Chapter 9

Pricing in terms of Market Probabilities: The Radon-Nikodym Theorem.

9.1 Radon-Nikodym Theorem

Theorem 1.27 (Radon-Nikodym) *Let P and* $\widetilde{I\!\!P}$ *be two probability measures on a space* $(\Omega, \mathcal{F})$*. Assume that for every* $A \in \mathcal{F}$ *satisfying* $I\!\!P(A) = 0$*, we also have* $\widetilde{I\!\!P}(A) = 0$*. Then we say that* $\widetilde{I\!\!P}$ *is* absolutely continuous *with respect to P. Under this assumption, there is a nonegative random variable* Z *such that*

$$\widetilde{I\!\!P}(A) = \int_A Z\,dI\!\!P, \quad \forall A \in \mathcal{F}, \tag{1.1}$$

and Z *is called the* Radon-Nikodym derivative *of* $\widetilde{I\!\!P}$ *with respect to P.*

Remark 9.1 Equation (1.1) implies the apparently stronger condition

$$\widetilde{I\!\!E}X = I\!\!E[XZ]$$

for every random variable X for which $I\!\!E|XZ| < \infty$.

Remark 9.2 If $\widetilde{I\!\!P}$ is absolutely continuous with respect to P, and P is absolutely continuous with respect to $\widetilde{I\!\!P}$, we say that P and $\widetilde{I\!\!P}$ are *equivalent.* P and $\widetilde{I\!\!P}$ are equivalent if and only if

$$I\!\!P(A) = 0 \text{ exactly when } \widetilde{I\!\!P}(A) = 0, \quad \forall A \in \mathcal{F}.$$

If P and $\widetilde{I\!\!P}$ are equivalent and Z is the Radon-Nikodym derivative of $\widetilde{I\!\!P}$ w.r.t. P, then $\frac{1}{Z}$ is the Radon-Nikodym derivative of P w.r.t. $\widetilde{I\!\!P}$, i.e.,

$$\widetilde{I\!\!E}X = I\!\!E[XZ] \;\; \forall X, \tag{1.2}$$

$$I\!\!EY = \widetilde{I\!\!E}[Y.\frac{1}{Z}] \;\; \forall Y. \tag{1.3}$$

(Let X and Y be related by the equation $Y = XZ$ to see that (1.2) and (1.3) are the same.)

Example 9.1 (Radon-Nikodym Theorem) Let $\Omega = \{HH, HT, TH, TT\}$, the set of coin toss sequences of length 2. Let P correspond to probability $\frac{1}{3}$ for H and $\frac{2}{3}$ for T, and let $\widetilde{\mathbb{P}}$ correspond to probability $\frac{1}{2}$ for H and $\frac{1}{2}$ for T. Then $Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$, so

$$Z(HH) = \frac{9}{4}, \quad Z(HT) = \frac{9}{8}, \quad Z(TH) = \frac{9}{8}, \quad Z(TT) = \frac{9}{16}.$$

■

9.2 Radon-Nikodym Martingales

Let Ω be the set of all sequences of n coin tosses. Let P be the market probability measure and let $\widetilde{\mathbb{P}}$ be the risk-neutral probability measure. Assume

$$\mathbb{P}(\omega) > 0, \ \widetilde{\mathbb{P}}(\omega) > 0, \ \forall \omega \in \Omega,$$

so that P and $\widetilde{\mathbb{P}}$ are equivalent. The Radon-Nikodym derivative of $\widetilde{\mathbb{P}}$ with respect to P is

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}.$$

Define the P-martingale

$$Z_k \stackrel{\triangle}{=} \mathbb{E}[Z|\mathcal{F}_k], \quad k = 0, 1, \ldots, n.$$

We can check that Z_k is indeed a martingale:

$$\begin{aligned} \mathbb{E}[Z_{k+1}|\mathcal{F}_k] &= \mathbb{E}\left[\mathbb{E}[Z|\mathcal{F}_{k+1}]|\mathcal{F}_k\right] \\ &= \mathbb{E}[Z|\mathcal{F}_k] \\ &= Z_k. \end{aligned}$$

