' - s \wedge s'), \end{aligned} \tag{2.17}$$

where we define $f(t,t') := \operatorname{cov}(F_{0,t}, F_{0,t'})$.

Define the maps $p_1, p_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$p_1(s,t,s',t') = t \wedge t' - s \wedge s', \tag{2.18}$$

$$p_2(s,t,s',t') = t \vee t' - s \wedge s', \tag{2.19}$$

and the map $p : \mathbb{R}^2 \times \mathbb{R}^2 \to H$ by $p(s,t,s',t') = (p_1, p_2)$; thus if Γ is the covariance structure of a stationary random field model, then $\Gamma \circ p^{-1}$ is well defined and equals f. Later, we will need the simple result that since F is continuous, so Γ, and hence also c and f, are continuous functions.

The HJM approach specifies an interest-rate model via the forward-rate volatilities, $\sigma(\cdot\,,\cdot)$, which amounts to specifying the covariance structure of the infinitesimal increments:

$$\lim_{\delta \to 0} \delta^{-1} \operatorname{cov}(F_{s+\delta,u} - F_{s,u}, F_{s+\delta,v} - F_{s,v}).$$

We will begin by showing that this limit exists for Gaussian random field models. Denote by $(\mathrm{D}f)(x,y)$ the 'diagonal' directional derivative, defined by

$$(\mathrm{D}f)(x,y) := \lim_{\delta \to 0} \delta^{-1}[f(x+\delta, y+\delta) - f(x,y)].$$

Proposition 2.2 *For each $t, t' \in \mathbb{R}$,*

$$\tau_s(t,t') := \frac{\partial}{\partial s} c(s,t,t') \tag{2.20}$$

exists for (Lebesgue) almost all $s \in \mathbb{R}$. Moreover, if F is stationary and $s, u, v \in \mathbb{R}$ are such that either $(\partial/\partial s)c(s,u,v)$ or $(\mathrm{D}f)(u-s, v-s)$ exists, then both derivatives exist and

$$\tau_s(u,v) = -(\mathrm{D}f)(u-s, v-s) = \lim_{\delta \to 0} \delta^{-1} \operatorname{cov}(F_{s+\delta,u} - F_{s,u}, F_{s+\delta,v} - F_{s,v}).$$

Proof For the first part, let $s \le s'$ and $t, t' \in \mathbb{R}$; from the independent increments property, we have

$$\begin{aligned}
\operatorname{var}(F_{s',t} + F_{s',t'}) &= \operatorname{var}\big(F_{s,t} + F_{s,t'} + (F_{s',t} - F_{s,t}) + (F_{s',t'} - F_{s,t'})\big) \\
&= \operatorname{var}(F_{s,t} + F_{s,t'}) + \operatorname{var}(F_{s',t} - F_{s,t} + F_{s',t'} - F_{s,t'}) \\
&\ge \operatorname{var}(F_{s,t} + F_{s,t'})
\end{aligned}$$

so $\operatorname{var}(F_{s,t} + F_{s,t'})$ is non-decreasing in s. Similarly, we can show that $\operatorname{var}(F_{s,t} - F_{s,t'})$ is non-decreasing in s. Since

$$\operatorname{cov}(F_{s,t}, F_{s,t'}) = \tfrac{1}{4}[\operatorname{var}(F_{s,t} + F_{s,t'}) - \operatorname{var}(F_{s,t} - F_{s,t'})]$$

we deduce that $c(s,t,t')$ is of finite variation in s, and hence for each t and t', the derivative $(\partial/\partial s)c(s,t,t')$ exists for (Lebesgue) almost all s (Dudley 1989, Section 7.2.7). Secondly, let $\delta > 0$ and consider the covariance structure of the increment in the forward-rate curve between times s and $s+\delta$. By independent increments we have

$$\begin{aligned}
\operatorname{cov}(F_{s+\delta,u} - F_{s,u}, F_{s+\delta,v} - F_{s,v}) &= \operatorname{cov}(F_{s+\delta,u}, F_{s+\delta,v}) - \operatorname{cov}(F_{s,u}, F_{s,v}) \\
&= c(s+\delta,u,v) - c(s,u,v)
\end{aligned}$$

and by stationarity,

$$\begin{aligned}
\operatorname{cov}(F_{s+\delta,u}, F_{s+\delta,v}) - \operatorname{cov}(F_{s,u}, F_{s,v}) &= \operatorname{cov}(F_{0,u-s-\delta}, F_{0,v-s-\delta}) - \operatorname{cov}(F_{0,u-s}, F_{0,v-s}) \\
&= f(u-s-\delta, v-s-\delta) - f(u-s, v-s). \qquad (2.21)
\end{aligned}$$

Dividing throughout by δ and letting $\delta \to 0$ completes the proof. □

From (2.21) we see that

$$f(u-s-\delta, u-s-\delta) - f(u-s, u-s) = \operatorname{var}(F_{s+\delta,u} - F_{s,u}), \tag{2.22}$$

so $f(t,t)$ is non-increasing in t. A straightforward consequence of this, together with the stationarity property is that the stochastic part of the 'long rate' is constant over time.

Corollary 2.3 *Suppose the limit $F_{s,\infty} = \lim_{t\to\infty} F_{s,t}$ exists for all s with probability one, then the process $F_{s,\infty}$ is constant in s with probability one.*

Proof First define

$$F^*_{s,t} = \sup_{x\in[s,s+1]} (F_{x,t} - F_{s,t})^2$$
$$F^*_{s,\infty} = \sup_{x\in[s,s+1]} (F_{x,\infty} - F_{s,\infty})^2,$$

and observe that if $(t_n)_{n=1}^{\infty}$ is any sequence of times tending to infinity, the event $\{F^*_{s,\infty} > \epsilon\}$ is certainly contained in $\{F^*_{s,t_n} \geq \epsilon \text{ i.o.}\}$ by the continuity of F. Now consider the sequence $t_n = n$ and note that for each $u \in \mathbb{R}$, $F_{u,n}$ is a martingale since it has independent increments and mean zero. By Doob's submartingale inequality and (2.22),

$$\begin{aligned}\mathbb{P}(F^*_{s,n} \geq \epsilon) &\leq \epsilon^{-1} \operatorname{var}(F_{s+1,n} - F_{s,n}) \\ &= \epsilon^{-1}[f(n-s-1, n-s-1) - f(n-s, n-s)].\end{aligned}$$

As $f(m-s, m-s)$ is positive and non-increasing in m, we have $\sum_n \mathbb{P}(F^*_{s,n} \geq \epsilon) < \infty$. Thus by the first Borel-Cantelli Lemma, $\mathbb{P}(F^*_{s,n} \geq \epsilon \text{ i.o.}) = 0$. Hence $\mathbb{P}(F^*_{s,\infty} \geq \epsilon) = 0$, and with probability one, $F_{s,\infty}$ is constant on $[s, s+1]$. As $\mathbb{R}$ is a countable union of unit intervals, it follows that with probability one, $F_{s,\infty}$ is constant on $\mathbb{R}$. □

We can also use the stationarity property to prove that if $\mu_{s,\infty} := \lim_{t\to\infty} \mu_{s,t}$ exists for all s, then $\mu_{s,\infty}$ is constant in s; thus long forward-rates are constant (cf., Dybvig, Ingersoll & Ross (1996)).

Remark 2.4 When the derivative $\mathrm{D}f$ exists everywhere, we can write

$$c(0, x, y) = c(-s, x, y) + \int_{-s}^{0} \tau_u(x, y)\, du.$$

If the limit $\lim_{s\to\infty} f(x+s, y+s)$ exists for all x and y, we can let $s \to \infty$ and deduce that

$$f(x, y) = \lim_{s\to\infty} f(x+s, y+s) + \int_0^{\infty} \tau_0(x+u, y+u)\, du.$$

This has the form

$$f(x, y) = \kappa(x, y) + \int_0^{\infty} \tau(x+u, y+u)\, du, \tag{2.23}$$

where $\tau : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a non-negative definite function satisfying

$$\int_x^{\infty} \tau(u, u)\, du < \infty \quad \text{for all } x \in \mathbb{R}, \tag{2.24}$$

and $\kappa : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the covariance structure of a stationary Gaussian process on $\mathbb{R}$. Conversely, given any such τ and κ, defining f via equation (2.23) and Γ by Remark 2.1 gives the covariance structure of a stationary field with independent increments. The Cauchy-Schwarz inequality combined with (2.24) ensure that f is finite. Thus F is the sum of a field which is constant in s and a field that tends to zero as $t \to \infty$.

2.2 Markov Properties

Recall the definition $\mathcal{F}_s = \sigma(F_{u,t} : u \leq s, t \in \mathbb{R})$, of the σ-algebra generated by the forward-rates up to time t, and define

$$\mathcal{G}_s := \sigma(F_{s,t} : t \in \mathbb{R})$$
$$\mathcal{H}_s := \sigma(F_{u,t} : u \geq s, t \in \mathbb{R}).$$

The natural interpretation of the Markov condition is that $\mathcal{F}_s$ and $\mathcal{H}_s$ be conditionally independent given $\mathcal{G}_s$. Another Markov property, which is a common feature of one-factor models, is for the short rate, $F_{s,s}$ to be Markov as a one-dimensional process. It is easy to show that the independent increments property (2.12) automatically implies the former, but the latter requires something stronger. In this section, we will consider stronger forms of the Markov property and investigate their consequences for the covariance structure of F. Kennedy (1997) also considers this problem; under his strongest formulation, the covariance structure of a stationary model is determined by just three parameters, and has the form (2.16). Recall that the Brownian Sheet is the zero-mean, continuous Gaussian field on $[0,\infty)^2$ with covariance structure

$$\text{cov}(X_{s_1,t_1}, X_{s_2,t_2}) = (s_1 \wedge s_2)(t_1 \wedge t_2).$$

This field satisfies the severe Markov property that for all $s_1 \leq s_2 \leq s_3$, $t_1 \leq t_2 \leq t_3$ and all $s_1 \geq s_2 \geq s_3$, $t_1 \geq t_2 \geq t_3$, we have $F_{s_1,t_1} \perp F_{s_3,t_3} \,|\, F_{s_2,t_2}$. Kennedy (1994) demonstrates how continuous injective transformations of the Brownian Sheet can be used to construct interesting random field models; here we will show that any stationary model which arises from a continuous injective transformation of a Brownian Sheet must be the three parameter model with covariance structure given by (2.16).

Before we define our Markov properties, we must introduce some more σ-algebras; define

$$\mathcal{F}^-_{s,t} := \sigma(F_{u,v} : u \leq s, v \leq t)$$
$$\mathcal{F}^+_{s,t} := \sigma(F_{u,v} : u \leq s, v \geq t)$$

and define $\mathcal{G}^\pm_{s,t}$ and $\mathcal{H}^\pm_{s,t}$ similarly.

Definition 2.5 We say that F has the *SWNE-Markov property* if for all s and t we have $\mathcal{F}^-_{s,t} \perp \mathcal{H}^+_{s,t} \,|\, \sigma(F_{s,t})$. Similarly, we say that F has the *NWSE-Markov property* if for all s and t we have $\mathcal{F}^+_{s,t} \perp \mathcal{H}^-_{s,t} \,|\, \sigma(F_{s,t})$.

Remarks (i) Using the independent increments property, we can replace $\mathcal{H}^+_{s,t}$ with $\mathcal{G}^+_{s,t}$ in the definition of SWNE-Markov and $\mathcal{H}^-_{s,t}$ with $\mathcal{G}^-_{s,t}$ in the definition of NWSE-Markov. An immediately consequence of the SWNE-Markov property is that forward-rates of fixed maturity are Ornstein-Ulhenbeck processes; in particular, the spot rate is Markov.

(ii) We can think of the NWSE-Markov property as imposing a form of independence between long and short rates; for example, suppose W_t and W'_t are independent Brownian motions and let $B_t = W_t$ for $t \geq 0$, $B_t = W'_{-t}$ for $t \leq 0$. The field F, defined by

$$F_{s,t} = (B_s - B_t)\,\mathrm{I}(s \geq t), \tag{2.25}$$

is NWSE-Markov, is zero on $s \leq t$ and non-deterministic on $s > t$.

Two related Markov properties are introduced in Kennedy (1997):

Definition 2.6 If for all $s_1 \leq s_2 \leq s_3$ and $t_1, t_2 \in \mathbb{R}$,

$$F_{s_1,t_1} \perp F_{s_3,t_2} \,|\, F_{s_2,t_1}, \tag{2.26}$$

we say that F has the *second Markov property (MP2)*. If $F_{0,t}$ is Markov as a one-dimensional process indexed by t, we say that F is *t-Markov*.

Remark 2.7 From the independent increments property, it is enough to consider the case $s_3 = s_2$ in (2.26).

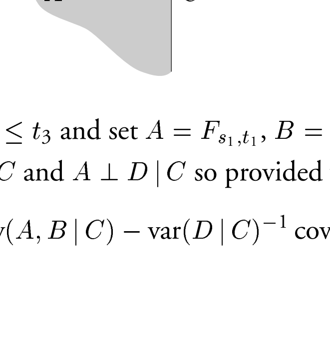
We will split MP2 into the *upper second (MP2.1)* and *lower second (MP2.2)* Markov properties by restricting (2.26) to $t_2 \geq t_1$ and $t_2 \leq t_1$ respectively.

Proposition 2.8

(i) The random field F is SWNE-Markov iff F is t-Markov and satisfies MP2.1.

(ii) The random field F is NWSE-Markov iff F is t-Markov and satisfies MP2.2.

Proof We will only prove the first statement; the proof of the second is very similar. The implication that if F is SWNE-Markov, then it is t-Markov and satisfies MP2.1 is immediate.



Conversely, let $s_1 \leq s_2,\ t_1 \leq t_2 \leq t_3$ and set $A = F_{s_1,t_1}$, $B = F_{s_2,t_3}$, $C = F_{s_2,t_1}$ and $D = F_{s_2,t_2}$. From MP2.1, we have $A \perp B \,|\, C$ and $A \perp D \,|\, C$ so provided $\mathrm{var}(D \,|\, C) > 0$ we have

$$\begin{aligned}\mathrm{cov}(A, B \,|\, C, D) &= \mathrm{cov}(A, B \,|\, C) - \mathrm{var}(D \,|\, C)^{-1}\,\mathrm{cov}(A, D \,|\, C)\,\mathrm{cov}(B, D \,|\, C)\\ &= 0.\end{aligned}$$

If $\mathrm{var}(D \mid C) = 0$, then D is a.s. a function of C and $\mathrm{cov}(A, B \mid C, D) = \mathrm{cov}(A, B \mid C) = 0$. We also have

$$\mathrm{cov}(A, B \mid C, D) = \mathrm{cov}(A, B \mid D) - \mathrm{var}(C \mid D)^{-1}\, \mathrm{cov}(A, C \mid D)\ \mathrm{cov}(B, C \mid D),$$

and $\mathrm{cov}(B, C \mid D) = 0$ by t-Markovness, implying $\mathrm{cov}(A, B \mid D) = 0$. (If $\mathrm{var}(C \mid D) = 0$, then C is a.s. a function of D; thus $\mathrm{cov}(A, B \mid C, D) = \mathrm{cov}(A, B \mid D)$ and $\mathrm{cov}(A, B \mid D) = 0$ again.) Thus $A \perp B \mid D$ and the result follows. □

Recall the definitions of $H = \{(x, y) : y \geq x\}$, and of the map $p : \mathbb{R}^2 \times \mathbb{R}^2 \to H$, given by $p(s_1, t_1, s_2, t_2) = (p_1, p_2)$ where p_1 and p_2 are defined by (2.18) and (2.19), and the fact that if F is stationary with covariance structure Γ, then $\Gamma \circ p^{-1}$ is well defined and equal to f, the covariance structure of $\{F_{0,t} : t \in \mathbb{R}\}$. We now introduce a property which will be central to the discussion of transformations of a Brownian sheet.

Definition 2.9 A random field F is said to *SWNE-factorise* if for all $(x, y) \in \mathbb{R}^2$, there exists $(s_1, t_1, s_2, t_2) \in p^{-1}(x, y)$ with $s_1 < s_2,\ t_1 < t_2$ and open neighbourhoods, U_1, U_2 of (s_1, t_1), (s_2, t_2) respectively such that for all $\alpha \in U_1$, $\beta \in U_2$, we have

$$\Gamma(\alpha, \beta) = \Gamma_1(\alpha)\Gamma_2(\beta)$$

for some functions $\Gamma_1 : U_1 \to \mathbb{R}$, $\Gamma_2 : U_2 \to \mathbb{R}$. A field is said to *NWSE-factorise* if it SWNE-factorises after reflection in the horizontal axis.

Theorem 2.10 *Suppose that F is a stationary random field model which is not deterministic everywhere. If F SWNE-factorises then for some constant μ,*

$$f(x, y) = \exp\bigl(g(x \wedge y) - \mu|x - y|\bigr), \qquad -2\mu \leq \frac{g(y) - g(x)}{y - x} \leq 0. \tag{2.27}$$

while if F NWSE-factorises then

$$f(x, y) = \exp\bigl(g(x \vee y) - \mu|x - y|\bigr), \qquad \frac{g(y) - g(x)}{y - x} \leq 2\mu \wedge 0, \tag{2.28}$$

where we allow g to take the value $-\infty$ in this case.

Proof We will break down the proof into several steps, but first observe that the restrictions on g and μ follow from (i) the fact that $f(x, x)$, and hence $g(x)$ is non-increasing in x, and (ii) the Cauchy-Schwarz inequality applied to $f(x, y)$.

Step 1 Consider the case $\Gamma > 0$. We will first show that f has the correct form locally. Suppose F SWNE-factorises (the NWSE case will be very similar). Let $x < y$ and $(s_1, t_1, s_2, t_2) \in p^{-1}(x, y)$ with $s_1 < s_2$, $t_1 < t_2$. Let U_1 and U_2 be open discs with centres $(s_1, t_1), (s_2, t_2)$ respectively, such that for all $\alpha \in U_1$, $\beta \in U_2$

$$\alpha_x < \beta_x, \quad \alpha_y < \beta_y. \tag{2.29}$$

Now let $\alpha \in U_1$, $\beta \in U_2$ be arbitrary. By SWNE-factorisation and Remark 2.1 we have

$$\begin{aligned}\Gamma(\alpha,\beta) &= \Gamma_1(\alpha)\Gamma_2(\beta)\\ &= f(\alpha_y-\alpha_x,\beta_y-\alpha_x),\end{aligned} \tag{2.30}$$

so $\Gamma_2(\beta)$ cannot depend on β_x. Since $\Gamma > 0$, w.l.o.g. Γ_1 and Γ_2 are both positive. Setting $s(\alpha)=\alpha_y-\alpha_x$ and writing $\Gamma_2(\beta_y)$ for $\Gamma_2(\beta)$, we have

$$\begin{aligned}\log\Gamma(\alpha,\beta) &= \log f(s,\beta_y-\alpha_x)\\ &= \log\Gamma_1(\alpha_x,\alpha_x+s)+\log\Gamma_2(\beta_y).\end{aligned} \tag{2.31}$$

Using that fact that for any continuous functions a, b and c satisfying $a(x)+b(y)=c(x-y)$ on a connected open subset of $\mathbb{R}^2$, c must be linear, condition (2.31) implies that $\log f(s,\beta_y-\alpha_x)$ is linear in $\beta_y-\alpha_x$ for each s. Now let r be the radius of U_1 and V be the open disc with centre (s_1,t_1) and radius $\frac{1}{\sqrt{2}}r$. Let $\beta^*=(s_2,t_2)$, and for $\alpha\in V$, let $\alpha^*=(s_1,s_1+s(\alpha))$. Restricting attention to $\alpha\in V$, we have $\alpha^*\in U_1$ and $s(\alpha)=s(\alpha^*)$ so

$$\log\Gamma_1(\alpha)+\log\Gamma_2(\beta)=\log f(\alpha_y-\alpha_x,\beta_y^*-\alpha_x^*)-\mu(s)[(\beta_y-\alpha_x)-(\beta_y^*-\alpha_x^*)]$$

for some function $\mu(s)$. Considering the dependence of both sides on β_y, and noting that s depends on α but not β, we see that $\mu(s)$ must independent of s. Thus we have

$$\begin{aligned}\log\Gamma(\alpha,\beta) = &\log f(\alpha_y-\alpha_x,\beta_y^*-\alpha_x^*)+\mu(\beta_y^*-\alpha_x^*)\\ &-\mu(\alpha_y-\alpha_x)-\mu(\beta_y-\alpha_y),\end{aligned} \tag{2.32}$$

for some constant μ. Defining

$$g(s) := \log f(s,\beta_y^*-\alpha_x^*)+\mu(\beta_y^*-\alpha_x^*)-\mu s$$

and noting that $\alpha_y-\alpha_x<\beta_y-\alpha_x$ in (2.30), we see from (2.32) that f has the correct form on $p(V\times U_2)$.

Since μ is uniquely determined by $f|_{p(V\times U_2)}$, so too is the function $g|_{p_1(V\times U_2)}$. As the set $\{(x,y): x<y\}$ is connected, f must take the required form on the whole of $\{(x,y): x<y\}$, for some constant μ and function g. As f is continuous and symmetric, it has the required form on all of $\mathbb{R}^2$. The proof in the NWSE case is identical, except that we replace (2.29) with $\alpha_x<\beta_x$, $\alpha_y>\beta_y$ and now have $\alpha_y-\alpha_x>\beta_y-\alpha_x$.

Step 2 Suppose that $\Gamma(\alpha,\alpha)=0$ for some $\alpha\in\mathbb{R}^2$. It follows that $F_{s,t}$ is deterministic for $t-s\geq\alpha_y-\alpha_x$. To see this, first note that stationarity implies that $F_{S(x)\alpha}$ is deterministic for all $x\in\mathbb{R}$. Now let $t-s\geq\alpha_y-\alpha_x$. By the independent increments property, $F_{S(t-\alpha_y)\alpha}-F_{s,t}\perp F_{s,t}$, thus $F_{s,t}$ is deterministic.

