Chapter 8

Random Walks

8.1 First Passage Time

Toss a coin infinitely many times. Then the sample space Ω is the set of all infinite sequences $\omega = (\omega_1, \omega_2, \ldots)$ of H and T. Assume the tosses are independent, and on each toss, the probability of H is $\frac{1}{2}$, as is the probability of T. Define

$$Y_j(\omega) = \begin{cases} 1 & \text{if } \omega_j = H, \\ -1 & \text{if } \omega_j = T, \end{cases}$$

$$\begin{aligned} M_0 &= 0, \\ M_k &= \sum_{j=1}^{k} Y_j, \ k = 1, 2, \ldots \end{aligned}$$

The process $\{M_k\}_{k=0}^{\infty}$ is a *symmetric random walk* (see Fig. 8.1) Its analogue in continuous time is *Brownian motion*.

Define

$$\tau = \min\{k \geq 0; M_k = 1\}.$$

If M_k never gets to 1 (e.g., $\omega = (TTTT\ldots)$), then $\tau = \infty$. The random variable τ is called the *first passage time to 1*. It is the first time the number of heads exceeds by one the number of tails.

8.2 τ is almost surely finite

It is shown in a Homework Problem that $\{M_k\}_{k=0}^{\infty}$ and $\{N_k\}_{k=0}^{\infty}$ where

$$\begin{aligned} N_k &= \exp\left\{\theta M_k - k \log\left(\frac{e^{\theta} + e^{-\theta}}{2}\right)\right\} \\ &= e^{\theta M_k} \left(\frac{2}{e^{\theta} + e^{-\theta}}\right)^k \end{aligned}$$

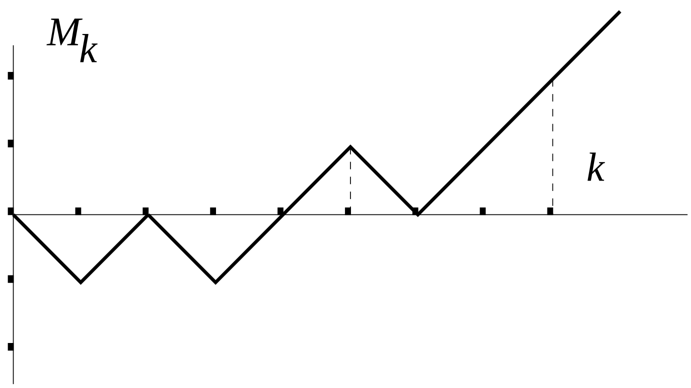

Figure 8.1: *The random walk process* M_k

Figure 8.2: *Illustrating two functions of* θ

are martingales. (Take $M_k = -S_k$ in part (i) of the Homework Problem and take $\theta = -\sigma$ in part (v).) Since $N_0 = 1$ and a stopped martingale is a martingale, we have

$$1 = I\!\!E N_{k\wedge\tau} = I\!\!E\left[e^{\theta M_{k\wedge\tau}}\left(\frac{2}{e^\theta + e^{-\theta}}\right)^{k\wedge\tau}\right] \tag{2.1}$$

for every fixed $\theta \in I\!\!R$ (See Fig. 8.2 for an illustration of the various functions involved). We want to let $k\to\infty$ in (2.1), but we have to worry a bit that for some sequences $\omega \in \Omega$, $\tau(\omega) = \infty$.

We consider fixed $\theta > 0$, so

$$\left(\frac{2}{e^\theta + e^{-\theta}}\right) < 1.$$

As $k\to\infty$,

$$\left(\frac{2}{e^\theta + e^{-\theta}}\right)^{k\wedge\tau} \to \begin{cases} \left(\frac{2}{e^\theta + e^{-\theta}}\right)^\tau & \text{if } \tau < \infty, \\ 0 & \text{if } \tau = \infty \end{cases}$$

Furthermore, $M_{k\wedge\tau} \le 1$, because we stop this martingale when it reaches 1, so

$$0 \le e^{\theta M_{k\wedge\tau}} \le e^\theta$$

and

$$0 \leq e^{\theta M_{k\wedge\tau}} \left(\frac{2}{e^{\theta}+e^{-\theta}}\right)^{k\wedge\tau} \leq e^{\theta}.$$