Lemma 2.28 *If X is $\mathcal{F}_k$-measurable, then $\widetilde{\mathbb{E}}X = \mathbb{E}[XZ_k]$.*

Proof:

$$\begin{aligned} \widetilde{\mathbb{E}}X &= \mathbb{E}[XZ] \\ &= \mathbb{E}\left[\mathbb{E}[XZ|\mathcal{F}_k]\right] \\ &= \mathbb{E}\left[X.\mathbb{E}[Z|\mathcal{F}_k]\right] \\ &= \mathbb{E}[XZ_k]. \end{aligned}$$

■

Note that Lemma 2.28 implies that if X is $\mathcal{F}_k$-measurable, then for any $A \in \mathcal{F}_k$,

$$\widetilde{\mathbb{E}}[I_A X] = \mathbb{E}[Z_k I_A X],$$

or equivalently,

$$\int_A X \, d\widetilde{\mathbb{P}} = \int_A X Z_k \, d\mathbb{P}.$$

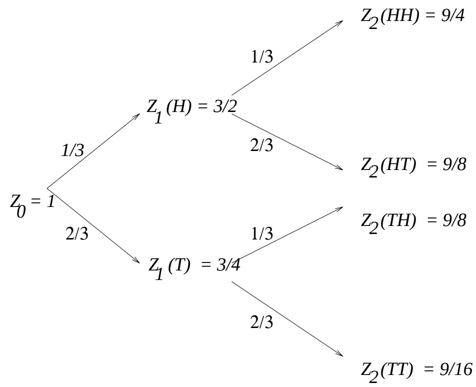

Figure 9.1: *Showing the Z_k values in the 2-period binomial model example. The probabilities shown are for P, not $\widetilde{\mathbb{P}}$.*

Lemma 2.29 *If X is $\mathcal{F}_k$-measurable and $0 \le j \le k$, then*

$$\widetilde{\mathbb{E}}[X|\mathcal{F}_j] = \frac{1}{Z_j}\mathbb{E}[XZ_k|\mathcal{F}_j].$$

Proof: Note first that $\frac{1}{Z_j}\mathbb{E}[XZ_k|\mathcal{F}_j]$ is $\mathcal{F}_j$-measurable. So for any $A \in \mathcal{F}_j$, we have

$$\begin{aligned}
\int_A \frac{1}{Z_j}\mathbb{E}[XZ_k|\mathcal{F}_j]\,d\widetilde{\mathbb{P}} &= \int_A \mathbb{E}[XZ_k|\mathcal{F}_j]\,d\mathbb{P} \quad \text{(Lemma 2.28)}\\
&= \int_A XZ_k\,d\mathbb{P} \quad \text{(Partial averaging)}\\
&= \int_A X\,d\widetilde{\mathbb{P}} \quad \text{(Lemma 2.28)}
\end{aligned}$$

■

Example 9.2 (Radon-Nikodym Theorem, continued) We show in Fig. 9.1 the values of the martingale Z_k. We always have $Z_0 = 1$, since

$$Z_0 = \mathbb{E}Z = \int_\Omega Z\,d\mathbb{P} = \widetilde{\mathbb{P}}(\Omega) = 1.$$

■

9.3 The State Price Density Process

In order to express the value of a derivative security in terms of the market probabilities, it will be useful to introduce the following *state price density process*:

$$\zeta_k = (1+r)^{-k}Z_k,\ k = 0,\ldots,n.$$

We then have the following pricing formulas: For a **Simple European derivative security** with payoff C_k at time k,

$$\begin{aligned} V_0 &= \widetilde{\mathbb{E}}\left[(1+r)^{-k}C_k\right] \\ &= \mathbb{E}\left[(1+r)^{-k}Z_kC_k\right] \quad \text{(Lemma 2.28)} \\ &= \mathbb{E}[\zeta_k C_k]. \end{aligned}$$