Step 3 Now consider the case of general Γ. Let $r\in[-\infty,\infty]$ be the unique r such that F is deterministic on the set $Z=\{(s,t): t-s\geq r\}$ and $\mathrm{var}(F)>0$ on $Z^c=\{(s,t): t-s<r\}$.

If $r = -\infty$ then F is deterministic everywhere, a case excluded in the statement of the theorem, so suppose $r > -\infty$. We know that $f = 0$ on $p(Z \times \mathbb{R}^2)$ and $p(\mathbb{R}^2 \times Z)$, so we now consider the form of f on $p(Z^c \times Z^c)$. Let $A = \{(\gamma, \gamma') \in Z^c \times Z^c : \Gamma(\gamma, \gamma') > 0\}$, which is open, and non-empty since $r > -\infty$. We note that $\{(\alpha, \alpha) : \alpha \in Z^c\}$ is a connected subset of A, and let B be the connected component of A containing $\{(\alpha, \alpha) : \alpha \in Z^c\}$. It will turn out that $p(B) = p(Z^c \times Z^c)$. As p is a continuous open mapping, $p(B)$ is open and connected. Our aim is to apply Step 1 to $f|_{p(B)}$, but we must check that we can choose U_1 and U_2 such that $U_1 \times U_2 \subseteq B$. If $(x, y) \in p(B)$, say $p(s_1, t_1, s_2, t_2) = (x, y)$, then $(s_1 \wedge s_2, t_1, s_1 \wedge s_2, t_2) \in B$. As B is open, we can find neighbourhoods U_1, U_2 such that $U_1 \times U_2 \subseteq B$ as required. Applying Step 1 to $f|_{p(B)}$, we deduce that f takes the required form on $p(B)$. Finally let $(\alpha, \beta) \in \partial B$, which cannot be empty unless $B = \mathbb{R}^2 \times \mathbb{R}^2$ when $\Gamma > 0$ and we are done by Step 1. We will show that $p(\alpha, \beta) \notin p(Z^c \times Z^c)$. Pick a sequence $\{(\alpha_x^n, \alpha_y^n, \beta_x^n, \beta_y^n)\}$ of points in B such that $(\alpha^n, \beta^n) \to (\alpha, \beta)$. If we are dealing with the SWNE case set

$$y_n = \alpha_y^n \wedge \beta_y^n - \alpha_x^n \wedge \beta_x^n,$$

and otherwise set

$$y_n = \alpha_y^n \vee \beta_y^n - \alpha_x^n \wedge \beta_x^n,$$

so that

$$\Gamma(\alpha^n, \beta^n) = \exp\big(g(y_n) - \mu|\beta_y^n - \alpha_y^n|\big).$$

As $(\alpha, \beta) \notin A$, Γ is continuous and $\mu > -\infty$ is constant throughout B, it follows that as $n \to \infty$, $g(y_n) \to -\infty$. Thus we must be dealing with NWSE rather than SWNE factorisation. Since $p_2(\alpha^n, \beta^n) = y_n$ and $\Gamma(\alpha^n, \beta^n) > 0$, we have $y_n < r$, so $(0, y_n, 0, y_n) \in B$. Now

$$\begin{aligned}
\operatorname{var}(F_{0,\alpha_y \vee \beta_y - \alpha_x \wedge \beta_x}) &= \lim_{n \to \infty} \operatorname{var}(F_{0,y_n}) \\
&= \lim_{n \to \infty} \exp\big(g(y_n)\big) \\
&= 0.
\end{aligned}$$

Thus we must have $\alpha_y \vee \beta_y - \alpha_x \wedge \beta_x \geq r$, so $p(\alpha, \beta) \notin p(Z^c \times Z^c)$, and so $p(\partial B) \subseteq H - p(Z^c \times Z^c)$. Since $p(B)$ is open and connected and $p(B) \subseteq p(Z^c \times Z^c)$ (which is also open and connected) we must have $p(B) = p(Z^c \times Z^c)$. Thus we have established that f has the correct form on all of $p(Z^c \times Z^c) = \{(x, y) : x \leq y < r\}$. Defining $g(x) = -\infty$ for $x \geq r$ gives the correct form for f on the whole of $\mathbb{R}^2$. □

Remarks (i) If F has either factorisation property, Γ is non-negative and interest-rates are positively correlated, a property usually observed in real interest-rates. In the SWNE-Markov case, Γ is strictly positive.

(ii) When F has both factorisation properties, f has the equivalent forms

$$\begin{aligned} f(x,y) &= \sigma^2 \exp\bigl(-\lambda x \wedge y - \mu|x-y|\bigr) \\ &= \sigma^2 \exp\bigl(-\lambda x \vee y - (\mu-\lambda)|x-y|\bigr) \end{aligned}$$

where $0 \le \lambda \le 2\mu$. This is the three parameter model of Kennedy (1997).

A consequence of the form of f given by Theorem 2.10 is that each factorisation property is equivalent to the corresponding Markov property:

Corollary 2.11 *Let F be a stationary random field model.*
(i) The field F is SWNE-Markov iff F SWNE-factorises.
(ii) The field F is NWSE-Markov iff F NWSE-factorises.

Proof We will only prove the first statement; the proof of the second is virtually identical. Let $s_1 < s_2 < s_3$, $t_1 < t_2 < t_3$ be arbitrary. To show F is SWNE-Markov, we must show that $F_{s_1,t_1} \perp F_{s_3,t_3} \,|\, F_{s_2,t_2}$, i.e., that

$$\Gamma\bigl((s_1,t_1),(s_2,t_2)\bigr)\Gamma\bigl((s_2,t_2),(s_3,t_3)\bigr) = \Gamma\bigl((s_1,t_1),(s_3,t_3)\bigr)\Gamma\bigl((s_2,t_2),(s_2,t_2)\bigr).$$

Using the form of f given by Theorem 2.10 we see both sides reduce to

$$\exp\bigl(g(t_1-s_1)+g(t_2-s_2)+\mu(t_3-t_1)\bigr).$$

Conversely, suppose $x<y$; set $r=(y-x)/4$ and choose U_1, U_2 as the open discs, radii r, with centres $(0,x), (y-x,y)$ respectively. Let γ be the point $(3r, x+r)$. For $\alpha \in U_1, \beta \in U_2$ we have $F_\alpha \perp F_\beta \,|\, F_\gamma$ by the SWNE-Markov property. Thus

$$\Gamma(\alpha,\beta) = \operatorname{var}(F_\gamma)^{-1}\,\Gamma(\alpha,F_\gamma)\,\Gamma(\beta,F_\gamma),$$

so F SWNE-factorises. (If $\operatorname{var}(F_\gamma)=0$, it is easy to show that $\Gamma(\alpha,\beta)=0$ for all $\alpha \in U_1$, $\beta \in U_2$ (see Step 2 of the proof of Theorem 2.10) so F trivially SWNE-factorises.) □

Remark 2.12 When F has both Markov properties, f has the form

$$f(x,y) = \sigma^2 \exp\bigl(-\lambda x \wedge y - \mu|x-y|\bigr).$$

Kennedy (1997) proves the equivalent result that a stationary t-Markov random field model satisfying MP2 has a covariance structure of this form. He also observes that this covariance structure arises as a continuous transformation of the Brownian Sheet

$$F_{s,t} = \sigma e^{-\mu t} X_{e^{\lambda s}, e^{(2\mu-\lambda)t}}.$$

In the next section we will show that this is the only continuous injective transformation of a Brownian Sheet to give rise to a stationary random field model.

2.2.1 Transformations of a Brownian sheet

Recall that the Brownian Sheet is the continuous Gaussian field on $[0,\infty)^2$ with mean zero and covariance structure

$$\mathrm{cov}(X_{s_1,t_1}, X_{s_2,t_2}) = (s_1 \wedge s_2)(t_1 \wedge t_2)$$

(see Adler (1981) or Rogers & Williams (1994)). In this section, we will show that the only stationary random field model which arises from an injective transformation of a Brownian Sheet, is the three parameter model shown in (2.16). Throughout this section, F will denote a stationary random field model of the form

$$F_{s,t} = K_{s,t} W_{\phi(s,t)} \tag{2.33}$$

where $K : \mathbb{R}^2 \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to [0,\infty)^2$ are continuous functions, and ϕ is injective. We let ϕ_x and ϕ_y denote the coordinate projections of ϕ, so

$$\phi(s,t) = (\phi_x(s,t), \phi_y(s,t)).$$

Lemma 2.13 *If* $\Gamma(\alpha,\beta) = 0$ *for some* α, β, *then* $F(\gamma) = 0$ *a.s. for all* $\gamma \in \mathbb{R}^2$.

Proof If $\Gamma(\alpha,\beta) = 0$ then either at least one of K_α and K_β is zero, or at least one of $\phi(\alpha)$ and $\phi(\beta)$ lies on the coordinate axes. Therefore, w.l.o.g., we may assume that $\mathrm{var}(F_\alpha) = 0$ implying that the field is zero a.s. on the diagonal upper half plane through α (see Step 2 in the proof of Theorem 2.10). Now suppose that there exists $\gamma \in \mathbb{R}^2$ with $\mathrm{var}(F_\gamma) \neq 0$. Let η be the point on $\{S(y)(\alpha) : y \in \mathbb{R}\}$ closest to γ, let $\gamma' = S(2(\eta_y - \gamma_y))\gamma$ and $\zeta = S(\eta_y - \gamma_y)\eta$. Since $F_\zeta = 0$ a.s. (by stationarity) and $F_\gamma \perp (F_{\gamma'} - F_\zeta)$, we have $F_\gamma \perp F_{\gamma'}$. Thus $\Gamma(\gamma,\gamma') = 0$, so by the previous argument, at least one of $\mathrm{var}(F_\gamma)$ and $\mathrm{var}(F_{\gamma'})$ is 0. Stationarity implies that both these variances are equal, and we conclude $\mathrm{var}(F_{\gamma'}) = \mathrm{var}(F_\gamma) = 0$, a contradiction. □

To exclude this trivial case, we will assume from now on that $\Gamma > 0$. We will now exploit the special Markov structure of the Brownian Sheet to show that F both SWNE and NWSE-factorises. Introduce the notation $T(d)$ for the translation parallel through a distance d parallel to the x-axis, $T(d)(s,t) = (s+d,t)$.

Theorem 2.14 *The field* F *both SWNE and NWSE-factorises.*

Proof We break up the proof in to several pieces.
Step 1 We first show that F SWNE-factorises. Let $x < y$, $\epsilon > 0$, and define

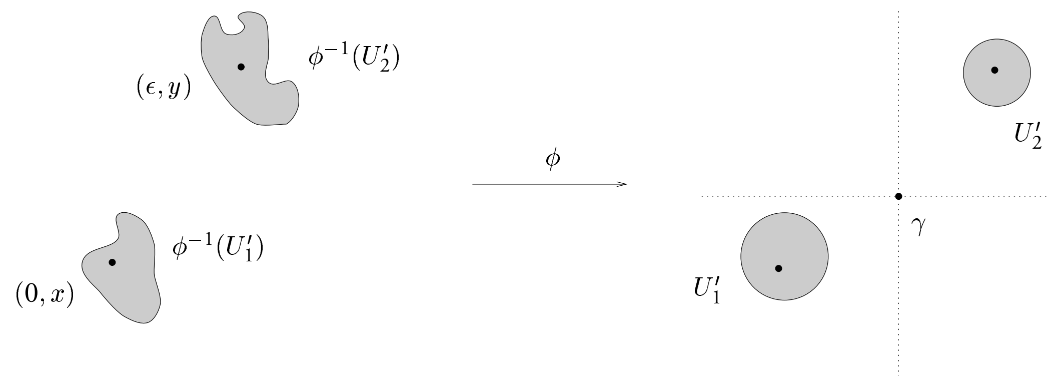
$$\mathrm{cro}(u,v) := \{(s,t) : s = u \text{ or } t = v\}$$

to be the 'cross' formed by the union of the horizontal and vertical lines through (u,v). Let $\gamma = \frac{1}{2}(\phi(0,x) + \phi(\epsilon,y))$. Note that either both $\phi(0,x)$ and $\phi(\epsilon,y)$ are contained in $\mathrm{cro}(\gamma)$ or neither is, so we have two cases to consider.

Case 1a If $\phi(0,x), \phi(\epsilon,y) \notin \text{cro}(\gamma)$, we consider the situation when

$$\phi_x(0,x) < \phi_x(\epsilon,y), \qquad \phi_y(0,x) < \phi_y(\epsilon,y),$$

(the other cases can be handled in a similar way). Let U_1', and U_2' be open neighbourhoods of $\phi(0,x)$, $\phi(\epsilon,y)$ respectively which do not intersect cro(γ).



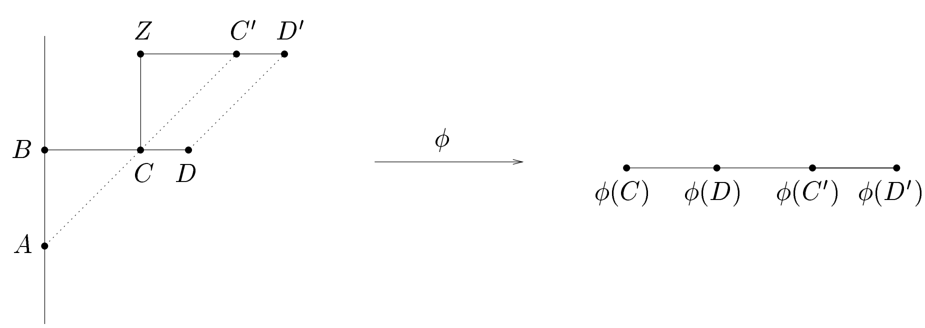
Now set $U_1 = \phi^{-1}(U_1')$, $U_2 = \phi^{-1}(U_2')$. By the Markov properties of the Brownian Sheet, we have

$$F_\alpha \perp F_\beta \mid X_\gamma \qquad \text{for } \alpha \in U_1, \beta \in U_2.$$

Thus F SWNE-factorises in a neighbourhood of (x,y).

Case 1bi Suppose $\phi(0,x), \phi(\epsilon,y) \in \text{cro}(\gamma)$, but $\phi(T(t)(0,y)) \notin \text{cro}(\phi(0,x))$ for some $t > 0$. Since $p((0,x), T(t)(0,y)) = (x,y)$ for all $t > 0$, a similar argument to the one used in Case 1a shows that F SWNE-factorises near (x,y).

Case 1bii Now suppose that $\phi(\{T(t)(0,y) : t > 0\}) \subseteq \text{cro}(\phi(0,x))$. Let $A = (0,x)$, $B = (0,y)$, $C = (y-x,y)$ and $D = (\frac{3}{2}(y-x), y)$. Since ϕ is injective, $\phi(C) \neq \phi(A)$, and by continuity, $\phi(C)$ and $\phi(D)$ lie in the same 'branch' of $\text{cro}(\phi(A))$. For all $s,t \geq 0$, $p(S(s)A, T(t)S(s)B) = (x,y)$. Thus, either F SWNE-factorises by an argument similar to Case 1bi, or $\phi(\{T(t)S(s)B : t \geq 0\}) \subseteq \text{cro}(\phi(S(s)A))$ for all $s \geq 0$. As $S(s)A$, $S(s)C$ and $S(s)D$ are distinct and ϕ is injective, $\phi(S(s)A)$, $\phi(S(s)C)$ and $\phi(S(s)D)$ are also distinct. From the continuity of ϕ, we deduce that $\phi(S(s)C)$ and $\phi(S(s)D)$ lie in the same branch of $\text{cro}(\phi(S(s)A))$ for every $s \geq 0$.



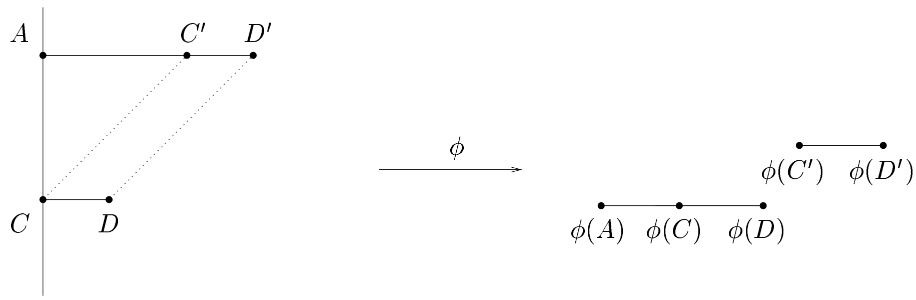
Now choose $s = y-x$, and let $C' = S(y-x)C$ and $D' = S(y-x)D$. As $S(y-x)A = C$, continuity of ϕ implies that $\phi(C)$, $\phi(D)$, $\phi(C')$ and $\phi(D')$ are collinear; in addition $\overrightarrow{\phi(C)\phi(D)}$ points in the same direction as $\overrightarrow{\phi(C')\phi(D')}$. Finally let $Z = (C_x, C_y + y - x)$. We have $F_D \perp F_{D'} \,|\, F_{C'}$, so $\phi(C') \in [\phi(D), \phi(D')]$, $F_C \perp F_{C'} \,|\, F_Z$, so $\phi(Z) \in [\phi(C), \phi(C')]$, and $F_Z \perp F_D \,|\, F_C$, so $\phi(C) \in [\phi(Z), \phi(D)]$, which imply $\phi(C) = \phi(Z)$. Hence ϕ is not injective, a contradiction.

Step 2 We now show NWSE-factorisation (the first two cases are very similar to Cases 1a and 1bi above). Let $\epsilon > 0$ and consider the points $\phi(\epsilon, x)$ and $\phi(0, y)$. Let $\gamma = \frac{1}{2}(\phi(\epsilon, x) + \phi(0, y))$.

Case 2a If $\phi(\epsilon, x), \phi(0, y) \notin \text{cro}(\gamma)$ we can prove that F NWSE-factorises using a very similar argument to Case 1a.

Case 2bi Suppose $\phi(0, x), \phi(\epsilon, y) \in \text{cro}(\gamma)$, but $\phi(T(t)(0, x)) \notin \text{cro}(\phi(0, y))$ for some $t > 0$. Since $p(T(t)(0, x), (0, y)) = (x, y)$ for all $t > 0$, the argument of Case 1bi shows that F NWSE-factorises near (x, y).

Case 2bii Now suppose $\phi(\{T(t)(0, x) : t > 0\}) \subseteq \text{cro}(\phi(0, y))$. Let $A = (0, y)$, $C = (0, x)$ and $D = (\frac{1}{2}(y-x), x)$ and note that $\phi(C)$ and $\phi(D)$ lie in the same branch of $\text{cro}(\phi(A))$.



Since $p(S(s)A, T(t)S(s)C) = (x, y)$ for all $s, t > 0$, either F NWSE-factorises by the argument of Case 2bi or $\phi(\{T(t)S(s)C : t \geq 0\}) \subseteq \text{cro}(\phi(S(s)A))$ for all $s > 0$. As ϕ is continuous and injective, $\phi(A)$, $\phi(C)$ and $\phi(D)$ are distinct and collinear. Let $C' = S(y-x)C$ and $D' = S(y-x)D$. As ϕ is continuous, $\overrightarrow{\phi(C)\phi(D)}$ is parallel to and points in the same direction as $\overrightarrow{\phi(C')\phi(D')}$. Finally, observe that $F_D \perp F_{D'} \,|\, F_{C'}$, $F_A \perp F_D \,|\, F_C$ and $F_C \perp F_{C'} \,|\, F_A$. Together these imply that $\phi(C) = \phi(A)$. Hence ϕ is not injective, giving a contradiction. □

Corollary 2.15 *The field F has the same law as*

$$\sigma e^{-\mu t} W_{e^{\lambda s}, e^{(2\mu - \lambda)t}} \tag{2.34}$$

where $0 \leq \lambda \leq 2\mu$ *and* W *is a Brownian Sheet.*

Proof Since F is NWSE-Markov it NWSE-factorises, by Corollary 2.11, and similarly, since it is SWNE-Markov, it SWNE-factorises. Thus by Remark (ii), on page 26, it has a covariance structure of the form

$$f(x, y) = \sigma^2 \exp\bigl(-\lambda(t_1 \wedge t_2 - s_1 \wedge s_2) - \mu |t_1 \vee t_2 - s_1 \wedge s_2|\bigr), \qquad 0 \leq \lambda \leq 2\mu,$$

where $f(x, y) = \mathrm{cov}(F_{0,x}, F_{0,y})$. But this is also the covariance structure of the field given by (2.34), so the result follows. $\square$

3

Option Pricing Techniques

3.1 Introduction

In this chapter, we consider the problem of calculating option prices in a generalisation of the Black-Scholes model. The normal Black-Scholes model assumes constant volatility and interest-rate (see Section 1.3), but here we will suppose that the volatility and interest-rate are time-dependent but deterministic. Specifically, we assume that the stock price obeys the SDE

$$dS_t = S_t[\sigma(t)\,dW_t + \mu(t)\,dt], \tag{3.1}$$

for a Brownian motion W_t, and deterministic functions $\sigma(t)$ and $\mu(t)$, where $\mu(t)$ is bounded, and $\sigma(t)$ is bounded away from zero. The instantaneous riskless interest-rate, $r(t)$, is also assumed to be deterministic and bounded, and we define $D(t,T)$, the discount factor for the period $[t,T]$ by

$$\log D(t,T) = -\int_t^T r(u)\,du.$$

Recall from Section 1.3 that the time-t value of an option, which confers the right (but not the obligation) to receive the amount X at time T, is given by

$$V_t = D(t,T)\,\mathbb{E}^{\mathbb{Q}}\big(X^+ \,\big|\, \mathcal{F}_t\big), \tag{3.2}$$

where $X^+ = \max(0,X)$, the filtration $\mathcal{F}$ is the filtration generated by W_t, and $\mathbb{Q}$ denotes the martingale measure—under which the discounted stock price, $D(0,t)S_t$, is an $\mathcal{F}$-martingale. Using the Cameron-Martin-Girsanov theorem, we can rewrite (3.1) as

$$dS_t = S_t[\sigma(t)\,d\tilde{W}_t + r(t)\,dt] \tag{3.3}$$

where $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. To calculate time-t prices, we will use the function $\tau(s)$, defined for fixed t, by

$$\tau(s) = 0 \qquad \text{for } s \le t, \tag{3.4}$$

$$\tau'(s) = \sigma(s)^2 \qquad \text{for } s \ge t, \tag{3.5}$$

and the function $\alpha(s)$, defined by

$$\alpha(s) = \log(S_s/S_t) \qquad \text{for } s \le t,$$
$$\alpha'(s) = r(s) - \tfrac{1}{2}\sigma(s)^2 \qquad \text{for } s \ge t.$$

Notice that with these definitions $\alpha(s)$ is $\mathcal{F}_t$-measurable and we can write the solution to (3.3) as

$$S_s = S_t \exp\bigl(B_{\tau(s)} + \alpha(s)\bigr) \tag{3.6}$$

where B is a $\mathbb{Q}$-Brownian motion, independent of $\mathcal{F}_t$.