In addition,

$$\lim_{k\to\infty} e^{\theta M_{k\wedge\tau}} \left(\frac{2}{e^{\theta}+e^{-\theta}}\right)^{k\wedge\tau} = \begin{cases} e^{\theta}\left(\frac{2}{e^{\theta}+e^{-\theta}}\right)^{\tau} & \text{if } \tau<\infty, \\ 0 & \text{if } \tau=\infty. \end{cases}$$

Recall Equation (2.1):

$$I\!\!E\left[e^{\theta M_{k\wedge\tau}} \left(\frac{2}{e^{\theta}+e^{-\theta}}\right)^{k\wedge\tau}\right] = 1$$

Letting $k\to\infty$, and using the Bounded Convergence Theorem, we obtain

$$I\!\!E\left[e^{\theta} \left(\frac{2}{e^{\theta}+e^{-\theta}}\right)^{\tau} I_{\{\tau<\infty\}}\right] = 1. \tag{2.2}$$

For all $\theta \in (0,1]$, we have

$$0 \leq e^{\theta} \left(\frac{2}{e^{\theta}+e^{-\theta}}\right)^{\tau} I_{\{\tau<\infty\}} \leq e,$$

so we can let $\theta\downarrow 0$ in (2.2), using the Bounded Convergence Theorem again, to conclude

$$I\!\!E\left[I_{\{\tau<\infty\}}\right] = 1,$$

i.e.,

$$I\!\!P\{\tau<\infty\} = 1.$$

We know there are paths of the symmetric random walk $\{M_k\}_{k=0}^{\infty}$ which never reach level 1. We have just shown that these paths *collectively* have no probability. (In our infinite sample space Ω, each path *individually* has zero probability). We therefore do not need the indicator $I_{\{\tau<\infty\}}$ in (2.2), and we rewrite that equation as

$$I\!\!E\left[\left(\frac{2}{e^{\theta}+e^{-\theta}}\right)^{\tau}\right] = e^{-\theta}. \tag{2.3}$$

8.3 The moment generating function for τ

Let $\alpha \in (0,1)$ be given. We want to find $\theta > 0$ so that

$$\alpha = \left(\frac{2}{e^{\theta}+e^{-\theta}}\right).$$

Solution:

$$\alpha e^{\theta} + \alpha e^{-\theta} - 2 = 0$$

$$\alpha (e^{-\theta})^2 - 2e^{-\theta} + \alpha = 0$$

$$e^{-\theta} = \frac{1 \pm \sqrt{1-\alpha^2}}{\alpha}.$$

We want $\theta > 0$, so we must have $e^{-\theta} < 1$. Now $0 < \alpha < 1$, so

$$0 < (1-\alpha)^2 < (1-\alpha) < 1-\alpha^2,$$

$$1-\alpha < \sqrt{1-\alpha^2},$$

$$1-\sqrt{1-\alpha^2} < \alpha,$$

$$\frac{1-\sqrt{1-\alpha^2}}{\alpha} < 1$$

We take the negative square root:

$$e^{-\theta} = \frac{1 - \sqrt{1-\alpha^2}}{\alpha}.$$

Recall Equation (2.3):

$$I\!\!E\left[\left(\frac{2}{e^{\theta}+e^{-\theta}}\right)^{\tau}\right] = e^{-\theta},\ \theta > 0.$$

With $\alpha \in (0,1)$ and $\theta > 0$ related by

$$\begin{aligned} e^{-\theta} &= \frac{1-\sqrt{1-\alpha^2}}{\alpha}, \\ \alpha &= \left(\frac{2}{e^{\theta}+e^{-\theta}}\right), \end{aligned}$$

this becomes

$$I\!\!E\alpha^{\tau} = \frac{1-\sqrt{1-\alpha^2}}{\alpha},\ 0 < \alpha < 1. \tag{3.1}$$

We have computed the *moment generating function* for the first passage time to 1.