More generally for $0 \le j \le k$,

$$\begin{aligned} V_j &= (1+r)^j\widetilde{\mathbb{E}}\left[(1+r)^{-k}C_k|\mathcal{F}_j\right] \\ &= \frac{(1+r)^j}{Z_j}\mathbb{E}\left[(1+r)^{-k}Z_kC_k|\mathcal{F}_j\right] \quad \text{(Lemma 2.29)} \\ &= \frac{1}{\zeta_j}\mathbb{E}[\zeta_k C_k|\mathcal{F}_j] \end{aligned}$$

Remark 9.3 $\{\zeta_j V_j\}_{j=0}^k$ is a martingale under P, as we can check below:

$$\begin{aligned} \mathbb{E}[\zeta_{j+1}V_{j+1}|\mathcal{F}_j] &= \mathbb{E}\left[\mathbb{E}[\zeta_k C_k|\mathcal{F}_{j+1}]|\mathcal{F}_j\right] \\ &= \mathbb{E}[\zeta_k C_k|\mathcal{F}_j] \\ &= \zeta_j V_j. \end{aligned}$$

Now for an **American derivative security** $\{G_k\}_{k=0}^n$:

$$\begin{aligned} V_0 &= \sup_{\tau\in T_0}\widetilde{\mathbb{E}}\left[(1+r)^{-\tau}G_\tau\right] \\ &= \sup_{\tau\in T_0}\mathbb{E}\left[(1+r)^{-\tau}Z_\tau G_\tau\right] \\ &= \sup_{\tau\in T_0}\mathbb{E}[\zeta_\tau G_\tau]. \end{aligned}$$

More generally for $0 \le j \le n$,

$$\begin{aligned} V_j &= (1+r)^j\sup_{\tau\in T_j}\widetilde{\mathbb{E}}\left[(1+r)^{-\tau}G_\tau|\mathcal{F}_j\right] \\ &= (1+r)^j\sup_{\tau\in T_j}\frac{1}{Z_j}\mathbb{E}\left[(1+r)^{-\tau}Z_\tau G_\tau|\mathcal{F}_j\right] \\ &= \frac{1}{\zeta_j}\sup_{\tau\in T_j}\mathbb{E}[\zeta_\tau G_\tau|\mathcal{F}_j]. \end{aligned}$$

Remark 9.4 Note that

(a) $\{\zeta_j V_j\}_{j=0}^n$ is a supermartingale under P,

(b) $\zeta_j V_j \ge \zeta_j G_j \ \forall j$,

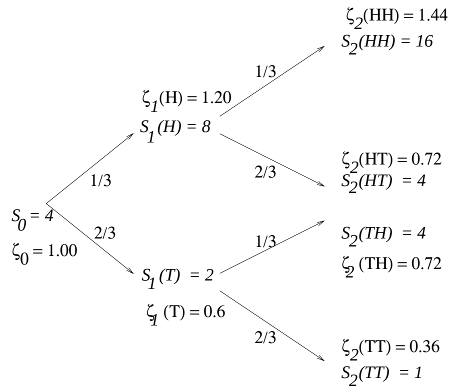

Figure 9.2: *Showing the state price values ζ_k. The probabilities shown are for P, not $\widetilde{I\!\!P}$.*

(c) $\{\zeta_j V_j\}_{j=0}^n$ is the smallest process having properties (a) and (b).

We interpret ζ_k by observing that $\zeta_k(\omega) I\!\!P(\omega)$ is the value at time zero of a contract which pays \$1 at time k if ω occurs.

Example 9.3 (Radon-NikodymTheorem, continued) We illustrate the use of the valuation formulas for European and American derivative securities in terms of market probabilities. Recall that $p = \frac{1}{3}$, $q = \frac{2}{3}$. The state price values ζ_k are shown in Fig. 9.2.