If, like all the options considered in Section 1.3, an option contract can only be exercised at a predetermined time, it is referred to as a *European* option. In contrast, an *American* option may be exercised at any point, up to some prearranged expiry time; for example, a standard American call option with strike price K, is worth $S_t - K$ if it is exercised at time t, but the holder may decide to wait if S_t is below or only just above K. If S_t never reaches K before the expiry time, the option will never be exercised. The new methods developed in this chapter deal with European, rather than American options.

In Section 3.2 we consider two examples of 'Asian' options. Here the payoff depends on the average of S_t over some time period,

$$A_t := \int_0^t S_u \, d\eta(u),$$

where η is a positive measure with $\eta[0,T] = 1$. In practice, the average is usually discrete, such as one sample per month, but the case of continuous averaging is sometimes simpler to handle theoretically. Note the time-t value of such an option will generally depend on A_t, in addition to t and S_t. The two Asian options we will consider are the fixed-strike call option, corresponding to the choice $X = A_T - K$ and the floating-strike call option, where $X = A_T - S_T$. We will derive accurate upper and lower bounds on the value of both kinds of option.

Another interesting type of option is the barrier option: we let G_t and F_t be deterministic functions with $G_0 < S_0 < F_0$ and consider an option containing a knock-out clause, cancelling the contract if S_t ever hits G_t or F_t. A standard example is the knock-out call option, where

$$X = (S_T - K)\,\mathrm{I}(G_t < S_t < F_t, 0 \le t \le T).$$

In Section 3.3 we derive bounds on the price of barrier options, which are accurate when F_t and G_t are twice-differentiable and approximately linear.

In Section 3.4, we modify our technique developed for Asian options to price options of the form $X = f(\sum L_i)$ where $\{\log L_i\}$ are joint normal under $\mathbb{Q}$. This covers many types of options: basket options, swaptions (see Section 1.4) and stock index options, provided the underlying stochastic processes (exchange-rates, interest-rates, stock prices etc.) are lognormal under $\mathbb{Q}$.

There is an extensive literature describing numerical methods for option pricing. The most robust and widely applicable approach to European options is Monte-Carlo simulation: simply draw a large number of samples of all the variables on which X depends from $\mathbb{Q}$, and take an average of the discounted payouts. General discussions of applying the Monte-Carlo method to option pricing can be found in Boyle (1977) and Boyle, Broadie & Glasserman (1997). Its main drawback is its slow convergence, but it can be used to get a rough estimate (or even an accurate estimate if speed is not a factor), and can handle very complex options and stock price models. Techniques of variance reduction can increase its accuracy considerably; see for example Newton (1994) and Newton (1997) for a discussion of these techniques in the context of the simulation of diffusion sample paths. Stochastic volatility models are natural candidates for a Monte-Carlo approach; Carverhill & Clewlow (1994) use control variates to speed up the valuation of look-back options, while Fournié, Lasry & Touzi (1997) use importance sampling to price European call options. Recently, methods based on so-called quasi-random sequences have been tried (see Berman (1997) and its references).

To get fast, accurate answers we leave aside Monte-Carlo methods. Two remaining methods, applicable to both European and American options are the tree method (see Baxter & Rennie (1996) for example) and the PDE method (Wilmott, Dewynne & Howison (1993) being the standard reference). Both of these methods are efficient for simple option pricing problems, but rely on discretising all the variables relevant to the price of the option, and as a result, the methods converge slowly for options with discontinuous payouts (such as barrier options which are discontinuous in the stock price). They also suffer from the so-called 'curse of dimensionality': if the option value depends on more than just time and stock price, but on time and the position of an n-dimensional Markov process (as is the case for an Asian option which depends on A_t), it becomes necessary to discretise a sufficiently large region of $\mathbb{R}^n$. If n is more than about two or three, this can be computationally difficult. If the option does not suffer either of these problems, the PDE and tree methods are effective ways of solving a wide range of option pricing problems. A recent development of the tree method is to allow random, rather than fixed time-steps, (see Rogers & Stapleton (1998) and Leisen (1997) for example); these methods appear to enjoy faster convergence than trees with fixed time-steps.

The original material of this chapter takes a different approach. Relying on simple optimisation techniques, we derive bounds on the value of several types of option, which turn out to be accurate for typical option pricing problems. (A similar approach has been applied to American call options with considerable success, see Broadie & Detemple (1995) and Broadie & Detemple (1996).)

Notational Note 3.1 Henceforth, unless otherwise indicated, all expectations and probabilities will refer to the martingale measure, which will now be denoted $\mathbb{P}$. We will also use the notation $\mathbb{E}_t(\,\cdot\,)$ for $\mathbb{E}(\,\cdot\,|\,\mathcal{F}_t)$ expectation under $\mathbb{P}_t$, the martingale measure conditional on the history up to time t, (and similarly, $\mathrm{cov}_t(\,\cdot,\cdot\,)$ and $\mathrm{var}_t(\,\cdot\,)$).

3.1.1 Tree methods

For simplicity, consider an option whose payout at time T, depends only on the final value of the stock, S_T. Setting $v = \mathbb{E}_t(X^+)$, it follows from the Markovian property of S_t, that v is a just function of S_t and t. By discretising time, by restricting to the set $\{0, \Delta t, \ldots, N\Delta t\}$, where $\Delta t = T/N$, we can calculate $v(S_t, t)$ recursively, working backwards from time T, using the relation

$$v(x,t) = \mathbb{E}[v(S_{t+\Delta t}, t + \Delta t) \mid S_t = x].$$

The option price at time 0 is then $D(0,T)v(S_0, 0)$.

In practice, since we can only store a finite set of values of $v(x,t)$ at each stage, we must restrict the range of $S_{t+\Delta t}$ (and we will now only have $v(x,t) \approx \mathbb{E}_t(X^+)$). The simplest solution, which works when the volatility and interest-rate are constant, is the binomial model of Cox, Ross & Rubinstein (1979). They assume that $S_{t+\Delta t}$ takes the value uS_t with probability p and dS_t with probability $1 - p$ for constants u, d and p. It is convenient to take $d = u^{-1}$ so that the tree 'recombines', and no more than $N + 1$ values need ever be stored. We determine u and p from the fact that $\log(S_{t+\Delta t}/S_t)$ has mean $(r - \frac{1}{2}\sigma^2)\Delta t$ and variance $\sigma^2 \Delta t$ under $\mathbb{Q}$. As $\Delta t \to 0$, the price produced by this procedure converges to the true value. (If $\sigma(t)$ and $r(t)$ are non-constant, we can use a 'trinomial' tree instead. This has the flexibility to cope with time-varying coefficients, even of the form $\sigma(S_t, t)$ and $r(S_t, t)$, and still give rise to a recombining tree.)

The tree method is simple to implement, but the condition $p \geq 0$ can force Δt to be very small. It also suffers from the usual problems associated with discretising the stock price (poor convergence for barrier options) and often exhibits an 'odd-even' bias for options with non-smooth payoffs, whereby the value produced by the method oscillates with the parity of N. For a European call option with $K = S_t$, for even values of N, the sequence of approximate option prices converges to the true price from above, while for odd N it converges from below. Some form of averaging or extrapolation is usually employed to counter these problems.

3.1.2 PDE methods

Like the tree method, the PDE method finds the time-t price of an option by first calculating the value of a related martingale. In this section, we consider the process $v_t = D(0,t)V_t$, where V_t is the time-t value of the option; from (3.2) it can be seen that this is indeed a martingale. If the option's payout is a function of the time-T stock price, then V will be a function of t and S_t alone. More generally, V could depend on t and the time-t position of a Markov semimartingale Y_t; for example, if we were pricing an Asian option, we could well have $Y_t = (S_t, A_t)$. Provided v is sufficiently smooth, we can use Itô's Lemma to derive a partial differential equation satisfied by V. To keep the notation simple, we will consider the case where Y_t is the one-dimensional process S_t, so $V = V(S_t, t)$. Applying Itô's Lemma to $D(0,t)V(S_t,t)$ and setting the finite variation term to zero, gives

$$\tfrac{1}{2}\sigma(t)^2 x^2 \frac{\partial^2 V}{\partial x^2} + r(t)x\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} - r(t)V = 0, \qquad \text{on } [0,T] \times \mathbb{R}, \tag{3.7}$$

the *Black-Scholes equation*, to be solved subject to the boundary condition that $V(x,T)$ must equal the payout of the option when $S_T = x$.

Numerical methods for solving PDE's of this form are well developed; here will give a quick description of three methods, and refer the reader to Wilmott et al. (1993) for further details. The general approach is to restrict (x,t) to a rectangular lattice

$$t \in \{n\Delta t : n = 0, 1, \ldots, N-1, N\}$$
$$x \in \{m\Delta x : m = -M, -M+1, \ldots, M-1, M\}$$

where $N\Delta t = T$ and $M\Delta x$ is 'large', about 3(var $S_T)^{1/2}$, and to interpret (3.7) as a system of linear equations, by replacing the partial derivatives with linear combinations of V evaluated at nearby points in the lattice. We must also specify boundary conditions on $\{x = \pm M\Delta x\}$, but this choice is generally not crucial provided $M\Delta x$ is large; if no asymptotic boundary condition is available, simply assuming that the stock grows at the risk-free interest-rate for the remaining period up to time T usually works.

The explicit method

If we use the approximations

$$\left.\frac{\partial V}{\partial t}\right|_{x,t} = (\Delta t)^{-1}[V(x, t+\Delta t) - V(x,t)]$$
$$\left.\frac{\partial V}{\partial x}\right|_{x,t} = (2\Delta x)^{-1}[V(x+\Delta x, t+\Delta t) - V(x-\Delta x, t+\Delta t)]$$
$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x,t} = (\Delta x)^{-2}[V(x+\Delta x, t+\Delta t) - 2V(x, t+\Delta t) + V(x-\Delta x, t+\Delta t)]$$

we can rearrange the resulting system of linear equations implied by (3.7), to give $V(x,t)$ explicitly in terms of $V(x+\Delta x, t+\Delta t)$, $V(x, t+\Delta t)$ and $V(x-\Delta x, t+\Delta t)$. This is really no more than a trinomial tree method, as shown by Brennan & Schwartz (1978). The standard objection to the explicit method is that it is only *stable* if $\Delta t \leq (\Delta x)^2/\sigma^2$, which can require a very small time-step. (This is similar to the problem of ensuring $p \geq 0$ in the binomial method.)

The implicit method

The implicit method uses the approximations

$$\begin{aligned}
\left.\frac{\partial V}{\partial t}\right|_{x,t} &= (\Delta t)^{-1}[V(x, t + \Delta t) - V(x, t)] \\
\left.\frac{\partial V}{\partial x}\right|_{x,t} &= (2\Delta x)^{-1}[V(x + \Delta x, t) - V(x - \Delta x, t)] \\
\left.\frac{\partial^2 V}{\partial x^2}\right|_{x,t} &= (\Delta x)^{-2}[V(x + \Delta x, t) - 2V(x, t) + V(x - \Delta x, t)].
\end{aligned}$$

This time $V(x, t)$ is not just a function of $V(\,\cdot\,, t + 1)$, and solving the system of equations resulting from (3.7) requires a matrix inversion. This can impose a significant memory burden, particularly if Y_t is actually two or three dimensional, when the curse of dimensionality becomes apparent. The advantage of the implicit method over the explicit method is that is unconditionally stable, so a much larger time-step can be used.

The Crank-Nicolson method

The only difference between the implicit and explicit methods is the discretisation of the partial derivatives. Other choices are possible; if we take the average of the implicit and explicit discretisations, we obtain the Crank-Nicolson scheme. Like the implicit method, this method is unconditionally stable and thus does not need a very small time-step. It also requires matrix inversion, but unlike the previous schemes, it has the advantage of being accurate to second order in Δt. (All three methods are accurate to second order in Δx.)

3.2 Asian options

The term 'Asian option' is a generic term covering any contract whose payout involves the average of some quantity over a period of time. The underlying quantity is often an exchange rate, though for our discussion, we will assume that it is a stock price (all we really require is lognormality under the martingale measure). Such options are usually less sensitive to sudden price movements than normal European call options. In consequence, they are generally easier to hedge, and they are more effective at reducing certain types of risk.

In this section we will consider two examples of Asian options: the fixed-strike call option and the floating-strike call option. A *fixed-strike* Asian call option on a stock price S_t, with averaging measure η, exercise time T and strike $K > 0$, is a contract with value $(A_T - K)^+$ at time T, where

$$A_t = \int_0^t S_u \, d\eta(u).$$

The measure η is assumed to be positive and satisfy $\eta[0, T] = 1$; typically it is concentrated in a set of equally spaced points or it is just proportional to Lebesgue measure, $d\eta(t) = dt/T$. We will usually assume the latter; it will normally be obvious how to extend the results to other cases. To simplify the notation we will assume without loss of generality that $T = 1$.

Using the expressions for S_u given in (3.6), and for the value of a general European option given by (3.2), we can write the time-t value of a fixed-strike Asian option as

$$D(t, 1) \, \mathbb{E}_t \left(\int_0^1 S_t \exp\bigl(B_{\tau(u)} + \alpha(u)\bigr) \, du - K \right)^+ \tag{3.8}$$

where $D(t, T)$ is the discount factor for the period $[t, T]$ and the expectation is performed under the martingale measure; the definitions of the functions τ and α may be found on page 31. Unfortunately, the distribution of the arithmetic mean of a collection of lognormal random variables does not have a simple form (though in Section 3.4 we will derive a very accurate approximation), and the expectation calculation in (3.8) is distinctly non-trivial. The other Asian option we will look at is the *floating-strike* call option, with a value of $(A_1 - S_1)^+$ at time 1.

Several different approaches to the problem of valuing Asian options have been tried; the first attempts were Monte-Carlo simulations (Kemna & Vorst 1990) followed by formulae for the approximate price, obtained by replacing the law of A_1 with a lognormal random variable with appropriate parameters by Levy (1992) and Turnbull & Wakeman (1991). (Levy & Turnbull (1992) compares many methods of this type.)

Since the time-t price of an Asian option generally depends on three variables: t, S_t and A_t, we might expect the PDE approach of Section 3.1.2 to be rather slow. For fixed-strike and floating-strike options however, we can reduce the problem to just two variables, making the PDE approach competitive. This was first observed for floating-strike options by Ingersoll (1987), and for fixed-strike options by Rogers & Shi (1992). We will describe this work in Section 3.2.1.

Although it is unlikely that simple pricing formulae will ever be found, exact pricing formulae have been discovered (see Yor (1992) and Geman & Yor (1993)), based on an inverse Laplace transform. This can be inverted numerically (Geman & Eydeland 1995) but this method is relatively slow and appears difficult to implement accurately.

Lower bounds on the price, in the form of single and double integrals have been presented by Curran (1992) and Rogers & Shi (1992); these formulae can be evaluated rapidly and are surprisingly accurate. In Section 3.2.2 we derive the lower bound of Rogers & Shi, and in Section 3.2.3 we give an alternative derivation, leading to a simpler formulae. In this form, the bound can be evaluated more quickly and is similar to the formula of Curran (1992). We examine the continuous-averaging limit of Curran's formula in Section 3.2.4; specifically, we give a derivation of his bound for fixed-strike options (Curran's article gives little indication of how his bound is derived) and show how a similar argument leads to a bound for floating-strike options. To complement these lower bounds, and to justify our earlier assertion of their accuracy, in Section 3.2.5 we find new upper bounds on the price of fixed-strike and floating-strike options; these upper bounds are rarely more than 0.5% above the lower bounds. (A numerical comparison of all these bounds may be found in Section 3.2.6.)

We will be interested in the time-t value of fixed-strike and floating-strike options for any $t < 1$, including times before the start of the averaging period ($t < 0$). Two observations reduce the number of case we need to consider:

Remark 3.2 If we are trying to calculate the time-t price of a fixed-strike option, we can assume, without loss of generality, that $t \leq 0$. To see this, let $t \in (0, 1)$, and write the payout of the option as

$$\begin{aligned} A_1 - K &= (A_1 - A_t) - (K - A_t) \\ &= (1-t)\left[\frac{1}{1-t}\int_t^1 S_u\, du - \frac{1}{1-t}(K - A_t)\right]. \end{aligned}$$

By time t, we can already observe A_t; thus the payout of the option is proportional to the payout of a fixed-strike option at the start of its averaging period, with strike $(1-t)^{-1}(K - A_t)$.

Remark 3.3 For floating-strike options we can assume that $t \geq 0$. For $t < 0$, we can write the time-t value of the option as

$$\begin{aligned} V_t &= D(t,1)\,\mathbb{E}_t\left[S_0\left(\int_0^1 \frac{S_u}{S_0}\, du - \frac{S_1}{S_0}\right)\right]^+ \\ &= D(t,1)\,\mathbb{E}_t(S_0)\,\mathbb{E}\left(\int_0^1 \frac{S_u}{S_0}\, du - \frac{S_1}{S_0}\right)^+ \end{aligned} \tag{3.9}$$

using the fact that $\{S_u/S_0 : u \geq 0\}$ is independent of $\{S_u : u \leq 0\}$, which follows from (3.6). Since the discounted stock price is a martingale under the martingale measure,

$$D(t,1)\,\mathbb{E}_t(S_0) = D(t,0)D(0,1)\,\mathbb{E}_t(S_0) = D(0,1)S_t.$$

Thus (3.9) is just the time-0 value of the same floating-strike option when the time-0 stock price equals S_t.

3.2.1 A PDE formulation for Asian options

We now describe the PDE method of Rogers & Shi (1992), showing that the problem of pricing fixed-strike and floating-strike Asian options can be reduced to solving a PDE in two variables. This is straightforward for the case of floating-strike options, as noted by several authors (e.g., Ingersoll (1987) and Wilmott et al. (1993)), but was first shown to hold in the fixed-strike case by Rogers & Shi (1992). For floating-strike options, observe that since $\{S_u/S_t : u \geq t\}$ is independent of $\mathcal{F}_t$, we have

$$\begin{aligned}\mathbb{E}_t(A_1 - S_1)^+ &= S_t\,\mathbb{E}_t\left(\int_0^t \frac{S_u}{S_t}\,du + \int_t^1 \frac{S_u}{S_t}\,du - \frac{S_1}{S_t}\right)^+ \\ &= S_t f(X_t, t)\end{aligned}$$

for some function $f(x,t)$, where $X_t = A_t/S_t$. The process X_t satisfies the SDE

$$\begin{aligned}dX_t &= \frac{1}{S_t}S_t\,dt - A_t\frac{1}{S_t^2}\,dS_t + A_t\frac{1}{S_t^3}\,d[S]_t \\ &= dt - X_t[\sigma(t)\,d\tilde{W}_t + r(t)\,dt] + X_t\sigma(t)^2\,dt \qquad (3.10)\end{aligned}$$

using the SDE for S_t given in (3.3), from which it may be seen that X_t is Markov. Now apply Itô's lemma to the martingale $S_t f(X_t, t)$ (assuming that f is sufficiently smooth) and equate the finite-variation term to zero, giving

$$r(t)f(x,t) + \frac{\partial f}{\partial t} + [1 - xr(t)]\frac{\partial f}{\partial x} + \tfrac{1}{2}x^2\sigma(t)^2\frac{\partial^2 f}{\partial x^2} = 0 \qquad (3.11)$$

with boundary condition

$$f(x,1) = (x-1)^+.$$

We now have a PDE in two variables, which could be solved using one of the finite-difference schemes described in Section 3.1. Once we have the function f, the time-t value of the option is just $D(t,1)S_t f(A_t, t)$. In the fixed-strike case, we have

$$\begin{aligned}\mathbb{E}_t\left(\int_0^1 S_u\,du - K\right)^+ &= S_t\,\mathbb{E}_t\left(\int_0^t \frac{S_u}{S_t}\,du + \int_t^1 \frac{S_u}{S_t}\,du - \frac{K}{S_t}\right)^+ \\ &= S_t\,g(X_t, t)\end{aligned}$$

for some function $g(x,t)$, where now $X_t = (A_t - K)/S_t$. Using (3.10), we see that X_t satisfies the SDE

$$\begin{aligned}dX_t &= dt - \left(X_t + \frac{K}{S_t}\right)[\sigma(t)\,d\tilde{W}_t + r(t)\,dt] + \left(X_t + \frac{K}{S_t}\right)\sigma(t)^2\,dt \\ &\qquad + \frac{K}{S_t^2}\,dS_t - \frac{K}{S_t^3}\,d[S]_t \\ &= dt - X_t[\sigma(t)\,d\tilde{W}_t + r(t)\,dt] + X_t\sigma(t)^2\,dt\end{aligned}$$

which is exactly the same as (3.10). Thus $g(x,t)$ satisfies PDE (3.11) but now with the boundary condition

$$g(x,1) = x^+.$$

With $g(x,t)$ determined, the time-t price of the option is $D(t,1)S_t\,g(X_t,t)$. Extending the method to handle more general averaging measures and to time-t prices for $t < 0$ is straightforward.