8.4 Expectation of τ

Recall that

$$I\!\!E\alpha^{\tau} = \frac{1-\sqrt{1-\alpha^2}}{\alpha}, \quad 0 < \alpha < 1,$$

so

$$\begin{aligned} \frac{d}{d\alpha} I\!\!E\alpha^{\tau} &= I\!\!E(\tau\alpha^{\tau-1}) \\ &= \frac{d}{d\alpha}\left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right) \\ &= \frac{1-\sqrt{1-\alpha^2}}{\alpha^2\sqrt{1-\alpha^2}}. \end{aligned}$$

Using the Monotone Convergence Theorem, we can let $\alpha \uparrow 1$ in the equation

$$\mathbb{E}(\tau\alpha^{\tau-1}) = \frac{1-\sqrt{1-\alpha^2}}{\alpha^2\sqrt{1-\alpha^2}},$$

to obtain

$$\mathbb{E}\tau = \infty.$$

Thus in summary:

$$\tau \stackrel{\triangle}{=} \min\{k; M_k = 1\},$$

$$\mathbb{P}\{\tau < \infty\} = 1,$$

$$\mathbb{E}\tau = \infty.$$

8.5 The Strong Markov Property

The random walk process $\{M_k\}_{k=0}^{\infty}$ is a Markov process, i.e.,

$$\begin{aligned} &\mathbb{E}\left[\text{ random variable depending only on } M_{k+1}, M_{k+2}, \ldots \mid \mathcal{F}_k\right] \\ &\quad = \mathbb{E}\left[\text{ same random variable } \mid M_k\right]. \end{aligned}$$

In discrete time, this Markov property implies the *Strong Markov property*:

$$\begin{aligned} &\mathbb{E}\left[\text{ random variable depending only on } M_{\tau+1}, M_{\tau+2}, \ldots \mid \mathcal{F}_\tau\right] \\ &\quad = \mathbb{E}\left[\text{ same random variable } \mid M_\tau\right]. \end{aligned}$$

for any almost surely finite stopping time τ.

8.6 General First Passage Times

Define

$$\tau_m \stackrel{\triangle}{=} \min\{k \geq 0; M_k = m\},\ m = 1, 2, \ldots$$

Then $\tau_2 - \tau_1$ is the number of periods between the first arrival at level 1 and the first arrival at level 2. The distribution of $\tau_2 - \tau_1$ is the same as the distribution of τ_1 (see Fig. 8.3), i.e.,

$$\mathbb{E}\alpha^{\tau_2 - \tau_1} = \frac{1-\sqrt{1-\alpha^2}}{\alpha},\ \alpha \in (0,1).$$

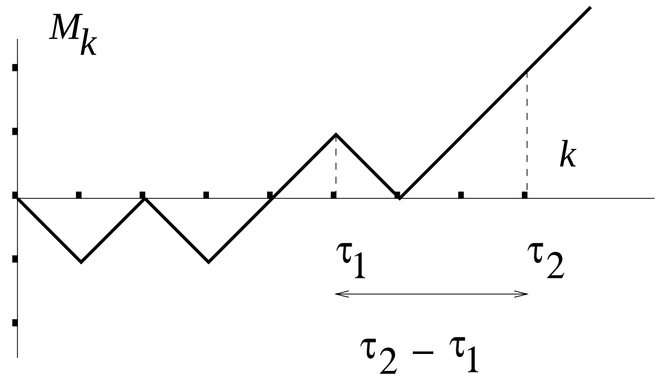

Figure 8.3: *General first passage times.*

For $\alpha \in (0,1)$,

$$
\begin{aligned}
I\!E\left[\alpha^{\tau_2} | \mathcal{F}_{\tau_1}\right] &= I\!E\left[\alpha^{\tau_1}\alpha^{\tau_2-\tau_1} | \mathcal{F}_{\tau_1}\right] \\
&= \alpha^{\tau_1} I\!E[\alpha^{\tau_2-\tau_1} | \mathcal{F}_{\tau_1}] \\
&\qquad \text{(taking out what is known)} \\
&= \alpha^{\tau_1} I\!E[\alpha^{\tau_2-\tau_1} | M_{\tau_1}] \\
&\qquad \text{(strong Markov property)} \\
&= \alpha^{\tau_1} I\!E[\alpha^{\tau_2-\tau_1}] \\
&\qquad (M_{\tau_1} = 1, \text{ not random }) \\
&= \alpha^{\tau_1}\left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right).
\end{aligned}
$$