For a **European Call** with strike price 5, expiration time 2, we have

$$V_2(HH) = 11,\ \zeta_2(HH)V_2(HH) = 1.44 \times 11 = 15.84.$$

$$V_2(HT) = V_2(TH) = V_2(TT) = 0.$$

$$V_0 = \frac{1}{3} \times \frac{1}{3} \times 15.84 = 1.76.$$

$$\frac{\zeta_2(HH)}{\zeta_1(HH)} V_2(HH) = \frac{1.44}{1.20} \times 11 = 1.20 \times 11 = 13.20$$

$$V_1(H) = \frac{1}{3} \times 13.20 = 4.40$$

Compare with the risk-neutral pricing formulas:

$$V_1(H) = \tfrac{2}{5} V_1(HH) + \tfrac{2}{5} V_1(HT) = \tfrac{2}{5} \times 11 = 4.40,$$

$$V_1(T) = \tfrac{2}{5} V_1(TH) + \tfrac{2}{5} V_1(TT) = 0,$$

$$V_0 = \tfrac{2}{5} V_1(H) + \tfrac{2}{5} V_1(T) = \tfrac{2}{5} \times 4.40 = 1.76.$$

Now consider an **American put** with strike price 5 and expiration time 2. Fig. 9.3 shows the values of $\zeta_k(5 - S_k)^+$. We compute the value of the put under various stopping times τ:

(0) Stop immediately: value is 1.

(1) If $\tau(HH) = \tau(HT) = 2,\ \tau(TH) = \tau(TT) = 1$, the value is

$$\frac{1}{3} \times \tfrac{2}{3} \times 0.72 + \tfrac{2}{3} \times 1.80 = 1.36.$$

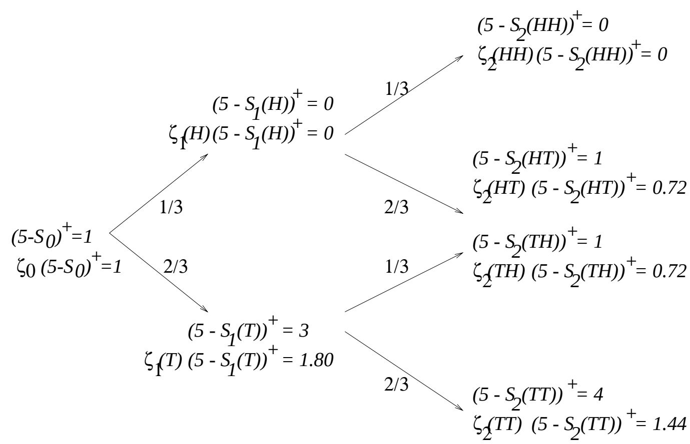

Figure 9.3: *Showing the values $\zeta_k(5 - S_k)^+$ for an American put. The probabilities shown are for P, not $\widetilde{I\!\!P}$.*

(2) If we stop at time 2, the value is

$$\frac{1}{3} \times \frac{2}{3} \times 0.72 + \frac{2}{3} \times \frac{1}{3} \times 0.72 + \frac{2}{3} \times \frac{2}{3} \times 1.44 = 0.96$$

We see that (1) is optimal stopping rule. ■

9.4 Stochastic Volatility Binomial Model

Let Ω be the set of sequences of n tosses, and let $0 < d_k < 1 + r_k < u_k$, where for each k, d_k, u_k, r_k are $\mathcal{F}_k$-measurable. Also let

$$\tilde{p}_k = \frac{1 + r_k - d_k}{u_k - d_k}, \quad \tilde{q}_k = \frac{u_k - (1 + r_k)}{u_k - d_k}.$$

Let $\widetilde{I\!\!P}$ be the risk-neutral probability measure:

$$\widetilde{I\!\!P}\{\omega_1 = H\} = \tilde{p}_0,$$

$$\widetilde{I\!\!P}\{\omega_1 = T\} = \tilde{q}_0,$$

and for $2 \leq k \leq n$,

$$\widetilde{I\!\!P}[\omega_{k+1} = H | \mathcal{F}_k] = \tilde{p}_k,$$

$$\widetilde{I\!\!P}[\omega_{k+1} = T | \mathcal{F}_k] = \tilde{q}_k.$$

Let P be the market probability measure, and assume $I\!\!P\{\omega\} > 0 \ \forall \omega \in \Omega$. Then P and $\widetilde{I\!\!P}$ are equivalent. Define

$$Z(\omega) = \frac{\widetilde{I\!\!P}(\omega)}{I\!\!P(\omega)} \quad \forall \omega \in \Omega,$$

$$Z_k = I\!\!E[Z|\mathcal{F}_k], \quad k = 0, 1, \ldots, n.$$

We define the *money market price process* as follows:

$$M_0 = 1,$$

$$M_k = (1 + r_{k-1})M_{k-1}, \; k = 1, \ldots, n.$$

Note that M_k is F_{k-1}-measurable.