3.2.2 The lower bounds of Rogers & Shi

Rogers & Shi (1992) obtain a lower bound on Asian option prices by using the inequality

$$\begin{aligned}\mathbb{E}X^+ &= \mathbb{E}\big[\mathbb{E}\big(X^+ \,|\, Y\big)\big] \\ &\geq \mathbb{E}\big[\mathbb{E}\big(X \,|\, Y\big)\big]^+ \end{aligned} \tag{3.12}$$

which holds for any random variables X and Y. To bound the time-t price of a fixed-strike option, we choose $X = A_1 - K$ and $Y = \int_0^1 B_{\tau(u)}\,du$, giving the bound

$$\begin{aligned} V_t &\geq D(t,1)\,\mathbb{E}_t[\mathbb{E}_t(A_1 - K \,|\, Y)]^+ \\ &= D(t,1)\int_{-\infty}^{\infty}\mathbb{P}_t(Y \in dy)[\mathbb{E}_t(A_1 - K \,|\, Y = y)]^+ \\ &= D(t,1)\int_{-\infty}^{\infty}\mathbb{P}_t(Y \in dy)\bigg(\int_0^1 \mathbb{E}_t(S_u \,|\, Y = y)\,du - K\bigg)^+. \end{aligned}$$

Since the conditional distribution of $B_{\tau(u)}$ given $Y = y$ is normal with mean $(c_u/v)y$ and variance $\tau(u) - c_u^2/v$, where $c_u = \mathrm{cov}_t\big(B_{\tau(u)}, Y\big)$ and $v = \mathrm{var}_t(Y)$, we have

$$\mathbb{E}_t\big(S_u \,|\, Y = y\big) = S_t \exp\big(\alpha(u) + (c_u/v)y + \tfrac{1}{2}[\tau(u) - c_u^2/v]\big).$$

Thus, a lower bound on the value of a fixed-strike option is

$$\begin{aligned} D(t,1)&\int_{-\infty}^{\infty}\frac{1}{\sqrt{v}}\phi\Big(\frac{y}{\sqrt{v}}\Big) \\ &\times\bigg(\int_0^1 S_t \exp\big(\alpha(u) + (c_u/v)y + \tfrac{1}{2}[\tau(u) - c_u^2/v]\big)\,du - K\bigg)^+ dy. \end{aligned} \tag{3.13}$$

Note that for $u \leq t$, we have $\tau(u) = c_u = 0$ and $\alpha(u) = \log(S_u/S_t)$. Thus if $t \geq 0$, the only dependence in (3.13) on $\{S_u : u \leq t\}$ is through (A_t, S_t) as we would expect.

Remark 3.4 In the Black-Scholes model with constant volatility σ, and interest-rate r, for $t \leq 0 \leq u$ we have

$$\alpha(u) = \big(r - \tfrac{1}{2}\sigma^2\big)(u - t) \tag{3.14}$$

$$\tau(u) = \sigma^2(u - t) \tag{3.15}$$

$$c_u = \sigma^2[u(1 - u/2) - t] \tag{3.16}$$

$$v = \sigma^2\big(\tfrac{1}{3} - t\big). \tag{3.17}$$

For $t \geq 0$, by Remark 3.2), we can write the value of the fixed-strike option in terms of the value of an option at the start of its averaging period, reducing the problem to the case $t = 0$.

To bound the time-t price of a floating-strike option, we set $X = A_1 - S_1$ and $Y = \int_0^1 B_{\tau(u)}\,du - B_{\tau(1)}$. With $c_u = \mathrm{cov}_t\big(B_{\tau(u)}, Y\big)$ and $v = \mathrm{var}_t(Y)$, the lower bound is

$$
\begin{aligned}
D(t,1)\int_{-\infty}^{\infty}\frac{1}{\sqrt{v}}\phi\Big(\frac{y}{\sqrt{v}}\Big)\Big(\int_0^1 & S_t\exp\big(\alpha(u)+(c_u/v)y+\tfrac12[\tau(u)-c_u^2/v]\big)\,du \\
& - S_t\exp\big(\alpha(1)+(c_1/v)y+\tfrac12[\tau(1)-c_1^2/v]\big)\Big)^{+}dy.
\end{aligned}
\tag{3.18}
$$

Remark 3.5 With the Black-Scholes assumptions of constant volatility σ, and interest-rate r, the functions α and τ are given by (3.14) and (3.15). If $t \le 0 \le u$, we have

$$
\begin{aligned}
c_u &= -\tfrac12\sigma^2u^2 \\
v &= \tfrac13\sigma^2,
\end{aligned}
$$

while for $t \ge 0$, $u \ge 0$ we have

$$
\begin{aligned}
c_u &= -\tfrac12\sigma^2(u+t)(u-t)^+ \\
v &= \tfrac13\sigma^2(1-t^3).
\end{aligned}
$$

3.2.3 An alternative derivation of the bounds of Rogers & Shi

The bounds (3.13) and (3.18) are slightly awkward to evaluate numerically since the outer integration involves a non-smooth integrand. In this section, we give simpler versions of (3.13) and (3.18) which are easier to evaluate; they are also similar to the formula given by Curran (1992) as an approximate price for fixed-strike options.

Fixed-strike options

We start off in a rather different direction, by trying to approximate the event that the option eventually pays off with something more tractable. Let $\mathcal{A} = \{A_1 > K\}$ be the event that the option makes a positive payout, and note that for any event $\mathcal{A}'$ we have

$$
\begin{aligned}
\mathbb{E}_t(A_1-K)^+ &\ge \mathbb{E}_t[(A_1-K)\,\mathrm{I}(\mathcal{A}')] \\
&= \int_0^1 \mathbb{E}_t(S_u-K\,;\mathcal{A}')\,du
\end{aligned}
\tag{3.19}
$$

with equality if $\mathcal{A}' = \mathcal{A}$. We will use

$$
\mathcal{A}' = \Big\{\int_0^1 B_{\tau(u)}\,du > \gamma\Big\},
$$

where B is the Brownian motion of (3.6), allowing us to write the expectation on the right-hand side of (3.19) as a Black-Scholes type formula involving γ. This just leaves us with a one-dimensional integral of a smooth integrand, which should be very fast. To determine the optimal value of γ, let $Y = \int_0^1 B_{\tau(u)}\,du$ and let $f_Y(y)$ denote the density function of Y, conditional on $\mathcal{F}_t$. Substituting $\mathcal{A}' = \{Y > \gamma\}$ into (3.19) and differentiating the right-hand side with respect to γ gives

$$\frac{\partial}{\partial\gamma}\int_0^1 \mathbb{E}_t(S_u - K\,;Y > \gamma)\,du = -\int_0^1 \mathbb{E}_t(S_u - K \,|\, Y = \gamma) f_Y(\gamma)\,du. \tag{3.20}$$

At a stationary point, we must have $\int_0^1 \mathbb{E}_t(S_u \,|\, Y = \gamma)\,du = K$, which, noting $t \leq 0$, gives

$$\int_0^1 S_t \exp\bigl(\alpha(u) + (c_u/v)\gamma + \tfrac{1}{2}[\tau(u) - c_u^2/v]\bigr)\,du = K \tag{3.21}$$

where $c_u = \mathrm{cov}_t\bigl(B_{\tau(u)}, Y\bigr)$ and $v = \mathrm{var}_t(Y)$. The problem of pricing the option when $K \leq 0$ is trivial (since the option is sure to payout) so we will assume that $K > 0$. Since

$$c_u = \int_0^1 \tau(u) \wedge \tau(s)\,ds$$

we see that $c_u = 0$ on $[0,t]$ and $c_u > 0$ on $(t,1]$. Thus, the left-hand side of (3.21) is strictly increasing in γ and (3.21) has a unique solution, γ^*. (Note that the monotonicity property also makes it very easy to find γ^* numerically.) Note also that

$$\mathbb{E}_t(A_1 \,|\, Y = \gamma) > K \qquad \text{if and only if} \qquad \gamma > \gamma^*.$$

From this observation, we can deduce that our bound is exactly the same as that of Rogers & Shi (1992), since

$$\begin{aligned}\mathbb{E}_t[\mathbb{E}_t(A_1 - K \,|\, Y)]^+ &= \mathbb{E}_t[\mathbb{E}_t(A_1 - K \,|\, Y)\,\mathrm{I}(Y > \gamma^*)]\\ &= \mathbb{E}_t[(A_1 - K)\,\mathrm{I}(Y > \gamma^*)].\end{aligned}$$

We now have a simpler version of (3.13)

$$D(t,1)\int_0^1 \mathbb{E}_t\Bigl[\Bigl(S_t\exp\bigl(B_{\tau(u)} + \alpha(u)\bigr) - K\Bigr)\,\mathrm{I}(Y > \gamma^*)\Bigr]\,du,$$

and it just remains to calculate the expectation. Recall the standard calculation:

$$\mathbb{E}[(\exp(N_1) - K)\,\mathrm{I}(N_2 > 0)] = \exp\bigl(\mu_1 + \tfrac{1}{2}\sigma_1^2\bigr)\,\Phi\left(\frac{\mu_2 + c}{\sigma_2}\right) - K\Phi\left(\frac{\mu_2}{\sigma_2}\right) \tag{3.22}$$

for normal random variables N_1, N_2 with respective means μ_1, μ_2, variances σ_1^2, σ_2^2 and covariance c. Substituting $N_1 = \log S_t + B_{\tau(u)} + \alpha(u)$ and $N_2 = Y - \gamma^*$ gives

$$D(t,1)\left[\int_0^1 S_t\exp\bigl(\alpha(u) + \tfrac{1}{2}\tau(u)\bigr)\,\Phi\left(\frac{-\gamma^* + c_u}{\sqrt{v}}\right)du - K\,\Phi\left(\frac{-\gamma^*}{\sqrt{v}}\right)\right] \tag{3.23}$$

where γ^* is the unique solution to (3.21), $c_u = \mathrm{cov}_t\bigl(B_{\tau(u)}, Y\bigr)$ and $v = \mathrm{var}_t(Y)$. Expressions for c_u and v in the usual Black-Scholes model are given in Remark 3.4.

Floating-strike options

We now derive a simpler version of (3.18), by choosing γ to maximise the lower bound

$$\mathbb{E}_t(A_1 - S_1)^+ \geq \int_0^1 \mathbb{E}_t[(S_u - S_1)\,\mathrm{I}(Y > \gamma)]\,du \tag{3.24}$$

where now we take $Y = \int_0^1 B_{\tau(u)}\,du - B_{\tau(1)}$. Differentiating the right-hand side of (3.24) with respect to γ, we obtain the condition for stationarity

$$\int_0^1 \mathbb{E}_t(S_u - S_1 \mid Y = \gamma)\,du = 0,$$

which gives

$$\begin{aligned}\int_0^1 S_t \exp\big(\alpha(u) + (c_u/v)\gamma + \tfrac{1}{2}[\tau(u) - c_u^2/v]\big)\,du \\ - S_t \exp\big(\alpha(1) + (c_1/v)\gamma + \tfrac{1}{2}[\tau(1) - c_1^2/v]\big) = 0\end{aligned} \tag{3.25}$$

where $c_u = \mathrm{cov}_t\big(B_{\tau(u)}, Y\big)$ and $v = \mathrm{var}_t(Y)$ as usual. We now show that (3.25) has a unique solution. For $0 \leq u \leq 1$, we have

$$\begin{aligned} c_u &= \mathrm{cov}\bigg(B_{\tau(u)}, \int_0^1 B_{\tau(s)}\,ds - B_{\tau(1)}\bigg) \\ &= \int_0^u \tau(s)\,ds + (1-u)\tau(u) - \tau(u) \\ &= \int_0^u \tau(s)\,ds - u\tau(u). \end{aligned}$$

Differentiating with respect to u gives

$$c'(u) = -u\sigma(u)^2\,\mathrm{I}(u \geq t).$$

As $c(0) = 0$ (by the independent increments property of Brownian motion), we have $c(u) < 0$ for $\max(t, 0) < u < 1$. Thus as $\gamma \to +\infty$, the left-hand side of (3.25) converges to a strictly positive quantity if $t > 0$ and to $0+$ if $t \leq 0$. As $\gamma \to -\infty$, it converges to $-\infty$ in both cases. Thus (3.25) has at least one solution. Now consider the gradient of the left-hand side of (3.25) a solution γ^*. Differentiating the left-hand side of (3.25) with respect to γ, and using the fact that γ^* satisfies (3.25), gives

$$\int_0^1 S_t[(c_u/v) - (c_1/v)]\exp\big(\alpha(u) + (c_u/v)\gamma^* + \tfrac{1}{2}[\tau(u) - c_u^2/v]\big)\,du. \tag{3.26}$$

As c_u is decreasing, and strictly decreasing on $(\max(t, 0), 1)$, it follows that (3.26) is strictly positive. Thus γ^* is unique. We also deduce that

$$\mathbb{E}_t(A_1 - S_1 \,\big|\, Y = \gamma) > 0 \qquad \text{if and only if} \qquad \gamma > \gamma^*.$$

A very similar argument to that used in the fixed-strike case now shows that the bound of this section is exactly the same as that of Rogers & Shi (1992). Since γ^* is unique, solving (3.25) numerically is straightforward. Using (3.22) again gives an alternative form for (3.18)

$$D(t,1)\left[\int_0^1 S_t e^{\alpha(u)+\frac{1}{2}\tau(u)}\Phi\left(\frac{-\gamma^*+c_u}{\sqrt{v}}\right)du - S_t e^{\alpha(1)+\frac{1}{2}\tau(1)}\Phi\left(\frac{-\gamma^*+c_1}{\sqrt{v}}\right)\right]. \tag{3.27}$$

Remark 3.6 Recall from Section 3.2 that the price of a floating-strike option at time $t < 0$ is just the time-0 price, assuming the time-0 stock price equals the current stock price. Since c_u and v are independent of t for $t \leq 0$ (see Remark 3.5 for explicit formulas in the Black-Scholes model), the solution to (3.25) is also the same all $t \leq 0$. From the definitions of α, τ and r (page 31), for $u \geq t$ we have

$$\alpha(u) + \tfrac{1}{2}\tau(u) = \int_t^u r(s)\,ds.$$

Thus for fixed S_t, the bound (3.27) also takes the same value for all $t < 0$.

3.2.4 Curran's approximation and an extension

Curran (1992) presents a formula for the approximate price of a fixed-strike option with discrete averaging. He starts from (3.12) but gives no details of his method. His formula is, in fact, the discrete-averaging version of (3.23), with a particular choice for γ^*; thus his formula is always a lower bound and can be viewed as providing an approximate solution to (3.21). It may be useful to have such an approximation available if we require answers very rapidly, and cannot afford to solve (3.21) numerically.

In this section we will give some justification of his approximate solution to (3.21), and show how the same method can be used to produce an approximate solution to (3.25), making available a fast lower bound for floating-strike options.

It is worth emphasising that these bounds are *extremely* close to those of Rogers & Shi (1992) (which arise from using the exact solution to (3.21)).

Fixed-strike options

We give a derivation of Curran's approximate solution to (3.21) in the limit of continuous averaging. Let

$$f(\gamma) = \mathbb{E}_t\left(\int_0^1 S_t \exp\bigl(B_{\tau(u)} + \alpha(u)\bigr)\,du \,\middle|\, Y = \gamma\right), \tag{3.28}$$

where $Y = \int_0^1 B_{\tau(u)}\,du$ and recall that we seek $\gamma^* = f^{-1}(K)$. We construct an approximation to f^{-1} by observing that an approximation to $f(\gamma)$ is given by

$$\begin{aligned}\tilde{f}(\gamma) &= \mathbb{E}_t\left[S_t \exp\left(\int_0^1 \left(B_{\tau(u)} + \alpha(u)\right) du\right) \,\middle|\, Y = \gamma\right] \\ &= S_t \exp\left(\gamma + \int_0^1 \alpha(u)\,du\right) \end{aligned} \tag{3.29}$$

obtained by interchanging the orders of integration and exponentiation in (3.28). We can now invert (3.29) easily, giving

$$\tilde{f}^{-1}(x) = \log(x/S_t) - \int_0^1 \alpha(u)\,du. \tag{3.30}$$

We now appeal to the fact that if $f \approx \tilde{f}$, then

$$f^{-1}(x) \approx \tilde{f}^{-1}\big(2x - f \circ \tilde{f}^{-1}(x)\big), \tag{3.31}$$

the approximation being exact when $f - \tilde{f}$ is constant. Applying (3.31) to (3.30), we deduce than an approximate solution to (3.21) is

$$\begin{aligned}\gamma^* \approx \log\bigg(\frac{2K}{S_t} - \int_0^1 &\exp\big(\alpha(u) + (c_u/v)\big[\log(K/S_t) - \textstyle\int_0^1 \alpha(s)\,ds\big] \\ &+ \tfrac{1}{2}[\tau(u) - c_u^2/v]\big)\,du\bigg) - \int_0^1 \alpha(u)\,du\end{aligned}$$

where $c_u = \mathrm{cov}_t\big(B_{\tau(u)}, Y\big)$ and $v = \mathrm{var}_t(Y)$. In the Black-Scholes model at time 0, this becomes

$$\begin{aligned}\gamma^* \approx \log\bigg(\frac{2K}{S_0} - \int_0^1 &\exp\big((r - \tfrac{1}{2}\sigma^2)u + 3u(1-u/2)[\log(K/S_0) - \tfrac{1}{2}(r - \tfrac{1}{2}\sigma^2)] \\ &+ \tfrac{1}{2}\sigma^2[u - 3u^2(1-u/2)^2]\big)\,du\bigg) - \tfrac{1}{2}(r - \tfrac{1}{2}\sigma^2)\end{aligned} \tag{3.32}$$

which is the continuous-averaging limit of Curran's formula.

Floating-strike options

We now try to derive an approximate solution to (3.25). Defining

$$f(\gamma) = \mathbb{E}_t\left(\int_0^1 \exp\big(B_{\tau(u)} + \alpha(u)\big)\,du \,\middle|\, Y = \gamma\right) \Big/ \, \mathbb{E}_t(S_1 \,|\, Y = \gamma)$$

where $Y = \int_0^1 B_{\tau(u)}\,du - B_{\tau(1)}$, we try to solve $f(\gamma^*) = 1$. As before, we will approximate f by interchanging the orders of integration and exponentiation, giving

$$\tilde{f}(\gamma) = \mathbb{E}_t\left[S_t \exp\left(\int_0^1 \left(B_{\tau(u)} + \alpha(u)\right) du\right) \,\middle|\, Y = \gamma\right] \Big/ \, \mathbb{E}_t(S_1 \,|\, Y = \gamma).$$

Since $\int_0^1 B_{\tau(u)}\,du = Y + B_{\tau(1)}$ this simplifies to

$$\tilde{f}(\gamma) = \exp\Big(\gamma + \int_0^1 \alpha(u)\,du - \alpha(1)\Big)$$

which can be inverted easily. Thus an approximate solution to (3.25) is

$$\begin{aligned}\gamma^* \approx \log\Big(2 - \exp\big(-\alpha(1) - (c_1/v)\big[\alpha(1) - \textstyle\int_0^1 \alpha(u)\,du\big] - \frac{1}{2}[\tau(1) - c_1^2/v]\big)\\ \times \int_0^1 \exp\big(\alpha(u) + (c_u/v)\big[\alpha(1) - \textstyle\int_0^1 \alpha(s)\,ds\big] + \frac{1}{2}[\tau(u) - c_u^2/v]\big)\,du\Big)\\ + \alpha(1) - \int_0^1 \alpha(u)\,du.\end{aligned} \tag{3.33}$$

3.2.5 Upper bounds

Rogers & Shi also derive an upper bound on the price of fixed-strike and floating-strike Asian options, by bounding the error made by their lower bound. Recall from Section 3.2.2 the inequality used by Rogers & Shi to obtain a lower bound: $\mathbb{E}X^+ \geq (\mathbb{E}X)^+$. To get an upper bound, they use the inequalities

$$\begin{aligned}\mathbb{E}X^+ - (\mathbb{E}X)^+ &= \tfrac{1}{2}(\mathbb{E}|X| - |\mathbb{E}X|)\\ &\leq \tfrac{1}{2}\mathbb{E}|X - \mathbb{E}X|\\ &\leq \tfrac{1}{2}[\mathrm{var}(X)]^{1/2}\end{aligned}$$

which hold for any random variable X. By subsequently bounding $\mathrm{var}(X)$, they obtain an upper bound.

In this section we will derive new upper bounds, which, judging by the numerical results on page 53, appear to be very accurate. For example, in a Black-Scholes model with parameters $\sigma = 0.3$, $r = 0.09$ and $S = 100$, the lower bound of Section 3.2.2 on the time-0 price of a fixed-strike option with strike 100 is 8.8276, while the upper bound of this section is 8.8333 (the upper bound of Rogers & Shi for this problem is 9.039).

The new bounds exploit the following simple inequality: let X be a random variable and let $f_u(\omega)$, $u \in [0,1]$ be a random function with $\int_0^1 f_u(\omega)\,du = 0$, for all $\omega \in \Omega$, and μ_u be a deterministic function with $\int_0^1 \mu_u\,du = 1$. Then

$$\begin{aligned}\mathbb{E}_t\Big(\int_0^1 S_u\,du - X\Big)^+ &= \mathbb{E}_t\Big(\int_0^1 [S_u - X(\mu_u + f_u)]\,du\Big)^+\\ &\leq \mathbb{E}_t \int_0^1 [S_u - X(\mu_u + f_u)]^+\,du\\ &= \int_0^1 \mathbb{E}_t[S_u - X(\mu_u + f_u)]^+\,du.\end{aligned} \tag{3.34}$$

Since this bound holds for all such $\{f_u, \mu_u\}$, we can try to optimise $\{f_u, \mu_u\}$ over some convenient set. With this idea in mind, we will try to minimise the Lagrangian

$$L\big(\lambda, \{f_u\}, \{\mu_u\}\big) = \int_0^1 \mathbb{E}_t[S_u - X(\mu_u + f_u)]^+ \, du - \lambda\bigg(1 - \int_0^1 \mu_u \, du\bigg).$$

Considering the stationarity of L at $(\lambda, \{f_u\}, \{\mu_u\})$ with respect to a small deterministic perturbation $\{\epsilon_u\}$ in $\{\mu_u\}$, gives the condition

$$\int_0^1 \big[\mathbb{E}_t\big(-X; S_u - X(\mu_u + f_u) \geq 0\big) + \lambda\big]\epsilon_u \, du = 0.$$

Since $\{\epsilon_u\}$ is arbitrary, we deduce that a necessary condition for $\{\mu_u\}$ to be optimal is

$$\mathbb{E}_t[X; S_u \geq X(\mu_u + f_u)] \text{ is independent of } u. \tag{3.35}$$

Our approach will be to pick $\{f_u\}$ based on our intuition, and rely on (3.35) to give an excellent choice for $\{\mu_u\}$.