Take expectations of both sides to get

$$
\begin{aligned}
I\!E\alpha^{\tau_2} &= I\!E\alpha^{\tau_1}.\left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right) \\
&= \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^2
\end{aligned}
$$

In general,

$$
I\!E\alpha^{\tau_m} = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^m, \ \alpha \in (0,1).
$$

8.7 Example: Perpetual American Put

Consider the binomial model, with $u = 2, d = \frac{1}{2}, r = \frac{1}{4}$, and payoff function $(5 - S_k)^+$. The risk neutral probabilities are $\tilde{p} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$, and thus

$$
S_k = S_0 u^{M_k},
$$

where M_k is a symmetric random walk under the risk-neutral measure, denoted by $\widetilde{I\!\!P}$. Suppose $S_0 = 4$. Here are some possible exercise rules:

Rule 0: Stop immediately. $\tau_0 = 0, V^{(\tau_0)} = 1$.

Rule 1: Stop as soon as stock price falls to 2, i.e., at time

$$\tau_{-1} \stackrel{\triangle}{=} \min\{k; M_k = -1\}.$$

Rule 2: Stop as soon as stock price falls to 1, i.e., at time

$$\tau_{-2} \stackrel{\triangle}{=} \min\{k; M_k = -2\}.$$

Because the random walk is symmetric under $\widetilde{I\!\!P}$, τ_{-m} has the same distribution under $\widetilde{I\!\!P}$ as the stopping time τ_m in the previous section. This observation leads to the following computations of value. **Value of Rule 1:**

$$\begin{aligned} V^{(\tau_{-1})} &= \widetilde{I\!\!E}\left[(1+r)^{-\tau_{-1}}(5 - S_{\tau_{-1}})^+\right] \\ &= (5-2)^+ I\!\!E\left[(\tfrac{4}{5})^{\tau_{-1}}\right] \\ &= 3.\frac{1 - \sqrt{1 - (\frac{4}{5})^2}}{\frac{4}{5}} \\ &= \frac{3}{2}. \end{aligned}$$

Value of Rule 2:

$$\begin{aligned} V^{(\tau_{-2})} &= (5-1)^+ \widetilde{I\!\!E}\left[(\tfrac{4}{5})^{\tau_{-2}}\right] \\ &= 4.(\tfrac{1}{2})^2 \\ &= 1. \end{aligned}$$

This suggests that the optimal rule is Rule 1, i.e., stop (exercise the put) as soon as the stock price falls to 2, and the value of the put is $\frac{3}{2}$ if $S_0 = 4$.

Suppose instead we start with $S_0 = 8$, and stop the first time the price falls to 2. This requires 2 down steps, so the value of this rule with this initial stock price is

$$(5-2)^+ \widetilde{I\!\!E}\left[(\tfrac{4}{5})^{\tau_{-2}}\right] = 3.(\tfrac{1}{2})^2 = \frac{3}{4}.$$

In general, if $S_0 = 2^j$ for some $j \geq 1$, and we stop when the stock price falls to 2, then $j-1$ down steps will be required and the value of the option is

$$(5-2)^+ \widetilde{I\!\!E}\left[(\tfrac{4}{5})^{\tau_{-(j-1)}}\right] = 3.(\tfrac{1}{2})^{j-1}.$$

We define

$$v(2^j) \stackrel{\triangle}{=} 3.(\tfrac{1}{2})^{j-1}, \; j = 1, 2, 3, \ldots$$

If $S_0 = 2^j$ for some $j \leq 1$, then the initial price is at or below 2. In this case, we exercise immediately, and the value of the put is

$$v(2^j) \stackrel{\triangle}{=} 5 - 2^j,\ j = 1, 0, -1, -2, \ldots$$

Proposed exercise rule: Exercise the put whenever the stock price is at or below 2. The value of this rule is given by $v(2^j)$ as we just defined it. Since the put is perpetual, the initial time is no different from any other time. This leads us to make the following:

Conjecture 1 *The value of the perpetual put at time k is $v(S_k)$.*

How do we recognize the value of an American derivative security when we see it?

There are three parts to the proof of the conjecture. We must show:

(a) $v(S_k) \geq (5 - S_k)^+ \ \forall k$,

(b) $\left\{(\frac{4}{5})^k v(S_k)\right\}_{k=0}^{\infty}$ is a supermartingale,

(c) $\{v(S_k)\}_{k=0}^{\infty}$ is the smallest process with properties (a) and (b).