We then define the *state price process* to be

$$\zeta_k = \frac{1}{M_k} Z_k, \; k = 0, \ldots, n.$$

As before the portfolio process is $\{\Delta_k\}_{k=0}^{n-1}$. The self-financing value process (wealth process) consists of X_0, the non-random initial wealth, and

$$X_{k+1} = \Delta_k S_{k+1} + (1 + r_k)(X_k - \Delta_k S_k), \; k = 0, \ldots, n-1.$$

Then the following processes are martingales under $\widetilde{I\!\!P}$:

$$\left\{ \frac{1}{M_k} S_k \right\}_{k=0}^{n} \quad \text{and} \quad \left\{ \frac{1}{M_k} X_k \right\}_{k=0}^{n},$$

and the following processes are martingales under P:

$$\{\zeta_k S_k\}_{k=0}^{n} \quad \text{and} \quad \{\zeta_k X_k\}_{k=0}^{n}.$$

We thus have the following pricing formulas:

Simple European derivative security with payoff C_k at time k:

$$\begin{aligned} V_j &= M_j \widetilde{I\!\!E} \left[\frac{C_k}{M_k} \middle| \mathcal{F}_j \right] \\ &= \frac{1}{\zeta_j} I\!\!E \left[\zeta_k C_k | \mathcal{F}_j \right] \end{aligned}$$

American derivative security $\{G_k\}_{k=0}^{n}$:

$$\begin{aligned} V_j &= M_j \sup_{\tau \in T_j} \widetilde{I\!\!E} \left[\frac{G_\tau}{M_\tau} \middle| \mathcal{F}_j \right] \\ &= \frac{1}{\zeta_j} \sup_{\tau \in T_j} I\!\!E \left[\zeta_\tau G_\tau | \mathcal{F}_j \right]. \end{aligned}$$

The usual hedging portfolio formulas still work.

9.5 Another Applicaton of the Radon-Nikodym Theorem

Let $(\Omega, \mathcal{F}, Q)$ be a probability space. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be a non-negative random variable with $\int_\Omega X \; dQ = 1$. We construct the conditional expectation (under Q) of X given $\mathcal{G}$. On $\mathcal{G}$, define two probability measures

$$\mathbb{P}(A) = Q(A) \quad \forall A \in \mathcal{G};$$

$$\widetilde{\mathbb{P}}(A) = \int_A X \, dQ \quad \forall A \in \mathcal{G}.$$

Whenever Y is a $\mathcal{G}$-measurable random variable, we have

$$\int_\Omega Y \; d\mathbb{P} = \int_\Omega Y \; dQ;$$

if $Y = \mathbf{1}_A$ for some $A \in \mathcal{G}$, this is just the definition of $\mathbb{P}$, and the rest follows from the "standard machine". If $A \in \mathcal{G}$ and $\mathbb{P}(A) = 0$, then $Q(A) = 0$, so $\widetilde{\mathbb{P}}(A) = 0$. In other words, the measure $\widetilde{\mathbb{P}}$ is absolutely continuous with respect to the measure $\widetilde{\mathbb{P}}$. The Radon-Nikodym theorem implies that there exists a $\mathcal{G}$-measurable random variable Z such that

$$\widetilde{\mathbb{P}}(A) \triangleq \int_A Z \; d\mathbb{P} \;\; \forall A \in \mathcal{G},$$

i.e.,

$$\int_A X \; dQ = \int_A Z \; d\mathbb{P} \;\; \forall A \in \mathcal{G}.$$

This shows that Z has the "partial averaging" property, and since Z is $\mathcal{G}$-measurable, it is the conditional expectation (under the probability measure Q) of X given $\mathcal{G}$. The existence of conditional expectations is a consequence of the Radon-Nikodym theorem.