Fixed-strike options

To bound the time-t price of a fixed-strike option with strike K, we will assume, without loss of generality, that $t \leq 0$ (see Remark 3.2) and put $X = K$ in (3.34). Using (3.35) to determine $\{\mu_u\}$ leaves us to to choose the random function $\{f_u\}$. The bound (3.34) is only likely to be good when the random variables $\{S_u - K(\mu_u + f_u)\}$ take the same sign with high probability, which suggests that we need f_u large when S_u is large, and f_u small when S_u is small. Since f_u must integrate to zero, a simple choice is $f_u = B_{\tau(u)} - \int_0^1 B_{\tau(s)} \, ds$ (where B is the Brownian motion of (3.6)). Writing

$$Y_u = B_{\tau(u)} - \int_0^1 B_{\tau(s)} \, ds,$$

a more general choice would be $f_u = \nu Y_u$, for some constant ν, but a brief numerical investigation suggests that the choice $\nu = 1$ is almost as good as any, at least for the parameter values considered here. [In Section 3.4 we suggest a more complex choice for f_u for a similar type of problem. Since the numerical results based on the simpler choice appear to be quite acceptable, we will not try the more complex alternative here.]

We now use (3.35) to determine $\{\mu_u\}$. Since $X = K$ is deterministic, we have the condition: $\mathbb{P}[S_u \geq K(\mu_u + \nu Y_u)]$ is independent of u. A similar but more manageable condition is: $\mathbb{P}(N_u > \mu_u)$is independent of u, where N is normal, since it can be re-arranged to give

$$\mathbb{E}(N_u) - \mu_u = \gamma\sqrt{\text{var}(N_u)}$$

for some constant γ. In view of this we will use the approximation

$$S_u \approx S_t \exp(\alpha(u))(1 + B_{\tau(u)})$$

to give

$$S_t \exp(\alpha(u)) - K\mu_u = \gamma\sqrt{\xi_u}$$

where

$$\begin{aligned} \gamma &= \left(S_t \int_0^1 \exp(\alpha(u))\, du - K\right) \Big/ \int_0^1 \sqrt{\xi_u}\, du. \\ \xi_u &= \operatorname{var}_t\left(\big(S_t \exp(\alpha(u)) - K\nu\big) B_{\tau(u)} + K\nu \int_0^1 B_{\tau(s)}\, ds\right) \end{aligned}$$

We perform the integral $\int_0^1 \sqrt{\xi_u}\, du$ numerically. We now have the function $\{\mu_u\}$ for our upper bound on the time-t option price:

$$D(t,1) \int_0^1 \mathbb{E}_t[S_u - K(\mu_u + \nu Y_u)]^+ \, du.$$

A simple way of dealing with the remaining expectation is to condition on $Y_u = y$ and use (3.22), but an alternative method, which was found to be more numerically efficient, is to condition on $B_{\tau(u)} = x$, giving

$$\begin{aligned} &\int_0^1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{\tau(u)}} \phi\left(\frac{x}{\sqrt{\tau(u)}}\right) \\ &\qquad \times \mathbb{E}_t\left[\left(S_t \exp\big(\alpha(u) + x\big) - K[\mu_u + \nu(x - Z)]\right)^+ \,\middle|\, B_{\tau(u)} = x\right] dx\, du \end{aligned} \tag{3.36}$$

where $Z = \int_0^1 B_{\tau(s)}\, ds$. Using the fact that for $N \sim N(0,1)$ and $b > 0$, $\mathbb{E}(a + bN)^+ = a\Phi(a/b) + b\phi(a/b)$, (3.36) equals

$$\int_0^1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{\tau(u)}} \phi\left(\frac{x}{\sqrt{\tau(u)}}\right) \left[a(u,x)\Phi\left(\frac{a(u,x)}{b(u,x)}\right) + b(u,x)\phi\left(\frac{a(u,x)}{b(u,x)}\right)\right] dx\, du$$

where

$$\begin{aligned} a(u,x) &= S_t \exp\big(\alpha(u) + x\big) - K(\mu_u + \nu x) + K\nu[c_u/\tau(u)]x \\ b(u,x) &= K\nu[v - c_u^2/\tau(u)]^{1/2} \end{aligned}$$

and where $c_u = \operatorname{cov}_t\big(B_{\tau(u)}, Z\big)$ and $v = \operatorname{var}_t(Z)$. In this form the integrand may be badly behaved near $u = t$; if $t \geq 0$, then $\tau(u) \to 0$ as $u \to t$, so we make the change of variables $\tilde{u} = \sqrt{\tau(u)}$, $w = x/\sqrt{\tau(u)}$ giving

$$\int_{\sqrt{\tau(0)}}^{\sqrt{\tau(1)}} \int_{-\infty}^{\infty} \frac{2\tilde{u}}{\sigma(u)^2} \phi(w) \left[a(u,x)\Phi\left(\frac{a(u,x)}{b(u,x)}\right) + b(u,x)\phi\left(\frac{a(u,x)}{b(u,x)}\right)\right] dw\, d\tilde{u}. \tag{3.37}$$

For the normal Black-Scholes model, expressions for α, τ, c_u and v are given by (3.14)–(3.17).

Floating-strike options

To bound the time-t price of a floating-strike option, we will assume that $t \geq 0$ (see Remark 3.3) and note that by writing $A = A_t/(1-t)$ and $X = 1/(1-t)$, we have

$$A_t - S_1 = (1-t)\left(A + \frac{1}{1-t}\int_t^1 S_u\,du - XS_1\right)$$

so we will bound the time-0 price of an option with payout $A + \int_0^1 S_u\,du - XS_1$ at time 1, for constants $A \geq 0$ and $X \geq 1$. If x_1 and x_2 satisfy $x_1 + x_2 = X$, a simple bound on the price of such an option is given by

$$\begin{aligned}\mathbb{E}_t\left(A + \int_0^1 S_u\,du - XS_1\right)^+ &= \mathbb{E}_t\left(A - x_1S_1 + \int_0^1 (S_u - x_2S_1)\,du\right)^+ \\ &\leq \mathbb{E}_t(A - x_1S_1)^+ + \int_0^1 \mathbb{E}_t(S_u - x_2S_1)^+\,du. \qquad (3.38)\end{aligned}$$

This bound will not be particularly good unless $A - x_1S_1$ and the random variables $\{S_u - x_2S_1\}$ take the same sign with high probability. By considering the very crude approximation $S_u \approx S_0$, we see that this bound is likely to be successful when (approximately) $x_1 \propto A$ and $x_2 \propto S_0$ i.e., $x_1 = AX/(A + S_0)$ and $x_2 = S_0X/(A + S_0)$. The bound is still quite poor however; we do much better by replacing the constants x_1 and x_2 with random variables, giving a bound similar to (3.34)

$$\begin{aligned}\mathbb{E}_t\left(A + \int_0^1 S_u\,du - XS_1\right)^+ \leq\ & \mathbb{E}_t[A - (\mu + f)S_1]^+ \\ & + \int_0^1 \mathbb{E}_t[S_u - (\mu_u + f_u)S_1]^+\,du\end{aligned} \qquad (3.39)$$

where μ and $\{\mu_u\}$ are deterministic with

$$\mu + \int_0^1 \mu_u\,du = X \qquad (3.40)$$

and f and f_u have mean zero and satisfy

$$f(\omega) + \int_0^1 f_u(\omega)\,du = 0 \qquad \text{for all } \omega \in \Omega. \qquad (3.41)$$

We will determine μ and $\{\mu_u\}$ from a condition analagous to (3.35), to be derived using Lagrangian methods, but first we choose f and f_u. Thinking about when (3.39) is tight, we want f to be large when $S_1 \ll A$ and f_u to be large when $S_1 \ll S_u$. This suggests using $f = -\nu B_{\tau(1)}$ and $f_u = B_{\tau(u)} - B_{\tau(1)}$ for some constant $\nu > 0$. However we also need to satisfy (3.41) so we use

$$\begin{aligned} f &= y_1 I - \nu B_{\tau(1)} \\ f_u &= y_2 I + B_{\tau(u)} - B_{\tau(1)} \end{aligned}$$

where $y_1 = A/(A + S_0)$, $y_2 = S_0/(A + S_0)$ (choices suggested by our comments after (3.38)), and I satisfies

$$I + \int_0^1 B_{\tau(s)}\, ds - (1 + \nu)B_{\tau(1)} = 0.$$

(Taking $\nu = A/S_0$ was found to work well in practice.) We now have the bound

$$\begin{aligned}\mathbb{E}_t\Big(A + \int_0^1 S_u\, du - XS_1\Big)^+ &\le \mathbb{E}_t\big[A - S_1\big(\mu + y_1 I - \nu B_{\tau(1)}\big)\big]^+ \\ &\quad + \int_0^1 \mathbb{E}_t\big[S_u - S_1\big(\mu_u + y_2 I + B_{\tau(u)} - B_{\tau(1)}\big)\big]^+ du. \qquad (3.42)\end{aligned}$$

We determine the constant μ and the function μ_u by trying to minimise the Lagrangian

$$\begin{aligned}L(\lambda, \mu, \{\mu_u\}) = \mathbb{E}_t\big[A - S_1\big(\mu + y_1 I - \nu B_{\tau(1)}\big)\big]^+ \\ + \int_0^1 \mathbb{E}_t\big[S_u - S_1\big(\mu_u + y_2 I + B_{\tau(u)} - B_{\tau(1)}\big)\big]^+ du \\ - \lambda\Big(X - \mu - \int_0^1 \mu_u\, du\Big).\end{aligned}$$

Imposing stationarity with respect to μ gives the condition

$$\mathbb{E}_t\big[-S_1; A \ge S_1\big(\mu + y_1 I - \nu B_{\tau(1)}\big)\big] + \lambda = 0, \qquad (3.43)$$

and if, in addition, we have stationarity with respect to $\{\mu_u\}$, then

$$\mathbb{E}_t\big[-S_1; S_u \ge S_1\big(\mu_u + y_2 I + B_{\tau(u)} - B_{\tau(1)}\big)\big] + \lambda = 0 \qquad \text{for all } u. \qquad (3.44)$$

Since it is easier to handle a condition of the form '$\mathbb{P}(N_u > \mu_u)$ is independent of u', we will replace (3.43) and (3.44) with the condition

$$\mathbb{P}_t\big[S_u \ge S_1\big(\mu_u + y_2 I + B_{\tau(u)} - B_{\tau(1)}\big)\big] = \lambda' \qquad \text{for all } u,$$

where

$$\mathbb{P}_t\big[A \ge S_1\big(\mu + y_1 I - \nu B_{\tau(1)}\big)\big] = \lambda'$$

and use the approximations $\exp(-B_{\tau(1)}) \approx 1 - B_{\tau(1)}$, $\exp(B_{\tau(u)} - B_{\tau(1)}) \approx 1 + B_{\tau(u)} - B_{\tau(1)}$, to give

$$\begin{aligned}\mathbb{P}\Big[&\exp(\alpha(u))\big(1 + B_{\tau(u)} - B_{\tau(1)}\big) \\ &\ge \exp(\alpha(1))\Big(\mu_u - y_2\int_0^1 B_{\tau(s)}\, ds + (y_2\nu - y_1)B_{\tau(1)} + B_{\tau(u)}\Big)\Big] = \lambda''\end{aligned} \qquad (3.45)$$

for all u, where λ'' also satisfies

$$\mathbb{P}\Big[A\big(1 - B_{\tau(1)}\big) - S_0 \exp(\alpha(1))\big(\mu + y_1 I - \nu B_{\tau(1)}\big) \geq 0\Big] = \lambda''. \tag{3.46}$$

We now use the familiar fact that a condition of the form '$\mathbb{P}(N_u > \mu_u)$ is independent of u' implies $\mathbb{E}(N_u) - \mu_u = \gamma\sqrt{\mathrm{var}(N_u)}$ for some γ. Applying this to equations (3.45), (3.46), and using (3.40) to determine γ, gives

$$\begin{aligned} A - S_0 \exp(\alpha(1))\mu &= \gamma\sqrt{\xi} \\ \exp(\alpha(u)) - \exp(\alpha(1))\mu_u &= \gamma\sqrt{\xi_u} \end{aligned}$$

where

$$\begin{aligned} \xi &= \mathrm{var}_t\left[-AB_{\tau(1)} - S_0 \exp(\alpha(1))\left(-y_1 \int_0^1 B_{\tau(s)}\,ds + (y_1 - \nu y_2)B_{\tau(1)}\right)\right] \\ \xi_u &= \mathrm{var}_t\Bigg[\exp(\alpha(u))\big(B_{\tau(u)} - B_{\tau(1)}\big) \\ &\qquad - \exp(\alpha(1))\left(-y_2 \int_0^1 B_{\tau(s)}\,ds + (y_2\nu - y_1)B_{\tau(1)} + B_{\tau(u)}\right)\Bigg] \\ \gamma &= \left(A/\big(S_0 e^{\alpha(1)}\big) + \exp(-\alpha(1)) \int_0^1 \exp(\alpha(u))\,du - X\right) \\ &\quad \times \left(\sqrt{\xi}/\big(S_0 e^{\alpha(1)}\big) + \exp(-\alpha(1)) \int_0^1 \sqrt{\xi_u}\,du\right)^{-1}. \end{aligned}$$

[It is possible to use (3.43) and (3.44) more directly, rather than (3.45) and (3.46) to determine $\{\mu_u\}$, but the corresponding expression for γ requires us to solve an equation involving an integral of Φ^{-1}. Though this can all be done numerically, and this approach will probably lead to a better bound, we will not pursue it here since the numerical results seem quite satisfactory anyway.]

The first term in (3.42) is

$$\mathbb{E}_t\left[A - S_0 e^{\alpha(1) + B_{\tau(1)}}\left(\mu - y_1 \int_0^1 B_{\tau(s)}\,ds + (y_1 - y_2\nu)B_{\tau(1)}\right)\right]^+.$$

To evaluate this, we condition on $B_{\tau(1)} = x$, giving

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\tau(1)}}\phi\left(\frac{x}{\sqrt{\tau(1)}}\right)\left[a(x)\Phi\left(\frac{a(x)}{b(x)}\right) + b(x)\phi\left(\frac{a(x)}{b(x)}\right)\right] dx$$

where

$$\begin{aligned} a(x) &= A - S_0 e^{\alpha(1)+x}\left[\mu + x(y_1 - y_2\nu) - xy_1 \mathrm{cov}_t\big(B_{\tau(1)}, Y\big)/\tau(1)\right] \\ b(x) &= S_0 e^{\alpha(1)+x} y_1 \left[\mathrm{var}_t(Y) - \mathrm{cov}_t\big(B_{\tau(1)}, Y\big)^2/\tau(1)\right]^{1/2} \end{aligned}$$

where $Y = \int_0^1 B_{\tau(s)}\,ds$. Expressions for $\mathrm{var}_t(Y)$ and $\mathrm{cov}_t\big(B_{\tau(u)}, Y\big)$ in the normal Black-Scholes model are given in Remark 3.4.

For the second term in (3.42) we have

$$\int_0^1 \mathbb{E}_t\left[S_u - S_1\left(\mu_u - y_2\int_0^1 B_{\tau(s)}\,ds + (y_2\nu - y_1)B_{\tau(1)} + B_{\tau(u)}\right)\right]^+ du. \tag{3.47}$$

By writing the expectation above as $S_0\mathbb{E}[\exp(N_1(u)) - \exp(N_3)N_2(u)]^+$ and conditioning on the event $N_2(u) = x$ we can perform the resulting expectation analytically using versions of (3.22). Let

$$\begin{aligned} m_i(u) &= \mathbb{E}N_i(u) \\ \sigma_{ij}(u) &= \mathrm{cov}[N_i(u), N_j(u)] \end{aligned}$$

and denote the conditional distribution given $N_2(u) = x$ by tildes:

$$\begin{aligned} \tilde{m}_i(u,x) &= m_i + (x - m_2)\sigma_{i2}/\sigma_{22} \\ \tilde{\sigma}_{ij}(u) &= \sigma_{ij}(u) - \sigma_{i2}(u)\sigma_{j2}(u)/\sigma_{22}(u). \end{aligned}$$

With this notation, (3.47) becomes

$$\begin{aligned} \int_0^1\int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_{22}}}\phi\left(\frac{x-m_2}{\sqrt{\sigma_{22}}}\right)\Bigg[&\exp\left(\tilde{m}_1 + \tfrac{1}{2}\tilde{\sigma}_{11}\right)\Phi\left(\frac{\tilde{m}_1 - \tilde{m}_3 - \log(x) + \tilde{\sigma}_{11} - \tilde{\sigma}_{13}}{(\tilde{\sigma}_{11} - 2\tilde{\sigma}_{13} + \tilde{\sigma}_{33})^{1/2}}\right) \\ &- x\exp\left(\tilde{m}_3 + \tfrac{1}{2}\tilde{\sigma}_{33}\right)\Phi\left(\frac{\tilde{m}_1 - \tilde{m}_3 - \log(x) + \tilde{\sigma}_{31} - \tilde{\sigma}_{33}}{(\tilde{\sigma}_{11} - 2\tilde{\sigma}_{13} + \tilde{\sigma}_{33})^{1/2}}\right)\Bigg]\,dx\,du, \end{aligned}$$

where we take $\log(x) = -\infty$ for $x \le 0$. The various variances and covariances required here boil down to $\mathrm{var}_t(Y)$ and $\mathrm{cov}_t\left(B_{\tau(u)}, Y\right)$ again.

3.2.6 Numerical Results

We now see how the bounds presented here perform in practice. For simplicity, we will consider a Black-Scholes model with constant parameters $\sigma(s) \equiv \sigma$ and $r(s) \equiv r$. Thus, to calculate a time-t price, for $s \ge t$ the functions $\tau(s)$ and $\alpha(s)$ are given by (3.14) and (3.15).

Fixed-strike options

In Table 3.1 we price a variety of fixed-strike options at the start of the averaging period. We give the lower bound of Curran (1992), the upper and lower bounds of Rogers & Shi (1992), the Monte-Carlo results of Levy & Turnbull (1992) and the upper bound (3.37). The details for the upper bound of Rogers & Shi are simply those reported in Rogers & Shi (1992). All calculations assume that $r = 0.09\%$ and $S_0 = 100$. For the Monte-Carlo results, the estimated standard error is shown in parentheses. In Table 3.2 we consider $t = -0.5$, corresponding to a time 6 months before the start of the averaging period; we assume that $S_{-0.5} = 100$ and give the lower and upper bounds (3.23) and (3.37). None of the bounds described in this section take long to compute; the formula of Curran takes about 0.0002 seconds, while the lower and upper bounds of Section (3.2.3) and Section (3.2.5), take approximately 0.002 and 0.03 seconds respectively.

Volatility	Strike	Curran	R-S lower	M-C result	New Upper	R-S Upper
	95	8.8088	8.8088	8.81 (0.00)	8.8089	8.821
0.05	100	4.3082	4.3082	4.31 (0.00)	4.3084	4.318
	105	0.9583	0.9583	0.95 (0.00)	0.9585	0.968
	95	8.9118	8.9118	8.91 (0.00)	8.9130	8.95
0.10	100	4.9151	4.9151	4.91 (0.00)	4.9154	5.10
	105	2.0699	2.0699	2.06 (0.00)	2.0704	2.34
	90	14.9828	14.9828	14.96 (0.01)	14.9928	15.194
0.30	100	8.8276	8.8276	8.81 (0.01)	8.8333	9.039
	110	4.6949	4.6949	4.68 (0.01)	4.7027	4.906
	90	18.1829	18.1829	18.14 (0.03)	18.2208	18.57
0.50	100	13.0225	13.0225	12.98 (0.03)	13.0568	13.69
	110	9.1179	9.1180	9.10 (0.03)	9.1560	9.97

Table 3.1 Comparison of various bounds on fixed-strike Asian option prices at the start of the averaging period. The initial stock price is 100, and the interest-rate 0.09%. Estimates of standard errors are in parentheses (from Curran (1992)).

Floating-strike options

In Table 3.3 we consider floating-strike options at the start of the averaging period, with $S_0 = 100$, and show how the upper and lower bounds of Rogers & Shi (1992) compare to the upper bound (3.42) and the generalisation to floating-strike options of Curran's lower bound using (3.33). In Table 3.4 we give examples of the lower and upper bounds (3.27) and (3.42) on the price of floating strike options 6 months into the averaging period. We assume that $S_{0.5} = 100$ and that the average stock price over the first 6 months is also 100. The approximate timings are 0.0002 seconds for the generalisation of Curran's formula, 0.004 seconds for the lower bound and 0.03 seconds for the upper bound.

Volatility σ	Strike K	R-S lower	New Upper
	95	12.6299	12.6303
0.05	100	8.2985	8.2988
	105	4.3173	4.3179
	95	12.8425	12.8436
0.10	100	8.9750	8.9757
	105	5.7151	5.7156
	90	20.2959	20.3023
0.30	100	14.6530	14.6595
	110	10.2466	10.2542
	90	25.6198	25.6511
0.50	100	20.9894	21.0233
	110	17.1213	17.1579

Table 3.2 Bounds on fixed-strike Asian option prices 6 months before the start of the averaging period. The initial stock price is 100, and the interest-rate 0.09%.

Volatility σ	Interest-rate r	Generalised	R-S lower	Upper bound	R-S upper
	0.05	1.2454	1.2454	1.2457	1.355
0.1	0.09	0.6992	0.6992	0.6997	0.825
	0.15	0.2516	0.2516	0.2525	0.415
	0.05	3.4044	3.4044	3.4067	3.831
0.2	0.09	2.6216	2.6216	2.6240	3.062
	0.15	1.7098	1.7098	1.7126	2.187
	0.05	5.6246	5.6246	5.6324	6.584
0.3	0.09	4.7382	4.7382	4.7461	5.706
	0.15	3.6085	3.6085	3.6170	4.604

Table 3.3 Comparison of bounds on floating-strike Asian option prices at the start of the averaging period. The parameter values are taken from Rogers & Shi (1992).