Note: To simplify matters, we shall only consider initial stock prices of the form $S_0 = 2^j$, so S_k is always of the form 2^j, with a possibly different j.

Proof: (a). Just check that

$$v(2^j) \stackrel{\triangle}{=} 3.(\tfrac{1}{2})^{j-1} \geq (5 - 2^j)^+ \ \text{ for } \ j \geq 1,$$

$$v(2^j) \stackrel{\triangle}{=} 5 - 2^j \geq (5 - 2^j)^+ \ \text{ for } \ j \leq 1.$$

This is straightforward. ■

Proof: (b). We must show that

$$\begin{aligned} v(S_k) &\geq \widetilde{I\!\!E}\left[\tfrac{4}{5} v(S_{k+1}) | \mathcal{F}_k\right] \\ &= \tfrac{4}{5}.\tfrac{1}{2} v(2S_k) + \tfrac{4}{5}.\tfrac{1}{2} v(\tfrac{1}{2} S_k). \end{aligned}$$

By assumption, $S_k = 2^j$ for some j. We must show that

$$v(2^j) \geq \tfrac{2}{5} v(2^{j+1}) + \tfrac{2}{5} v(2^{j-1}).$$

If $j \geq 2$, then $v(2^j) = 3.(\frac{1}{2})^{j-1}$ and

$$\begin{aligned} &\tfrac{2}{5} v(2^{j+1}) + \tfrac{2}{5} v(2^{j-1}) \\ &= \tfrac{2}{5}.3.(\tfrac{1}{2})^j + \tfrac{2}{5}.3.(\tfrac{1}{2})^{j-2} \\ &= 3.\left[\tfrac{2}{5}.\frac{1}{4} + \tfrac{2}{5}\right](\tfrac{1}{2})^{j-2} \\ &= 3.\tfrac{1}{2}.(\tfrac{1}{2})^{j-2} \\ &= v(2^j). \end{aligned}$$

If $j = 1$, then $v(2^j) = v(2) = 3$ and

$$\begin{aligned} &\tfrac{2}{5}v(2^{j+1}) + \tfrac{2}{5}v(2^{j-1}) \\ &= \tfrac{2}{5}v(4) + \tfrac{2}{5}v(1) \\ &= \tfrac{2}{5}.3.\tfrac{1}{2} + \tfrac{2}{5}.4 \\ &= 3/5 + 8/5 \\ &= 2\tfrac{1}{5} < v(2) = 3 \end{aligned}$$

There is a gap of size $\frac{4}{5}$.
If $j \le 0$, then $v(2^j) = 5 - 2^j$ and

$$\begin{aligned} &\tfrac{2}{5}v(2^{j+1}) + \tfrac{2}{5}v(2^{j-1}) \\ &= \tfrac{2}{5}(5 - 2^{j+1}) + \tfrac{2}{5}(5 - 2^{j-1}) \\ &= 4 - \tfrac{2}{5}(4+1)2^{j-1} \\ &= 4 - 2^j < v(2^j) = 5 - 2^j. \end{aligned}$$

There is a gap of size 1. This concludes the proof of (b). ■

Proof: (c). Suppose $\{Y_k\}_{k=0}^n$ is some other process satisfying:

(a') $Y_k \ge (5 - S_k)^+ \; \forall k$,

(b') $\{(\frac{4}{5})^k Y_k\}_{k=0}^\infty$ is a supermartingale.

We must show that

$$Y_k \ge v(S_k) \; \forall k. \tag{7.1}$$

Actually, since the put is perpetual, every time k is like every other time, so it will suffice to show

$$Y_0 \ge v(S_0), \tag{7.2}$$

provided we let S_0 in (7.2) be any number of the form 2^j. With appropriate (but messy) conditioning on $\mathcal{F}_k$, the proof we give of (7.2) can be modified to prove (7.1).
For $j \le 1$,

$$v(2^j) = 5 - 2^j = (5 - 2^j)^+,$$

so if $S_0 = 2^j$ for some $j \le 1$, then (a') implies

$$Y_0 \ge (5 - 2^j)^+ = v(S_0).$$

Suppose now that $S_0 = 2^j$ for some $j \ge 2$, i.e., $S_0 \ge 4$. Let

$$\begin{aligned} \tau &= \min\{k; S_k = 2\} \\ &= \min\{k; M_k = j - 1\}. \end{aligned}$$