Volatility σ	Interest-rate r	R-S Lower bound	Upper bound
	0.05	1.3291	1.3307
0.05	0.09	0.8562	0.8573
	0.15	0.4014	0.4021
	0.05	3.3919	3.3961
0.10	0.09	2.7687	2.7722
	0.15	1.9962	1.9988
	0.05	5.4916	5.4990
0.30	0.09	4.8037	4.8103
	0.15	3.8917	3.8973

Table 3.4 Bounds on floating-strike Asian option prices 6 months into the averaging period.

3.3 Barrier options

The standard example of the barrier option is the knock-out call option—a European call option with an extra clause, cancelling the contract if the stock price hits a specified boundary. More complex is the 'double barrier' option, with two knock-out boundaries, and yet another possibility is the 'knock-in' version, which only pays out if the stock price hits the barrier. In this section we will consider single and double barrier knock-out options; the value of the corresponding knock-in can be found by subtracting the value of the knock-out from the normal value of the option. Other types of barrier option which we will not consider include 'protected' barrier options, where the barrier clause is only effective for part of the time, and 'rainbow' options, where the barrier clause refers to the price of a second stock. Carr (1995) describes the pricing of these types of options in the normal Black-Scholes model with constant volatility and interest-rate, and reviews much of the literature on other types of barrier options.

Unlike Asian options, for which pricing is difficult in the normal Black-Scholes model, one-sided barrier options with a constant barrier, or more generally, with a barrier of the form $c_1 \exp(c_2 t)$, do have fairly simple pricing formulae (see Goldman, Sosin & Gatto (1979) or Musiela & Rutkowski (1997) for example), but with two knock-out barriers, or in a model with time-varying volatility, no simple formula exists.

For these cases, numerical methods based on trees and two-dimensional lattices are very popular (see Section 3.1), but the discontinuity of the payoff function can lead to slow convergence (see Boyle & Lau (1994) for a discussion), unless some modification of the basic method is employed (see Rogers & Zane (1997) and Rogers & Stapleton (1998) for very successful methods).

Less numerical methods, leaving the answer as an infinite series or as a single or double integral can also be effective solutions. In Kunitomo & Ikeda (1992), a formula in the form of a rapidly convergent infinite series is derived for the case of two barriers of the form $c_1 \exp(c_2 t)$, in the normal Black-Scholes model. Another approach, via Laplace transforms, is pursued by Geman & Yor (1996) and Jamshidian (1997), also for the case of two constant barriers. Methods applicable to problems with more general barriers include those of Roberts & Shortland (1997), using bounds on the hazard rate of the first hitting time, and Lo (1997), who uses a clever modification of the method of images to obtain a simple formula for the approximate price, together with a bound on the error; we will review the work of Lo (1997) in Section 3.3.1. In principle, the method of Lo (1997) can be used to produce formulae with arbitrary accuracy; it also has the advantage of giving a simple pricing formulae, rather than leaving the answer as an integral, though her method requires judicious parameter choices to give very narrow bounds.

Our original contribution in this section is a pair of bounds on the price of single and double barrier options with twice-differentiable barriers. These can be evaluated rapidly and are accurate for typical numerical examples.

Notational Note 3.7 We will assume, without loss of generality, that the option is to be priced at time 0, that it has not already been knocked-out (otherwise the problem is trivial) and that it expires at time 1.

Amongst the various one-sided barrier options, we will concentrate on the 'up-and-out' call option. This has a single knock-out barrier $\{F_t : t \in [0,1]\}$, with $F_0 > S_0$: the contract is cancelled if $S_t \geq F_t$ for some $0 \leq t \leq 1$, and is otherwise worth $(S_1 - K)^+$ at time 1. It is more convenient to have the barrier expressed in terms of the Brownian motion B, of (3.6), so we define $f_t = \log\big(F_{\tau^{-1}(t)}/S_0\big) - \alpha\big(\tau^{-1}(t)\big)$ and $T = \tau(1)$. With these definitions, the up-and-out option is cancelled if $B_t \geq f_t$ for some $0 \leq t \leq T$. (The 'down-and-out' option, where the knock-out barrier is initially below the stock price can be handled by a similar method.)

Turning to the two-sided case, we now have two barriers, G_t and F_t, with $G_0 < S_0 < F_0$ and $G_t < F_t$ for all $t \in [0,1]$, and cancel the contract if $S_t \notin (G_t, F_t)$ for some $t \in [0,1]$. Defining f_t as above, and g_t by $g_t = \log\big(G_{\tau^{-1}(t)}/S_0\big) - \alpha\big(\tau^{-1}(t)\big)$, the contract is cancelled if $B_t \notin (g_t, f_t)$ for some $0 \leq t \leq T$.

As a shorthand, we will let $X = \big[S_0 \exp\big(B_T + \alpha(1)\big) - K\big]^+$, the payout of the standard European call option at time 1.

3.3.1 Lo's method of images

In her PhD thesis, Lo (1997) considers the problem of calculating approximations to the *taboo density*, $\mathbb{P}(B_T \in dx, H_f > T)$ where $H_f = \inf\{t : B_t \geq f_t\}$, for an arbitrary continuous boundary f_t, with $f_0 > 0$. She suggests an approximation of the form $\mathbb{P}(B_T \in dx, H_f > T) \approx h(T, x)$ where

$$h(t,x) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) - \sum_{i=1}^{m} \mathrm{I}(t > t_i) \int_0^\infty \frac{1}{\sqrt{t-t_i}}\phi\left(\frac{x-\theta}{\sqrt{t-t_i}}\right) dF_i(\theta) \tag{3.48}$$

for $0 = t_1 < t_2 < \cdots < t_m \leq T$, an increasing sequence of fixed times and $\{F_i\}$ a collection of signed σ-finite measures on $\mathbb{R}^+$.

To determine the times $\{t_i\}$ and measures $\{F_i\}$, we use the fact that the right-hand side of (3.48) is the exact taboo density for the boundary $\tilde{f}_t$, defined by $\tilde{f}_t = \inf\{x : h(t,x) < 0\}$. This follows from the fact that h satisfies the heat equation, $\partial h/\partial t = \frac{1}{2}\partial^2 h/\partial x^2$, on $\{(t,x) : x < \tilde{f}_t\}$ with boundary conditions $h(t, \tilde{f}_t) = 0$ and $h(0,x) = \delta_0(x)$, where $\delta_y(x)$ denotes the Dirac measure at y. To ensure that $h(T,x)$ gives a good approximation to the required taboo density, we aim to make $\tilde{f}_t \approx f_t$. For example, if a and b are positive constants, and f_t is the linear boundary

$$f_t = b + \left(\frac{\log a}{2b}\right)t,$$

which has the has taboo density

$$\mathbb{P}(B_T \in dx, H_f > T) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) - a^{-1}\frac{1}{\sqrt{t}}\phi\left(\frac{x-2b}{\sqrt{t}}\right),$$

we can take $m = 1$ and $F_1 = a^{-1}\delta_{2b}$; in this case the approximation is exact.

To choose the measures $\{F_i\}$, it is convenient to restrict attention to those of the form $F_i = \sum_{j=1}^{n_i} a_{ij}\delta_{\theta_{ij}}$ with θ_{ij} increasing in j, but even with this restriction, choosing $\{\theta_{ij}\}$ sensibly is a tricky problem. Lerche (1986) proves that $\lim_{t\to 0} \tilde{f}_t = \frac{1}{2}\theta_{11}$, which suggests the choice $\theta_{11} = 2f_0$; beyond this we must use a heuristic approach. Once $\{\theta_{ij}\}$ and $\{t_i\}$ have been determined, since $h(t,x)$ is a linear function of $\{a_{ij}\}$, it is relatively easy to determine $\{a_{ij}\}$ by interpolating f, or some of its derivatives, at a number of points in $[0, T]$. For example, interpolating f at t implies $h(f_t, t) = 0$, which is a linear equation in $\{a_{ij}\}$. Alternatively, we could interpolate f' instead: since $h(t, \tilde{f}_t) = 0$, we have $\partial h/\partial t + (\partial h/\partial x)\tilde{f}'_t = 0$. Imposing $\tilde{f}'_t = f'_t$, this becomes $\partial h/\partial t + (\partial h/\partial x)f'_t = 0$, which again is linear in $\{a_{ij}\}$. With several conditions of this form, we can solve the set of linear equations to determine $\{a_{ij}\}$. An advantage of using (3.48) with $m \gg 1$ is that for $i > j$, the choice of F_i does not effect $h(t,x)$ for $t \leq t_j$; thus we can select F_i with the knowledge of $h(t,x)$—and hence $\tilde{f}_t$—for $0 \leq t < t_i$.

Once we have determined the approximation $h(T,x)$ we can proceed to price knock-out options. The time-0 value of an up-and-out call option is proportional to $\mathbb{E}_0[X\,\mathrm{I}(B_s < f_s, 0 \leq s \leq T)]$, where $X = \left[S_0 \exp\left(B_T + \alpha(1)\right) - K\right]^+$ as usual; thus we can calculate the approximate value of the option by integrating $\left[S_0 \exp\left(x + \alpha(1)\right) - K\right]^+$ against $h(T,x)$. As $h(T,x)$ is a linear combination of normal densities, the expression for the option price will just be a linear combination of Black-Scholes formulae.

Lo's method can also be extended to handle problems involving two-sided boundaries; we refer the reader to Lo (1997) for further details.

3.3.2 One-sided barrier options

In this section, we try to calculate the time-0 value of an up-and-out call option with an arbitrary twice-differentiable barrier. We will derive upper and lower bounds, discuss how to handle the up-and-out put option and options with more general terminal payout functions, and suggest ways of improving on these bounds if necessary. Finally, we will look at a numerical example, and compare our bounds to those of Roberts & Shortland (1997) and Lo (1997) for this problem.

Recall the definition $X = \left[S_0 \exp\left(B_T + \alpha(1)\right) - K\right]^+$, the payout of the standard European call option, and that the time-0 value of the up-and-out call option is given by

$$D(0,1)\mathbb{E}_0[X\,\mathrm{I}(B_s < f_s, 0 \le s \le T)]$$

(see Section 3.1). Setting $E_1 = \mathbb{E}_0[X\,\mathrm{I}(B_s < f_s, 0 \le s \le T)]$, we will try to bound E_1.

Let $\tilde{B}_t = B_t - f_t + f_0$ and define the probability measure $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\int_0^T f'_t\, dB_t - \tfrac{1}{2}\int_0^T (f'_t)^2\, dt\right).$$

By the Cameron-Martin-Girsanov Theorem we have

$$E_1 = \tilde{\mathbb{E}}_0\left[e^{-\int_0^T f'_t\, d\tilde{B}_t - \frac{1}{2}\int_0^T (f'_t)^2\, dt} X\,\mathrm{I}\left(\tilde{B}_s < f_0, 0 \le s \le T\right)\right]$$

where $\tilde{B}$ is a $\tilde{\mathbb{P}}$-Brownian motion. Integrating $f'_t\, d\tilde{B}_t$ by parts, gives

$$E_1 = e^{-\frac{1}{2}\int_0^T (f'_t)^2\, dt}\tilde{\mathbb{E}}_0\left[e^{\int_0^T f''_t \tilde{B}_t\, dt - f'_T \tilde{B}_T} X\,\mathrm{I}\left(\tilde{B}_s < f_0, 0 \le s \le T\right)\right]. \tag{3.49}$$

Now recall Jensen's inequality for the function $y \mapsto \exp(y)$: for any random variable Y, we have $\mathbb{E}\exp(Y) \ge \exp(\mathbb{E}Y)$, and for any function a, $\int_0^1 \exp\left(a(t)\right) dt \ge \exp\left(\int_0^1 a(t)\, dt\right)$ with equality if and only if Y (respectively a) is almost surely constant. Notice that if β is a non-negative random variable with $\mathbb{E}\beta > 0$ we have

$$\mathbb{E}[\exp(Y)\beta] \ge (\mathbb{E}\beta)\exp\left(\mathbb{E}(Y\beta)/\mathbb{E}\beta\right)$$

and for $T \in \mathbb{R}$, we have $T^{-1}\int_0^T \exp\left(Ta(t)\right) dt \ge \exp\left(\int_0^T a(t)\, dt\right)$. Using these forms of Jensen's inequality, we can bound the expectation on the right-hand-side of (3.49) as follows:

$$\left(\tilde{\mathbb{E}}_0\beta\right)e^{\left(\int_0^T \tilde{\mathbb{E}}_0(\gamma_t\beta)\, dt\right)/\tilde{\mathbb{E}}_0\beta} \le \tilde{\mathbb{E}}_0\left(e^{\int_0^T \gamma_t\, dt}\beta\right) \le \frac{1}{T}\int_0^T \tilde{\mathbb{E}}_0\left(e^{T\gamma_t}\beta\right) dt, \tag{3.50}$$

where $\gamma_t = f''_t \tilde{B}_t$ and $\beta = \exp\left(-f'_T \tilde{B}_T\right) X\,\mathrm{I}\left(\tilde{B}_s < f_0, 0 \le s \le T\right)$. Note that both of these bounds will be exact if and only if $f''_t = 0$ for almost all $t \in [0,T]$, i.e., when f is linear.

Defining

$$G(a, S_0, K, u, t) = \mathbb{E}\big[e^{aW_t}\big(S_0 e^{W_t} - K\big)^+ \,\mathrm{I}(W_s < u, 0 \le s \le t)\big] \tag{3.51}$$

for a $\mathbb{P}$-Brownian motion W, and writing X as $\big[S_0 \exp\big(\tilde{B}_T + f_T - f_0 + \alpha(1)\big) - K\big]^+$, we see that $\tilde{\mathbb{E}}_0\beta = G\big(-f'_T, S_0 \exp\big(f_T - f_0 + \alpha(1)\big), K, f_0, T\big)$. The function G can be expressed in terms of the normal distribution function Φ, as

$$\begin{aligned}
&G(a, S_0, K, u, t)\\
&\quad = S_0 e^{\frac{1}{2}(a+1)^2 t}\bigg\{\bigg[\Phi\bigg(\frac{u-(a+1)t}{\sqrt{t}}\bigg) - \Phi\bigg(\frac{l-(a+1)t}{\sqrt{t}}\bigg)\bigg]\\
&\qquad\qquad + e^{2(a+1)u}\bigg[\Phi\bigg(\frac{u+(a+1)t}{\sqrt{t}}\bigg) - \Phi\bigg(\frac{2u-l+(a+1)t}{\sqrt{t}}\bigg)\bigg]\bigg\}\\
&\quad\quad - K e^{\frac{1}{2}a^2 t}\bigg\{\bigg[\Phi\bigg(\frac{u-at}{\sqrt{t}}\bigg) - \Phi\bigg(\frac{l-at}{\sqrt{t}}\bigg)\bigg]\\
&\qquad\qquad + e^{2au}\bigg[\Phi\bigg(\frac{u+at}{\sqrt{t}}\bigg) - \Phi\bigg(\frac{2u-l+at}{\sqrt{t}}\bigg)\bigg]\bigg\}
\end{aligned}$$

for $l < u$, $u > 0$ and 0 otherwise, where $l = \log(K/S_0)$. Now consider the expressions $\tilde{\mathbb{E}}_0(\gamma_t\beta)$ and $\tilde{\mathbb{E}}_0[\exp(T\gamma_t)\beta]$. Conditioning on $\tilde{B}_t = x$ we have

$$\begin{aligned}
\tilde{\mathbb{E}}_0(\gamma_t\beta) &= \tilde{\mathbb{E}}_0\big[f''_t \tilde{B}_t e^{-f'_T \tilde{B}_T} X\, \mathrm{I}\big(\tilde{B}_s < f_0, 0 \le s \le T\big)\big]\\
&= f''_t \int_{-\infty}^{f_0} x \tilde{\mathbb{P}}_0\big(\tilde{B}_s < f_0, 0 \le s \le t, \text{ and } \tilde{B}_t \in dx\big)\\
&\qquad\qquad \times \tilde{\mathbb{E}}_0\big[e^{-f'_T \tilde{B}_T} X\, \mathrm{I}(\tilde{B}_s < f_0, t \le s \le T) \,\big|\, \tilde{B}_t = x\big],
\end{aligned} \tag{3.52}$$

and for $\tilde{\mathbb{E}}_0[\exp(T\gamma_t)\beta]$, we have

$$\begin{aligned}
\tilde{\mathbb{E}}_0\big(e^{T\gamma_t}\beta\big) &= \tilde{\mathbb{E}}_0\big[e^{Tf''_t \tilde{B}_t} e^{-f'_T \tilde{B}_T} X\, \mathrm{I}(\tilde{B}_s < f_0, 0 \le s \le T)\big]\\
&= \int_{-\infty}^{f_0} e^{Txf''_t} \tilde{\mathbb{P}}_0\big(\tilde{B}_s < f_0, 0 \le s \le t, \text{ and } \tilde{B}_t \in dx\big)\\
&\qquad\qquad \times \tilde{\mathbb{E}}_0\big[e^{-f'_T \tilde{B}_T} X\, \mathrm{I}\big(\tilde{B}_s < f_0, t \le s \le T\big) \,\big|\, \tilde{B}_t = x\big].
\end{aligned} \tag{3.53}$$

To evaluate the integrands, recall the expression for the taboo density of Brownian motion with a linear boundary:

$$\tilde{\mathbb{P}}\big(\tilde{B}_s < f_0, 0 \le s \le t, \text{ and } \tilde{B}_t \in dx\big) = \mathrm{I}(x < f_0)\frac{1}{\sqrt{t}}\bigg[\phi\bigg(\frac{x}{\sqrt{t}}\bigg) - \phi\bigg(\frac{2f_0 - x}{\sqrt{t}}\bigg)\bigg]\, dx, \tag{3.54}$$

and observe that we can write the expectation on the right-hand side of (3.52) and (3.53) as

$$\begin{aligned}
&\tilde{\mathbb{E}}_0\big[e^{-f'_T(x+\tilde{W}_{T-t})}\big(S_0 e^{x+\tilde{W}_{T-t}+f_T-f_0+\alpha(1)} - K\big)^+ \,\mathrm{I}\big(\tilde{W}_s < f_0 - x, 0 \le s \le T-t\big)\big]\\
&\qquad = e^{-xf'_T} G\big(-f'_T, S_0 e^{\alpha(1)+x+f_T-f_0}, K, f_0 - x, T - t\big)
\end{aligned} \tag{3.55}$$

where $\tilde{W}_s = \tilde{B}_{s+t} - \tilde{B}_t, 0 \le s \le T - t$ is a $\tilde{\mathbb{P}}$-Brownian motion.

Other terminal payoff functions

To bound the value of an up-and-out put option or an option with a more general terminal payoff function, note that the form of the terminal payout only enters through the function G. Thus for an up-and-out put, where the terminal payout is $\left[K - S_0 \exp\left(B_T + \alpha(1)\right)\right]^+ \mathrm{I}(B_s < f_s, 0 \le s \le T)$, we just replace (3.51) with

$$G_{\text{put}}(a, S_0, K, u, t) = \mathbb{E}\left[e^{aW_t}\left(K - S_0 e^{W_t}\right)^+ \mathrm{I}(W_s < u, 0 \le s \le t)\right]$$

and hence

$$\begin{aligned} G_{\text{put}}(a, S_0, K, u, t) = {} & K e^{\frac{1}{2}a^2 t}\left\{\Phi\left(\frac{l - at}{\sqrt{t}}\right) + e^{2au}\left[\Phi\left(\frac{2u - l + at}{\sqrt{t}}\right) - 1\right]\right\} \\ & - S_0 e^{\frac{1}{2}(a+1)^2 t}\left\{\Phi\left(\frac{l - (a+1)t}{\sqrt{t}}\right)\right. \\ & \left. + e^{2(a+1)u}\left[\Phi\left(\frac{2u - l + (a+1)t}{\sqrt{t}}\right) - 1\right]\right\} \end{aligned}$$

for $u > 0$ and 0 otherwise, where $l = \min\left(\log(K/S_0), u\right)$.

For a claim with more general terminal payoff function, we could condition on both $\tilde{B}_T$ and $\tilde{B}_t$ in (3.52), leading to a three-dimensional integral for the price of the option, but it is probably better to approximate or bound the payoff with a polynomial of high degree (noting that for a call option, this approximation need only be accurate over $[K, F_1]$), when (3.52) becomes a sum of one-dimensional integrals.

Improved bounds

A simple improvement on the upper bound may be obtained by letting μ_t be a deterministic function satisfying $\int_0^T \mu_t\, dt = 0$ and replacing the upper bound in (3.50) by

$$\tilde{\mathbb{E}}_0\left(e^{\int_0^T \gamma_t\, dt}\beta\right) \le \frac{1}{T}\int_0^T \tilde{\mathbb{E}}_0\left(e^{T(\gamma_t - \mu_t)}\beta\right) dt. \tag{3.56}$$

Minimising the right-hand side of (3.56) over $\{\mu_t\}$, and recalling the definition $\gamma_t = f_t'' \tilde{B}_t$, gives $\mu_t = C + \log \tilde{\mathbb{E}}_0\left[\exp\left(T f_t'' \tilde{B}_t\right)\beta\right]$ where C is chosen so that $\int_0^T \mu_t\, dt = 0$. To implement this, we evaluate $\log \tilde{\mathbb{E}}_0\left[\exp\left(T f_t'' \tilde{B}_t\right)\beta\right]$ at a fixed number of points in $[0, T]$, and use cubic interpolation to determine $\mu_t - C$ for the remaining $t \in [0, T]$. The interpolating function is then easy to integrate, allowing us to calculate C. For our numerical example however, this did not lead to a substantial improvement.

3.3.3 A numerical example of a one-sided barrier problem

Roberts & Shortland (1997) consider an up-and-in call option with a constant knock-in barrier F, on a stock with constant volatility but a non-constant interest-rate. Specifically, they assume that $r(t)$ has been perturbed from some equilibrium level r_∞, to which it returns via an exponential decay: $r(t) = r_\infty + (r_0 - r_\infty)\exp(-ct)$, for some constant c. For a time-0 calculation, the functions τ and α, defined by (3.5), are given by $\tau(t) = \sigma^2 t$ and $\alpha(t) = r_\infty t + (r_0 - r_\infty)[1 - \exp(-ct)]/c - \frac{1}{2}\sigma^2 t$ for $t \geq 0$. In terms of the underlying Brownian motion, the barrier is

$$\begin{aligned} f_t &= \log(F/S_0) - \alpha(t/\sigma^2) \\ &= \log(F/S_0) - r_\infty t/\sigma^2 - (r_0 - r_\infty)[1 - \exp(-ct/\sigma^2)]/c + \tfrac{1}{2}t. \end{aligned}$$

The specific problem which Roberts & Shortland (1997) examine has the parameters $S_0 = 10, \sigma = 0.1, K = 11, r_0 = 0.15, r_\infty = 0.1, c = 1, F = 12$. As remarked earlier, the price of an up-and-in option is just the difference between the price of a standard European option and an up-and-out option with the same barrier. A comparison of the method described here, together with the methods of Roberts & Shortland (1997) and Lo (1997) is shown in Table 3.5.

Method	Lower bound	Upper bound	Width
Roberts & Shortland	0.516758	0.517968	0.23%
Lo	0.516243	0.5175570	0.25%
New	0.516369	0.517159	0.15%

Table 3.5 Bounds on the value of a one-sided barrier option.

3.3.4 Two-sided barrier options

Recall that the time-0 value of a two-sided knock-out call option is given by

$$D(0,1)\mathbb{E}_0[X\,\mathrm{I}(g_s < B_s < f_s, 0 \leq s \leq T)].$$

To generalise the method of the previous section, we let $E_2 = \mathbb{E}_0[X\,\mathrm{I}(g_s < B_s < f_s, 0 \leq s \leq T)]$ and try to bound E_2. First, we will transform the process B_s to make both knock-out barriers constant. Define the process $Y_s = (B_s - g_s)/(f_s - g_s)$, $0 \leq s \leq T$ so that

$$dY_s = \frac{dB_s}{f_s - g_s} - \frac{ds}{f_s - g_s}[g'_s + Y_s(f'_s - g'_s)],$$

and the time change υ_t, by $\psi_{\upsilon_t} = t$ where $\psi_0 = 0$, $\psi'_s = (f_s - g_s)^{-2}$. Finally, set $Z_t = Y_{\upsilon_t}$ and $\tilde{T} = \psi_T$. The process Z_t is then a diffusion on $[0, \tilde{T}]$ with SDE

$$dZ_t = dW_t + (\zeta_t + \xi_t Z_t)\, dt,$$

with $Z_0 = -g_0/(f_0 - g_0)$, for some Brownian motion W_t, where, with $s = \upsilon_t$, we have

$$\begin{aligned}\zeta_t &= -(f_s - g_s)g'_s \\ \xi_t &= -(f_s - g_s)(f'_s - g'_s).\end{aligned}$$

We now follow the same path as Section 3.3.2. Define the probability measure $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}}/d\mathbb{P} = \exp\big(-\int_0^{\tilde{T}} (\zeta_t + \xi_t Z_t)\, dW_t - \frac{1}{2}\int_0^{\tilde{T}} (\zeta_t + \xi_t Z_t)^2\, dt\big)$ and use the Cameron-Martin-Girsanov Theorem to give

$$E_2 = \tilde{\mathbb{E}}_0 \Big[e^{\int_0^{\tilde{T}} (\zeta_t + \xi_t Z_t)\, dZ_t - \frac{1}{2}\int_0^{\tilde{T}} (\zeta_t + \xi_t Z_t)^2\, dt} X\, \mathrm{I}\big(0 < Z_s < 1, 0 \le s \le \tilde{T}\big) \Big]$$

where $Z_t - Z_0$ is now a Brownian motion under $\tilde{\mathbb{P}}$. From Itô's Lemma, we have $d(\zeta_t Z_t) = \zeta'_t Z_t\, dt + \zeta_t\, dZ_t$ and $d(\xi_t Z_t^2) = \xi'_t Z_t^2\, dt + 2\xi_t Z_t\, dZ_t + \xi_t\, dt$, so

$$E_2 = e^{-\zeta_0 Z_0 - \frac{1}{2}\xi_0 Z_0^2 - \frac{1}{2}\int_0^{\tilde{T}} (\zeta_t^2 + \xi_t)\, dt}\, \tilde{\mathbb{E}}_0 \Big(e^{\int_0^{\tilde{T}} \gamma_t\, dt} \beta \Big) \tag{3.57}$$

where, setting $s = \upsilon_t$, we have

$$\begin{aligned}\gamma_t &= -(\zeta'_t + \zeta_t \xi_t) Z_t - \tfrac{1}{2}\big(\xi'_t + \xi_t^2\big) Z_t^2 \\ &= (f_s - g_s)^3 \big(g''_s Z_t + \tfrac{1}{2}(f''_s - g''_s) Z_t^2\big)\end{aligned} \tag{3.58}$$

and

$$\beta = e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}\big(0 < Z_s < 1, 0 \le s \le \tilde{T}\big).$$

We now use inequalities (3.50), replacing T with $\tilde{T}$, to bound the expectation on the right-hand side of (3.57). From (3.58) we can see that these bounds will be exact if and only if both f and g are linear.

It remains to compute $\tilde{\mathbb{E}}_0 \beta$, $\tilde{\mathbb{E}}_0(\gamma_t \beta)$ and $\tilde{\mathbb{E}}_0\big[\exp(\tilde{T}\gamma_t)\beta\big]$. For this, we use the result that if W is a Brownian motion, then the function $P(l, u, t, x)$, defined by

$$P(l, u, t, x) = \mathbb{P}(l < W_s < u, 0 \le s \le t, \text{ and } W_t \in dx)$$

is given by

$$P(l, u, t, x) = \mathrm{I}(l < 0 < u)\, \mathrm{I}(l < x < u) \tag{3.59}$$

$$\times \frac{1}{\sqrt{t}} \sum_{-\infty}^{\infty} \left[\phi\left(\frac{x + 2n(u - l)}{\sqrt{t}} \right) - \phi\left(\frac{x - 2u + 2n(u - l)}{\sqrt{t}} \right) \right] dx \tag{3.60}$$

(see Revuz & Yor (1994), page 106 for example). (It is worth remarking that this infinite series converges very rapidly.)

Since the terminal payout X, can be expressed as

$$X = \big[S_0 \exp\big(\alpha(1) + g_T + (f_T - g_T)Z_{\tilde{T}}\big) - K\big]^+,$$

we can write $\tilde{\mathbb{E}}_0\beta = \bar{G}\big(\zeta_{\tilde{T}}, \frac{1}{2}\xi_{\tilde{T}}, S_0\exp(\alpha(1)+g_T), f_T - g_T, K, \tilde{T}, Z_0\big)$, where we define

$$\begin{aligned}&\bar{G}(a, \bar{a}, S_0, \sigma, K, t, w_0)\\ &\quad = \mathbb{E}\big[e^{aW_t + \bar{a}W_t^2}\big(S_0 e^{\sigma W_t} - K\big)^+ \,\mathrm{I}(0 < W_s < 1, 0 \le s \le t) \,\big|\, W_0 = w_0\big]\end{aligned} \tag{3.61}$$

for a $\mathbb{P}$-Brownian motion $W - w_0$. Using (3.60) we find that

$$\bar{G}(a, \bar{a}, S_0, \sigma, K, t, w_0) = \mathrm{I}(0 < w_0 < 1)\,\mathrm{I}(m < 1)\frac{1}{\sqrt{t}}\sum_{-\infty}^{\infty} h_n(a, \bar{a}, S_0, \sigma, K, t, w_0) \tag{3.62}$$

where $m = \max\big(\sigma^{-1}\log(K/S_0), 0\big)$ and

$$\begin{aligned}h_n(a, \bar{a}, S_0, \sigma, K, t, w_0) = S_0\big(&H(m, \bar{a}, a+\sigma, -2n + w_0, 2t)\\ &- H(m, \bar{a}, a+\sigma, 2(1-w_0) - 2n + w_0, 2t)\big)\\ - K\big(&H(m, \bar{a}, a, -2n + w_0, 2t)\\ &- H(m, \bar{a}, a, 2(1-w_0) - 2n + w_0, 2t)\big)\end{aligned}$$

and

$$H(a, \bar{a}, b, c, d) = \frac{1}{\sqrt{2\pi}}\int_a^1 e^{bx + \bar{a}x^2 - (x-c)^2/d}\, dx.$$

The function H is given by

$$\begin{aligned}H(a, \bar{a}, b, c, d) = \sqrt{D}\exp&\big(2D[(b + \bar{a}c)c/d + b^2/4]\big)\\ &\times \left[\Phi\left(\sqrt{D}\left(\frac{2c}{d} + b\right) - \frac{a}{\sqrt{D}}\right) - \Phi\left(\sqrt{D}\left(\frac{2c}{d} + b\right) - \frac{1}{\sqrt{D}}\right)\right]\end{aligned}$$

where $D = \frac{1}{2}d/(1 - \bar{a}d)$, provided $D > 0$, which is the case for all our numerical examples. (If $D < 0$, a similar expression can be given involving Dawson's integral: $\int_0^x \exp(u^2)\, du$.)

We can calculate $\mathbb{E}_0(\gamma_t\beta)$ and $\mathbb{E}_0\big[\exp(\tilde{T}\gamma_t)\beta\big]$ by conditioning on $B_t = x$, as we did in Section 3.3.2. Thus for $\mathbb{E}_0(\gamma_t\beta)$ we have

$$\begin{aligned}\mathbb{E}_0(\gamma_t\beta) &= \tilde{\mathbb{E}}_0\big[(f_s - g_s)^3\big(g_s'' Z_t + \tfrac{1}{2}(f_s'' - g_s'')Z_t^2\big)\\ &\qquad\quad \times e^{\zeta_{\tilde{T}}Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}}Z_{\tilde{T}}^2} X\,\mathrm{I}\big(0 < Z_s < 1, 0 \le s \le \tilde{T}\big)\big]\\ &= \int_0^1 (f_s - g_s)^3\big(g_s'' x + \tfrac{1}{2}(f_s'' - g_s'')x^2\big)\\ &\qquad\quad \times \tilde{\mathbb{P}}_0(0 < Z_s < 1, 0 \le s \le t, \text{ and } Z_t \in dx)\\ &\qquad\quad \times \tilde{\mathbb{E}}_0\big[e^{\zeta_{\tilde{T}}Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}}Z_{\tilde{T}}^2} X\,\mathrm{I}(0 < Z_s < 1, t \le s \le \tilde{T}) \,\big|\, Z_t = x\big]\end{aligned}$$

and for $\mathbb{E}_0\big[\exp(\tilde{T}\gamma_t)\beta\big]$ we have

$$\begin{aligned}\mathbb{E}_0\big[\exp(\tilde{T}\gamma_t)\beta\big] &= \tilde{\mathbb{E}}_0\big[\exp\big(\tilde{T}(f_s - g_s)^3\big(g_s'' Z_t + \tfrac{1}{2}(f_s'' - g_s'')Z_t^2\big)\big) \\ &\qquad \times e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}\big(0 < Z_s < 1, 0 \le s \le \tilde{T}\big)\big] \\ &= \int_0^1 \exp\big(\tilde{T}(f_s - g_s)^3\big(g_s'' x + \tfrac{1}{2}(f_s'' - g_s'')x^2\big)\big) \\ &\qquad \times \tilde{\mathbb{P}}_0(0 < Z_s < 1, 0 \le s \le t, \text{ and } Z_t \in dx) \\ &\qquad \times \tilde{\mathbb{E}}_0\big[e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}\big(0 < Z_s < 1, t \le s \le \tilde{T}\big) \,\big|\, Z_t = x\big].\end{aligned}$$

In each case, we can write the integrand in terms of $\bar{G}$ and P, since

$$\tilde{\mathbb{P}}_0(0 < Z_s < 1, 0 \le s \le t, \text{ and } Z_t \in dx) = P\big(-Z(0), 1 - Z(0), t, x - Z(0)\big)$$

and

$$\begin{aligned}&\tilde{\mathbb{E}}_0\big[e^{\zeta_{\tilde{T}} Z_{\tilde{T}} + \frac{1}{2}\xi_{\tilde{T}} Z_{\tilde{T}}^2} X\, \mathrm{I}(0 < Z_s < 1, t \le s \le \tilde{T}) \,\big|\, Z_t = x\big] \\ &\qquad\qquad = G\big(\zeta_{\tilde{T}}, \tfrac{1}{2}\xi_{\tilde{T}}, S_0 \exp(\alpha(1) + g_T), f_T - g_T, K, \tilde{T} - t, x\big).\end{aligned}$$

3.3.5 Numerical examples of two-sided barrier problems

We now consider the numerical examples of Rogers & Zane (1997), Geman & Yor (1996), Kunitomo & Ikeda (1992) and Rogers & Stapleton (1998). Working in the Black-Scholes model with constant parameters, $\sigma(t) \equiv \sigma$ and $r(t) \equiv r$, they consider three types of knock-out barriers: (i) constant barriers, $L < S_t < U$; (ii) exponential barriers, $L \exp(\delta_L t) < S_t < U \exp(\delta_U t)$, and (iii) linear barriers, $L + \delta_L t < S_t < U + \delta_U t$. Problems (i) and (ii) have exact formulae in the form of infinite series (see Kunitomo & Ikeda (1992), or in terms of an inverse Laplace transform (see Geman & Yor (1996)), and as remarked above, our bounds are also exact in these cases (though we must also allow for the potential error introduced by the numerical integration). For these problems, the alternative methods, particularly Kunitomo & Ikeda (1992) are much quicker. It is the type (iii) problems we are more interested in, where no such formula exists.

In Table 3.7, the figure quoted by Rogers & Zane (1997) for problem (i-2) using the infinite series approach of Kunitomo & Ikeda (1992) is 0.017856, which is not consistent with our lower bound. We note that our own implementation of their method gave the figure 0.01785702, which is consistent. For this type of problem the computation time of our bounds is approximately 0.2 seconds.

In Table 3.9 Kunitomo & Ikeda (1992) give a figure of 10.86 for problem (ii-8), also not consistent with our bounds. Our implementation of their method gives 10.83. Here, the computation time is approximately 0.3 seconds.

Problem	σ	r	K	L	U	S	T
(i-1)	0.2	0.02	2	1.5	2.5	2	1
(i-2)	0.5	0.05	2	1.5	3	2	1
(i-3)	0.5	0.05	1.75	1	3	2	1

Table 3.6 Parameter values for type (i) double-barrier problems (constant barriers)

Problem	RZ	RS	GY	KI	Lower	Upper
(i-1)	0.041079	N/A	0.0411	0.041089	0.041088	0.041090
(i-2)	0.017837	N/A	0.0178	0.017856	0.0178568	0.0178573
(i-3)	0.076147	N/A	0.07615	0.076172	0.0761714	0.0761732

Table 3.7 Numerical results for type (i) double-barrier problems.

Since the bounds will not be exact for type (iii) problems, in Table 3.11 we report the result of performing the numerical integration to high accuracy, as well as the results using the quicker, less accurate code used for the type (i) and type (ii) problems. We also consider some more extreme numerical examples, in order to demonstrate that the bounds are not always accurate. The computation time is about 0.4 seconds, while the accurate answers take about 3 seconds.

In Table 3.11, the figures reported in Rogers & Zane (1997) for problems (iii-4) and (iii-5) are significantly different from our bounds. We implemented the algorithm of Rogers & Zane (1997) as described in their paper as closely as possible (using Mathematica to check intermediate quantities) and report the results in the final column.

3.4 Basket options and stock-index options

In this section, we try to calculate expectations of the form

$$\mathbb{E}\Big(\sum e^{\mu_i+N_i}-K\Big)^+ \tag{3.63}$$

where μ_i, $i=1,\dots,n$, are constants and N_i, $i=1,\dots,n$, are joint normal and mean zero. More generally, we will try to find an approximation to the density of $\sum \exp(\mu_i+N_i)$, enabling us to estimate $\mathbb{E}g(X)$ for an arbitrary function g when $X=\sum \exp(\mu_i+N_i)$.

Problem	σ	r	T	S	K	δ_U	δ_L	L	U
(ii-1)	0.2	0.05	0.5	1000	1000	0.1	−0.1	500	1500
(ii-2)	0.2	0.05	0.5	1000	1000	0.1	−0.1	600	1400
(ii-3)	0.2	0.05	0.5	1000	1000	0.1	−0.1	700	1300
(ii-4)	0.2	0.05	0.5	1000	1000	0.1	−0.1	800	1200
(ii-5)	0.2	0.05	0.5	1000	1000	−0.1	0.1	500	1500
(ii-6)	0.2	0.05	0.5	1000	1000	−0.1	0.1	600	1400
(ii-7)	0.2	0.05	0.5	1000	1000	−0.1	0.1	700	1300
(ii-8)	0.2	0.05	0.5	1000	1000	−0.1	0.1	800	1200
(ii-9)	0.25	0.1	1	95	100	0.1	−0.1	90	160

Table 3.8 Parameter values for type (ii) double-barrier problems (exponential barriers)

Problem	RZ	RS	KI	Lower	Upper
(ii-1)	67.7834	N/A	67.78	67.70	67.86
(ii-2)	64.6401	N/A	64.63	64.56	64.71
(ii-3)	55.1992	N/A	55.20	55.13	55.26
(ii-4)	34.5713	N/A	34.58	34.54	34.62
(ii-5)	62.7532	N/A	62.75	62.68	62.82
(ii-6)	52.5021	N/A	52.50	52.44	52.56
(ii-7)	33.4429	N/A	33.45	33.41	33.49
(ii-8)	10.8217	N/A	10.86	10.81	10.85
(ii-9)	5.3680	5.3672	5.3679	5.362	5.374

Table 3.9 Numerical results for type (ii) double-barrier problems

Problem	σ	r	T	S	K	U	L	δ_U	δ_L
(iii-1)	0.25	0.1	1	95	100	160	90	20	−20
(iii-2)	0.25	0.1	1	95	100	160	90	15	−15
(iii-3)	0.25	0.1	1	95	100	160	90	10	−10
(iii-4)	0.25	0.1	1	95	100	160	90	5	−5
(iii-5)	0.25	0.1	1	95	100	160	90	−5	5
(iii-6)	0.25	0.1	1	95	100	160	90	−10	10
(iii-7)	0.25	0.1	1	95	100	160	90	−15	15
(iii-8)	0.25	0.1	1	95	100	160	90	−20	20

Table 3.10 Parameter values for type (iii) double-barrier problems (linear barriers)

Problem	RZ	Lower	Upper	Approximate
(iii-1)	N/A	6.396 (6.40296)	6.609 (6.60181)	6.40999
(iii-2)	N/A	5.746 (5.75179)	5.789 (5.78319)	5.75352
(iii-3)	N/A	5.031 (5.03665)	5.045 (5.03948)	5.03683
(iii-4)	4.3438	4.263 (4.26790)	4.273 (4.26798)	4.26779
(iii-5)	2.5438	2.635 (2.63779)	2.641 (2.63782)	2.63747
(iii-6)	N/A	1.829 (1.83154)	1.834 (1.83167)	1.83117
(iii-7)	N/A	1.089 (1.09035)	1.092 (1.09082)	1.09001
(iii-8)	N/A	0.490 (0.49097)	0.493 (0.49199)	0.49069

Table 3.11 Numerical results for type (iii) double-barrier problems (linear barriers).

Expression (3.63) arises in a variety of option pricing problems. For example, fixed-strike Asian options, which are discussed at length in Section 3.2; swaptions (see Section 1.4); stock-index options, and currency basket options, to name a few; if the underlying stochastic processes (stock prices, bond prices, exchange-rates etc.) are joint lognormal, all these option pricing problems are essentially just (3.63).

Several previous attempts have been made at this type of problem; Rubinstein (1991) uses an elementary lattice-based approach, while Gentle (1993) replaces $\sum \exp(\mu_i + N_i)$ with an appropriate lognormal random variable (similar to the approach of Kemna & Vorst (1990) for Asian options, see also Musiela & Rutkowski (1997)), and Huynh (1994) approximates the density of $\sum \exp(\mu_i + N_i)$ using a generalised Edgeworth expansion about a lognormal density. While considering a slightly different problem, Lamberton & Lapeyre (1993) use an Edgeworth expansion about a normal density to approximate the density of $\sum \exp(\mu_i + N_i)$, in the case where $\{N_i\}$ are uncorrelated. (The idea of Edgeworth expansion was first applied to the problem of option pricing in Jarrow & Rudd (1982).)

Rather than look for another approximate formula, we will derive lower and upper bounds on (3.63), and demonstrate their accuracy with a few numerical examples. Our lower bound is the same as in Section 3.2.3 and turns out to be very accurate for all the problems we consider. For an upper bound, we will first use the bound of Section 3.2.5 for problems where $\{N_i\}$ are reasonably well correlated, or n is small (which applies to Asian options, swaptions and currency basket options) and derive a new bound to cope with the case when n is large, and $\{N_i\}$ are weakly correlated (as is the case for stock-index options). We will also derive an approximation to the density of $\sum \exp(\mu_i + N_i)$, using ideas from Curran (1992) and Rogers & Shi (1992). This approximate density can be used to give a lower bound on $\mathbb{E}g(X)$ for any convex function g; judging by our numerical examples, it is also very accurate.

Since the first two of these are described in some detail in Section 3.2.3 and Section 3.2.5, we will give only partial derivations.