Then

$$\begin{aligned} v(S_0) &= v(2^j) = 3.(\tfrac{1}{2})^{j-1} \\ &= I\!\!E\left[(\tfrac{4}{5})^\tau (5 - S_\tau)^+\right]. \end{aligned}$$

Because $\{(\frac{4}{5})^k Y_k\}_{k=0}^\infty$ is a supermartingale

$$Y_0 \geq I\!\!E\left[(\tfrac{4}{5})^\tau Y_\tau\right] \geq I\!\!E\left[(\tfrac{4}{5})^\tau (5 - S_\tau)^+\right] = v(S_0).$$

■

Comment on the proof of (c): If the candidate value process is the actual value of a particular exercise rule, then (c) will be automatically satisfied. In this case, we constructed v so that $v(S_k)$ is the value of the put at time k if the stock price at time k is S_k and *if we exercise the put the first time (k, or later) that the stock price is 2 or less.* In such a situation, we need only verify properties (a) and (b).

8.8 Difference Equation

If we imagine stock prices which can fall at any point in $(0, \infty)$, not just at points of the form 2^j for integers j, then we can imagine the function $v(x)$, defined for all $x > 0$, which gives the value of the perpetual American put when the stock price is x. This function should satisfy the conditions:

(a) $v(x) \geq (K - x)^+, \ \forall x,$

(b) $v(x) \geq \frac{1}{1+r}\left[\tilde{p}v(ux) + \tilde{q}v(dx)\right], \ \forall x,$

(c) At each x, either (a) or (b) holds with equality.

In the example we worked out, we have

$$\text{For } \ j \geq 1 : v(2^j) = 3.(\tfrac{1}{2})^{j-1} = \frac{6}{2^j};$$

$$\text{For } \ j \leq 1 : v(2^j) = 5 - 2^j.$$

This suggests the formula

$$v(x) = \begin{cases} \frac{6}{x}, & x \geq 3, \\ 5 - x, & 0 < x \leq 3. \end{cases}$$

We then have (see Fig. 8.4):

(a) $v(x) \geq (5 - x)^+; \forall x,$

(b) $v(x) \geq \frac{4}{5}\left[\frac{1}{2}v(2x) + \frac{1}{2}v(\frac{x}{2})\right]$ for every x except for $2 < x < 4$.

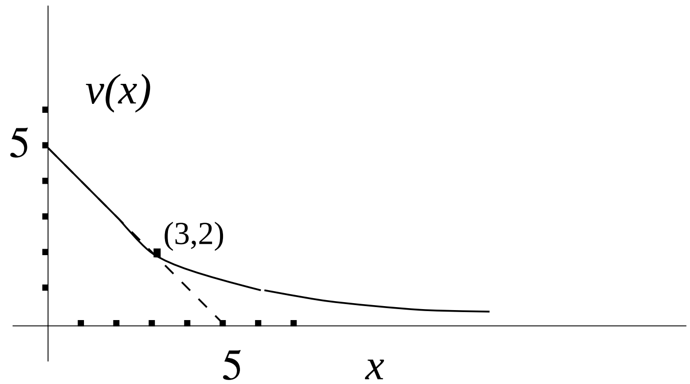

Figure 8.4: *Graph of* $v(x)$.

Check of condition (c):

- If $0 < x \leq 3$, then (a) holds with equality.
- If $x \geq 6$, then (b) holds with equality:

$$\tfrac{4}{5}\left[\tfrac{1}{2}v(2x) + \tfrac{1}{2}v(\frac{x}{2})\right] = \tfrac{4}{5}\left[\tfrac{1}{2}\frac{6}{2x} + \tfrac{1}{2}\frac{12}{x}\right] = \frac{6}{x}.$$

- If $3 < x < 4$ or $4 < x < 6$, then both (a) and (b) are strict. This is an artifact of the discreteness of the binomial model. This artifact will disappear in the continuous model, in which an analogue of (a) or (b) holds with equality at every point.