3.4.1 A lower bound

For the lower bound, observe that if w_j, $j = 1, \dots, n$, and γ are constants, then

$$\mathbb{E}\Big(\sum e^{\mu_i + N_i} - K\Big)^+ \geq \sum_i \mathbb{E}\big(e^{\mu_i + N_i} - K \,;\, \textstyle\sum_j w_j N_j > \gamma\big). \tag{3.64}$$

To determine the optimal value of γ, differentiate the right-hand side of (3.64) with respect to γ, and set the gradient equal to zero, giving

$$\sum_i \mathbb{E}\big(e^{\mu_i + N_i} \,\big|\, \textstyle\sum_j w_j N_j = \gamma\big) = K. \tag{3.65}$$

Setting $c_{ij} = \operatorname{cov}(N_i, N_j)$, $v = \operatorname{var}\big(\sum_j w_j N_j\big) = \sum_{j,k} w_j c_{jk} w_k$ and $v_i = \operatorname{cov}\big(N_i, \sum_j w_j N_j\big) = \sum_j c_{ij} w_j$, (3.65) reduces to

$$\sum \exp\big(\mu_i + \gamma v_i/v + \tfrac{1}{2}[c_{ii} - v_i^2/v]\big) = K. \tag{3.66}$$

For the constants $\{w_j\}$, the simple choice $w_j = \exp(\mu_j)$ is very plausible and seems to work well, though $w_j = \exp(\mu_j)(1 + \frac{1}{2}c_{jj})$ is slightly better for our examples.

For all the numerical examples here, solving (3.66) is straightforward, since the left-hand side turns out to be strictly increasing in γ over a large range. (Alternatively, we could use the method of Curran (1992), as described in Section 3.2.4, to obtain an approximate solution.)

Writing γ^* for the solution to (3.66), we can use (3.22) to perform the expectation calculation in (3.64), giving the bound

$$\mathbb{E}\Big(\sum e^{\mu_i + N_i} - K\Big)^+ \geq \sum e^{\mu_i + \frac{1}{2}c_{ii}}\Phi\left(\frac{-\gamma^* + v_i}{\sqrt{v}}\right) - K\Phi\left(\frac{-\gamma^*}{\sqrt{v}}\right). \tag{3.67}$$

3.4.2 Upper bounds

We now look for upper bounds to complement the lower bounds of the previous section.

Currency basket options

When n is small or $\{N_i\}$ are positively correlated, we suggest a variant of the bound of Section 3.2.5,

$$\mathbb{E}\Big(\sum e^{\mu_i + N_i} - K\Big)^+ \leq \sum \mathbb{E}\big(e^{\mu_i + N_i} - (m_i + y_i I + w_i N_i)\big)^+ \tag{3.68}$$

where m_i, y_i, and w_i are deterministic with $\sum m_i = K$, $\sum y_i = 1$ and $I = -\sum w_i N_i$.

We now try to minimise the right-hand side of (3.68) over all $\{m_i, y_i\}$ such that $\sum m_i = K$ and $\sum y_i = 1$. The usual Lagrangian analysis (see Section 3.2.5) leads to the optimality conditions

$$\begin{aligned} \mathbb{P}(e^{\mu_i + N_i} \geq m_i + y_i I + w_i N_i) &= \lambda \quad \text{for all } i, \\ \mathbb{E}(I\ ;\, e^{\mu_i + N_i} \geq m_i + y_i I + w_i N_i) &= \tilde{\lambda} \quad \text{for all } i, \end{aligned}$$

where λ and $\tilde{\lambda}$ are the Lagrange multipliers for the constraints $\sum m_i = K$ and $\sum y_i = 1$, respectively. We will set $w_i = \exp(\mu_i)$, and repeat the method of Section 3.2.5, by using an approximation of the form $\exp(\mu_i + N_i) - w_i N_i \approx d_i$ for some constant d_i. Unfortunately, this approximation reduces both of the above conditions to

$$d_i - m_i = \gamma y_i \qquad \text{for some } \gamma, \tag{3.69}$$

from which we cannot determine both m_i and y_i. To proceed further, we will make the arbitrary choice $y_i \propto \exp(\mu_i)c_{ii}$. Since $w_i = \exp(\mu_i)$, the choice $d_i = \exp(\mu_i)$ is quite natural, and we can now use (3.69) to give $m_i = d_i - \gamma y_i$, where $\gamma = \sum d_i - K$. With the notation $c_{ij} = \mathrm{cov}(N_i, N_j)$, $v_i = \mathrm{var}(y_i I + w_i N_i)$, and $c_i = \mathrm{cov}(N_i, y_i I + w_i N_i)$, the upper bound can be writing as

$$\mathbb{E}\Big(\sum e^{\mu_i+N_i} - K\Big)^+ \leq \sum \int_{-\infty}^{\infty} \frac{1}{\sqrt{c_{ii}}}\phi\left(\frac{x}{\sqrt{c_{ii}}}\right)\left[a_i\Phi\left(\frac{a_i}{b_i}\right) + b_i\phi\left(\frac{a_i}{b_i}\right)\right] dx \tag{3.70}$$

where $a_i = a_i(x) = \exp(\mu_i + x) - (m_i + xc_i/c_{ii})$ and $b_i = b_i(x) = \big(v_i - c_i^2/c_{ii}\big)^{1/2}$.

Stock-index options

For stock-index options, the bound of Section 3.4.2 is poor, due to the large number of assets and low correlations between them. For this type of option, we will derive a new upper bound from scratch, using a method similar to that of Rogers & Shi (1992) (see Section 3.2.5).

First note that for any random variable X we have

$$\begin{aligned}\mathbb{E}X^+ &= \tfrac{1}{2}(\mathbb{E}X + \mathbb{E}|X|) \\ &\leq \tfrac{1}{2}\mathbb{E}X + \tfrac{1}{2}\big(\mathbb{E}X^2\big)^{1/2}.\end{aligned} \tag{3.71}$$

Moreover, this bound is the best possible which only depends on $\mathbb{E}X$ and $\mathbb{E}X^2$; if $m_2 \geq m_1^2$, the random variable with distribution

$$\begin{aligned}\mathbb{P}(X = -\sqrt{m_2}) &= \tfrac{1}{2}\left(1 - \frac{m_1}{\sqrt{m_2}}\right) \\ \mathbb{P}(X = +\sqrt{m_2}) &= \tfrac{1}{2}\left(1 + \frac{m_1}{\sqrt{m_2}}\right)\end{aligned}$$

has $\mathbb{E}X = m_1$ and $\mathbb{E}X^2 = m_2$, and the inequality in (3.71) is tight in this case. We will put $X = \sum \exp(\mu_i + N_i) - K$ and first condition on $Y = \sum_i w_i N_i$, where $w_i = \exp(\mu_i)$, before applying (3.71) to the conditional distribution of X given Y. Thus, writing $S = \sum \exp(\mu_i + N_i)$, we have

$$\mathbb{E}(S-K)^+ \leq \tfrac{1}{2}\mathbb{E}(S-K) + \tfrac{1}{2}\mathbb{E}\big[\mathbb{E}\big((S-K)^2 \,\big|\, Y\big)\big]^{1/2}. \tag{3.72}$$

To evaluate the final term, we write $\mathbb{E}(S^2 \mid Y)$ as a double sum of exponentials.

3.4.3 A density approximation

In this section we will derive an approximation to the density of $\sum \exp(\mu_i + N_i)$ using the ideas of Rogers & Shi (1992).

Let w_j, $j = 1, \ldots, n$, be constants and let $Y = \sum w_j N_j$. Define the functions

$$x(y) = \mathbb{E}\big(\sum \exp(\mu_i + N_i) \,\big|\, Y = y\big)$$

and $y(\,\cdot\,) = x^{-1}(\,\cdot\,)$, the inverse function to $x(y)$. We will assume that $x : [0, \infty) \to \mathbb{R}$ is 1-1 and onto, and strictly increasing in y; for our numerical examples, the intuitive choice $w_j = \exp(\mu_j)$ appears to work. Let $f_Y(y)$ denote the density of Y at y, and define the function

$$\tilde{f}(x) = f_Y\big(y(x)\big)y'(x) \tag{3.73}$$

Note that $\tilde{f} \geq 0$ and $\int_0^\infty \tilde{f}(x)\,dx = \int_0^\infty f_Y(y(x))y'(x)\,dx = \int_{-\infty}^\infty f_Y(y)\,dy = 1$, so $\tilde{f}$ is a probability density. To evaluate $\tilde{f}$ in practice, it is quicker to first evaluate x' and then use $y'(u) = 1/x'\big(y(u)\big)$.

Proposition 3.8 *Let* $X = \sum \exp(\mu_i + N_i)$ *and let* $g(x)$ *be an arbitrary convex function on* $[0, \infty)$*; then* $\mathbb{E}g(X) \geq \int_0^\infty g(x)\tilde{f}(x)\,dx$.

Proof Using Jensen's inequality, we have

$$\begin{aligned}
\mathbb{E}g(X) &= \mathbb{E}\big[\mathbb{E}\big(g(X) \,\big|\, Y\big)\big] \\
&\geq \mathbb{E}g\big(\mathbb{E}(X \mid Y)\big) \\
&= \int_{-\infty}^{\infty} g\big(x(y)\big)\, f_Y(y)\,dy \\
&= \int_0^{\infty} g(x) f_Y\big(y(x)\big)y'(x)\,dx \\
&= \int_0^{\infty} g(x)\tilde{f}(x)\,dx.
\end{aligned}$$

□

Remarks (i) If $g(x) = (x - K)^+$, and we use the same $\{w_j\}$ as in Section 3.4.1, the lower bound on $\mathbb{E}g(X)$ given by $\int_0^\infty g(x)\tilde{f}(x)\,dx$ is just the lower bound of Section 3.4.1: Let $C(K) = \mathbb{E}(X - K)^+$, then $C'(K) = -\mathbb{P}(X \geq K)$ and $C''(K)$ is the density of X at K. If we have an approximation, $C(K) \approx \tilde{C}(K)$, we could use $\tilde{C}''(K)$ as an approximation to the density. If $\tilde{C}(K) = \mathbb{E}\big(\mathbb{E}(X \mid Y) - K\big)^+$, the lower bound of Section 3.4.1, we have $\tilde{C}'(K) = -\mathbb{P}\big(x(Y) \geq K\big) = \mathbb{P}\big(Y \geq y(K)\big)$, and thus $\tilde{C}''(K) = f_Y\big(y(K)\big)y'(K)$, the expression above.

(ii) With bounds of the form

$$\tilde{C}(K) \leq \mathbb{E}(X - K)^+ \leq \hat{C}(K) \qquad \text{for all } K$$

we can derive upper and lower bounds on the price of options with more general payoff functions. If $g'(x)\tilde{C}(x) \to 0$, and $g(x)\tilde{C}'(x) \to 0$ as $x \to \infty$, we can use

$$g(x) = \int_0^\infty (x - K)^+ g''(K)\,dK + g(0) + xg'(0)$$

to give the lower bound

$$\mathbb{E}g(X) \geq \int_0^\infty g(x)\tilde{C}''(x)\,dx + g(0)[1 + \tilde{C}'(0)] + g'(0)[\mathbb{E}X - \tilde{C}(0)].$$

when g is convex (and similarly an upper bound in terms of $\hat{C}$). Since any function whose first derivative has finite variation is the sum of a convex function and a concave function, we can bound $\mathbb{E}g(X)$ from above and below for a wide range of functions (in particular for any twice-differentiable function).

3.4.4 Numerical results

We now apply our bounds to a number of numerical examples.

Currency basket option

First, we will apply the bounds (3.67) and (3.70), and the density approximation (3.73) to the problem of Huynh (1994), Rubinstein (1991) and Gentle (1993)—a call option on a currency basket involving two currencies. The actual calculation required is

$$e^{-rt}\,\mathbb{E}\Big(\sum V_i X_i e^{t(r-r_i-\frac{1}{2}C_{ii})+N_i} - K\Big)^+$$

where r is the domestic interest-rate, C is the covariance matrix of the infinitesimal increments in the logarithm of the exchange-rates, (N_1, N_2) is bivariate normal with mean zero and covariance matrix cov$(N_i, N_j) = tC_{ij}$, and for currency i: r_i, V_i and X_i denote the interest-rate, the number of units of the currency in the basket, and the initial exchange-rate, respectively. For our problem, the parameters are r =0.04, $r_1 = 0.035$, $r_2 = 0.1$, $V_1X_1 = 10{,}000$, $V_2X_2 = 20{,}000$, $C_{11} = 0.12$, $C_{22} = 0.1$, $C_{12} = C_{21} = \rho\sqrt{C_{11}C_{12}}$ and $t = 0.5$. We consider a range of correlations ρ, and strikes K.

In Table 3.12 we show how the results of Huynh (1994), Rubinstein (1991), Gentle (1993) and our lower and upper bounds, compare to the exact answer, obtained by numerical integration; beneath each entry we show the percentage deviation from the exact answer. Computation times are not available for all the bounds shown; for those derived here, the lower bound takes approximately 0.005 seconds, and the upper bound 0.01 seconds.

Table 3.13 shows the first nine moments of $\sum V_i X_i \exp\big(t(r - r_i - \frac{1}{2}C_{ii}) + N_i\big)$ for the case $\rho = -0.5$, together with the lower bound obtained by integrating against the approximate density (see Proposition 3.8).

Figure 3.1 shows how the density of $\sum V_i X_i \exp\big(t(r - r_i - \frac{1}{2}C_{ii}) + N_i\big)$ compares to the approximate density (the curves are virtually identical).

ρ	K	Exact	Rubinstein	GKV	Huynh	Lower	Upper
	27,000	2,392.33	2,402.19	2,402.19	2,404.11	2,392.14	2,392.67
			0.412	0.412	0.492	−0.008	0.014
−0.5	29,400	486.559	492.74	490.68	492.3	485.617	486.855
			1.270	0.847	1.180	−0.194	0.061
	31,000	60.3734	61.69	59.21	59.81	59.7874	60.8504
			2.181	−1.927	−0.933	−0.971	0.013
	27,000	2,423.21	2,432.7	2,432.7	2,434.28	2,423.1	2,423.73
			0.392	0.392	0.457	−0.005	0.021
0.0	29,400	648.097	654.74	652.97	654.24	647.72	648.222
			1.025	0.752	0.973	−0.05	0.019
	31,000	150.521	152.83	151.90	152.47	150.261	150.989
			1.534	0.916	1.295	−0.173	0.311
	27,000	2,467.45	2,476.98	2,476.71	2,478.03	2,467.42	2,467.81
			0.386	0.375	0.429	−0.001	0.015
0.5	29,400	775.988	782.63	781.75	782.23	775.906	776.017
			0.856	0.743	0.804	−0.011	0.004
	31,000	241.041	244.28	243.66	243.82	240.977	241.291
			1.344	1.087	1.153	−0.027	0.103

Table 3.12 A comparison of bounds and approximations on values of a currency basket, with percentage deviations from the exact answer. Here ρ is the correlation between the exchange-rates and the K the strike price.

Moment	Exact	Lower bound
1	29433.9	29433.9
2	8.6781×10^{8}	8.6779×10^{8}
3	2.5629×10^{13}	2.5628×10^{13}
4	7.5814×10^{17}	7.5811×10^{17}
5	2.2465×10^{22}	2.2463×10^{22}
6	6.6679×10^{26}	6.6672×10^{26}
7	1.9825×10^{31}	1.9822×10^{31}
8	5.9042×10^{35}	5.9028×10^{35}
9	1.7613×10^{40}	1.7608×10^{40}

Table 3.13 Moments of the distribution of the final value of the currency basket (under the martingale measure), with the lower bounds derived from the approximate density.

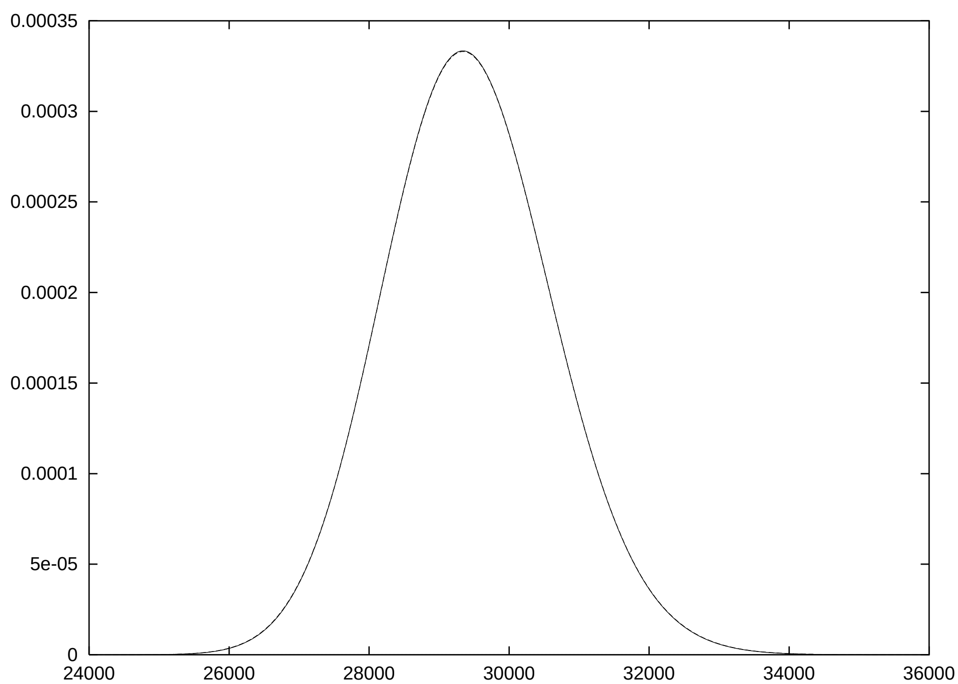


Figure 3.1 Density of the final value of a currency basket (under the martingale measure) consisting of two currencies with negatively correlated exchange-rates.

Stock-index option

Finally, we consider a stock-index option involving the constituent assets of the FTSE-100. The data, kindly supplied by Martin Baxter of Nomura, consists of $\{C_{ij}\}$, the covariance matrix of the instantaneous increments in the logarithm of the 103 stocks present in the FTSE-100 on May 20th 1998.

The precise calculation is

$$V = e^{-rt}\,\mathbb{E}\Big(\sum e^{t(r-q-\frac{1}{2}C_{ii})+N_i} - K\Big)^+$$

where $r = 7.25\%$ is the risk-free interest-rate, $q = 2\%$ is the dividend yield (assumed the same for each i), $\text{cov}(N_i, N_j) = tC_{ij}$, $K = 108.552$ (corresponding to an at-the-money option) and $t = 1$ year.

With these parameters, we can evaluate (3.67) and (3.72) giving bounds of

$$5.6259 \leq V \leq 5.6597$$

in about a second (with almost all the time spent calculating the upper bound). A Monte-Carlo simulation based on 60,060,000 samples gives a price of 5.6331 and a standard error of 0.00117, showing that the errors made by the lower and upper bounds are close to -0.13% and 0.47% respectively.

Table 3.14 shows the moments of the distribution of $\sum \exp\big(t(r-q-\frac{1}{2}C_{ii})+N_i\big)$, obtained via a Monte-Carlo method, together with the lower bound obtained using the approximate density. In Figure 3.2 we show the actual density of $\sum \exp\big(t(r-q-\frac{1}{2}C_{ii})+N_i\big)$, together with the analytic approximation.

Moment	Lower bound	Monte-Carlo result	(std. dev.)
1	108.552	108.552	(0.003)
2	12016.9	12017.5	(0.7)
3	1.3568×10^{6}	1.35704×10^{6}	(120)
4	1.5627×10^{8}	1.56326×10^{8}	(2×10^{4})
5	1.8363×10^{10}	1.83734×10^{10}	(3×10^{6})
6	2.2017×10^{12}	2.20357×10^{12}	(5×10^{8})
7	2.6939×10^{14}	2.69714×10^{14}	(7×10^{10})
8	3.3642×10^{16}	3.36963×10^{16}	(1×10^{13})
9	4.2886×10^{18}	4.29756×10^{18}	(2×10^{15})

Table 3.14 Moments of the payout of a stock-index option.

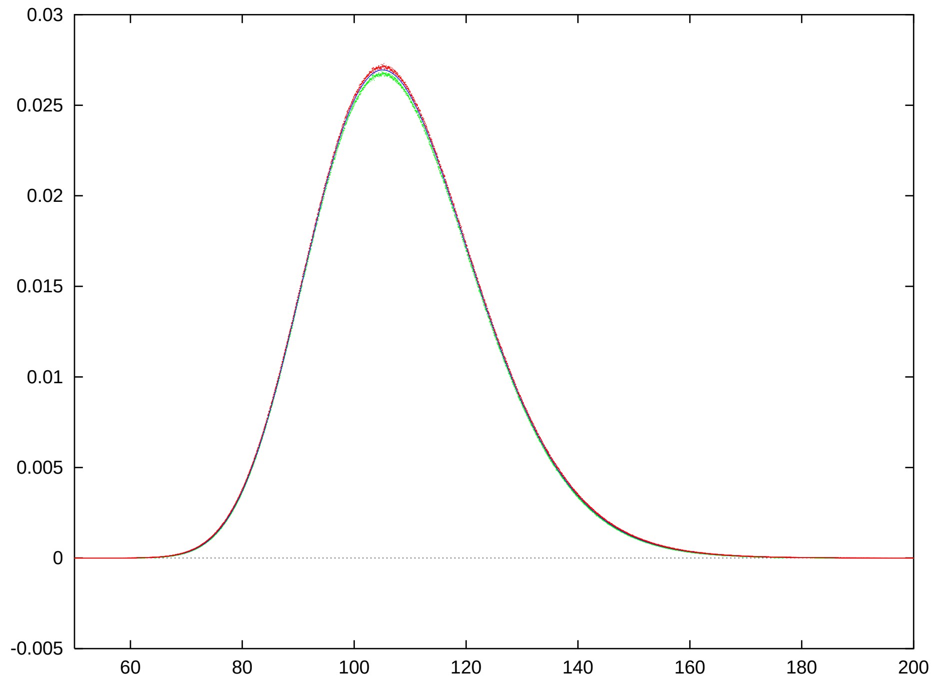


Figure 3.2 A Monte-Carlo estimate of the density of the payout of a stock-index option (confidence intervals of ± 3 standard deviations), together with an analytic approximation.

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