8.9 Distribution of First Passage Times

Let $\{M_k\}_{k=0}^{\infty}$ be a symetric random walk under a probability measure $I\!\!P$, with $M_0 = 0$. Defining

$$\tau = \min\{k \geq 0; M_k = 1\},$$

we recall that

$$I\!\!E\alpha^{\tau} = \frac{1 - \sqrt{1 - \alpha^2}}{\alpha}, \quad 0 < \alpha < 1.$$

We will use this moment generating function to obtain the distribution of τ. We first obtain the Taylor series expasion of $I\!\!E\alpha^{\tau}$ as follows:

$$\begin{aligned}
f(x) &= 1-\sqrt{1-x}, \quad f(0)=0 \\
f'(x) &= \tfrac{1}{2}(1-x)^{-\frac{1}{2}}, \quad f'(0)=\tfrac{1}{2} \\
f''(x) &= \frac{1}{4}(1-x)^{-\frac{3}{2}}, \quad f''(0)=\frac{1}{4} \\
f'''(x) &= \frac{3}{8}(1-x)^{-\frac{5}{2}}, \quad f'''(0)=\frac{3}{8} \\
&\cdots \\
f^{(j)}(x) &= \frac{1\times 3\times \ldots \times (2j-3)}{2^j}(1-x)^{-\frac{(2j-1)}{2}}, \\
f^{(j)}(0) &= \frac{1\times 3\times \ldots \times (2j-3)}{2^j} \\
&= \frac{1\times 3\times \ldots \times (2j-3)}{2^j}.\frac{2\times 4\times \ldots \times (2j-2)}{2^{j-1}(j-1)!} \\
&= \left(\tfrac{1}{2}\right)^{2j-1}\frac{(2j-2)!}{(j-1)!}
\end{aligned}$$

The Taylor series expansion of $f(x)$ is given by

$$\begin{aligned}
f(x) &= 1-\sqrt{1-x} \\
&= \sum_{j=0}^{\infty}\frac{1}{j!}f^{(j)}(0)x^j \\
&= \sum_{j=1}^{\infty}\left(\tfrac{1}{2}\right)^{2j-1}\frac{(2j-2)!}{j!(j-1)!}x^j \\
&= \frac{x}{2}+\sum_{j=2}^{\infty}\left(\tfrac{1}{2}\right)^{2j-1}\frac{1}{(j-1)}\binom{2j-2}{j}x^j.
\end{aligned}$$

So we have

$$\begin{aligned}
\mathbb{E}\alpha^\tau &= \frac{1-\sqrt{1-\alpha^2}}{\alpha} \\
&= \frac{1}{\alpha}f(\alpha^2) \\
&= \frac{\alpha}{2}+\sum_{j=2}^{\infty}\left(\frac{\alpha}{2}\right)^{2j-1}\frac{1}{(j-1)}\binom{2j-2}{j}.
\end{aligned}$$

But also,

$$\mathbb{E}\alpha^\tau = \sum_{j=1}^{\infty}\alpha^{2j-1}\mathbb{P}\{\tau = 2j-1\}.$$

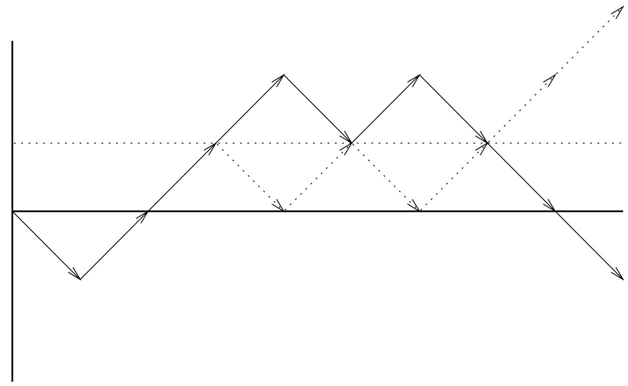

Figure 8.5: *Reflection principle.*

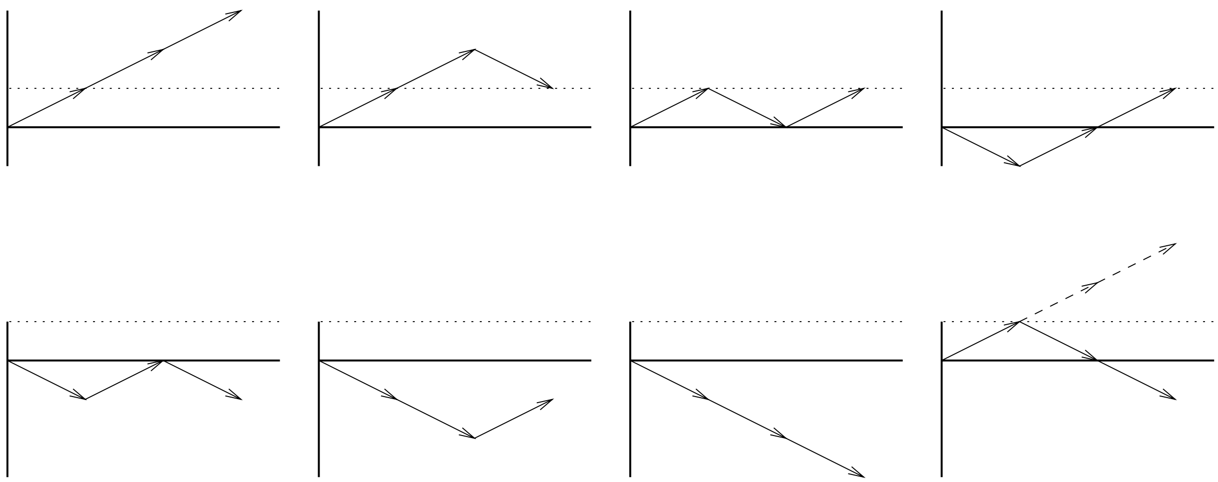

Figure 8.6: *Example with $j = 2$.*

Therefore,

$$\begin{aligned} I\!P\{\tau = 1\} &= \tfrac{1}{2}, \\ I\!P\{\tau = 2j-1\} &= \left(\frac{1}{2}\right)^{2j-1} \frac{1}{(j-1)} \binom{2j-2}{j}, \quad j = 2, 3, \ldots \end{aligned}$$

8.10 The Reflection Principle

To count how many paths reach level 1 by time $2j-1$, count all those for which $M_{2j-1} = 1$ and double count all those for which $M_{2j-1} \geq 3$. (See Figures 8.5, 8.6.)

In other words,

$$\begin{aligned}\mathbb{P}\{\tau \le 2j-1\} &= \mathbb{P}\{M_{2j-1} = 1\} + 2\mathbb{P}\{M_{2j-1} \ge 3\} \\ &= \mathbb{P}\{M_{2j-1} = 1\} + \mathbb{P}\{M_{2j-1} \ge 3\} + \mathbb{P}\{M_{2j-1} \le -3\} \\ &= 1 - \mathbb{P}\{M_{2j-1} = -1\}.\end{aligned}$$

For $j \ge 2$,

$$\begin{aligned}\mathbb{P}\{\tau = 2j-1\} &= \mathbb{P}\{\tau \le 2j-1\} - \mathbb{P}\{\tau \le 2j-3\} \\ &= [1 - \mathbb{P}\{M_{2j-1} = -1\}] - [1 - \mathbb{P}\{M_{2j-3} = -1\}] \\ &= \mathbb{P}\{M_{2j-3} = -1\} - \mathbb{P}\{M_{2j-1} = -1\} \\ &= \left(\tfrac{1}{2}\right)^{2j-3} \frac{(2j-3)!}{(j-1)!(j-2)!} - \left(\tfrac{1}{2}\right)^{2j-1} \frac{(2j-1)!}{j!(j-1)!} \\ &= \left(\tfrac{1}{2}\right)^{2j-1} \frac{(2j-3)!}{j!(j-1)!} [4j(j-1) - (2j-1)(2j-2)] \\ &= \left(\tfrac{1}{2}\right)^{2j-1} \frac{(2j-3)!}{j!(j-1)!} [2j(2j-2) - (2j-1)(2j-2)] \\ &= \left(\tfrac{1}{2}\right)^{2j-1} \frac{(2j-2)!}{j!(j-1)!} \\ &= \left(\tfrac{1}{2}\right)^{2j-1} \frac{1}{(j-1)} \binom{2j-2}{j}.\end{aligned}$$