Chapter 7

Jensen's Inequality

7.1 Jensen's Inequality for Conditional Expectations

Lemma 1.23 *If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and $\mathbb{E}|\varphi(X)| < \infty$, then*

$$\mathbb{E}[\varphi(X)|\mathcal{G}] \geq \varphi(\mathbb{E}[X|\mathcal{G}]).$$

For instance, if $\mathcal{G} = \{\phi, \Omega\}$, $\varphi(x) = x^2$:

$$\mathbb{E}X^2 \geq (\mathbb{E}X)^2.$$

Proof: Since φ is convex we can express it as follows (See Fig. 7.1):

$$\varphi(x) = \max_{\substack{h \leq \varphi \\ h \text{ is linear}}} h(x).$$

Now let $h(x) = ax + b$ lie below φ. Then,

$$\begin{aligned} \mathbb{E}[\varphi(X)|\mathcal{G}] &\geq \mathbb{E}[aX + b|\mathcal{G}] \\ &= a\mathbb{E}[X|\mathcal{G}] + b \\ &= h(\mathbb{E}[X|\mathcal{G}]) \end{aligned}$$

This implies

$$\begin{aligned} \mathbb{E}[\varphi(X)|\mathcal{G}] &\geq \max_{\substack{h \leq \varphi \\ h \text{ is linear}}} h(\mathbb{E}[X|\mathcal{G}]) \\ &= \varphi(\mathbb{E}[X|\mathcal{G}]). \end{aligned}$$

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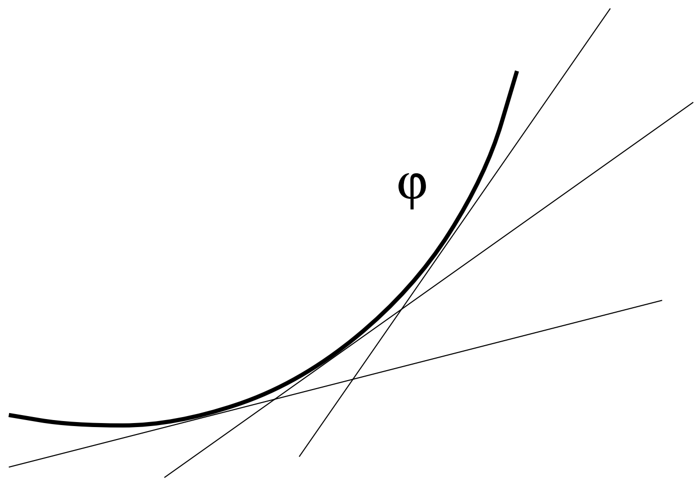

Figure 7.1: *Expressing a convex function as a max over linear functions.*

Theorem 1.24 *If $\{Y_k\}_{k=0}^n$ is a martingale and ϕ is convex then $\{\varphi(Y_k)\}_{k=0}^n$ is a submartingale.*

Proof:

$$\begin{aligned} I\!\!E[\varphi(Y_{k+1})|\mathcal{F}_k] &\geq \varphi(I\!\!E[Y_{k+1}|\mathcal{F}_k]) \\ &= \varphi(Y_k). \end{aligned}$$

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7.2 Optimal Exercise of an American Call

This follows from Jensen's inequality.

Corollary 2.25 *Given a convex function $g : [0,\infty)\rightarrow I\!\!R$ where $g(0) = 0$. For instance, $g(x) = (x-K)^+$ is the payoff function for an American call. Assume that $r \geq 0$. Consider the American derivative security with payoff $g(S_k)$ in period k. The value of this security is the same as the value of the simple European derivative security with final payoff $g(S_n)$, i.e.,*

$$\widetilde{I\!\!E}\left[(1+r)^{-n} g(S_n)\right] = \max_{\tau} \widetilde{I\!\!E}\left[(1+r)^{-\tau} g(S_\tau)\right],$$

where the LHS is the European value and the RHS is the American value. In particular $\tau = n$ is an optimal exercise time.

Proof: Because g is convex, for all $\lambda \in [0,1]$ we have (see Fig. 7.2):

$$\begin{aligned} g(\lambda x) &= g(\lambda x + (1-\lambda).0) \\ &\leq \lambda g(x) + (1-\lambda).g(0) \\ &= \lambda g(x). \end{aligned}$$

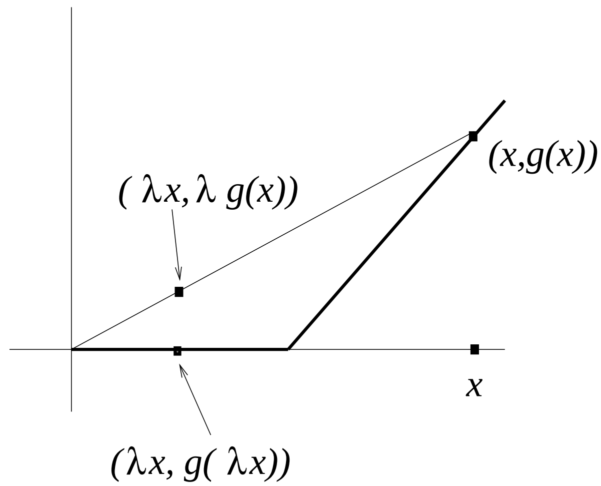

Figure 7.2: *Proof of Cor. 2.25*

Therefore,

$$g\left(\frac{1}{1+r}S_{k+1}\right) \le \frac{1}{1+r}g(S_{k+1})$$

and

$$\begin{aligned}\widetilde{IE}\left[(1+r)^{-(k+1)}g(S_{k+1})|\mathcal{F}_k\right] &= (1+r)^{-k}\widetilde{IE}\left[\frac{1}{1+r}g(S_{k+1})|\mathcal{F}_k\right] \\ &\ge (1+r)^{-k}\widetilde{IE}\left[g\left(\frac{1}{1+r}S_{k+1}\right)|\mathcal{F}_k\right] \\ &\ge (1+r)^{-k}g\left(\widetilde{IE}\left[\frac{1}{1+r}S_{k+1}|\mathcal{F}_k\right]\right) \\ &= (1+r)^{-k}g(S_k),\end{aligned}$$

So $\{(1+r)^{-k}g(S_k)\}_{k=0}^{n}$ is a submartingale. Let τ be a stopping time satisfying $0 \le \tau \le n$. The optional sampling theorem implies

$$(1+r)^{-\tau}g(S_\tau) \le \widetilde{IE}\left[(1+r)^{-n}g(S_n)|\mathcal{F}_\tau\right].$$

Taking expectations, we obtain

$$\begin{aligned}\widetilde{IE}\left[(1+r)^{-\tau}g(S_\tau)\right] &\le \widetilde{IE}\left(\widetilde{IE}\left[(1+r)^{-n}g(S_n)|\mathcal{F}_\tau\right]\right) \\ &= \widetilde{IE}\left[(1+r)^{-n}g(S_n)\right].\end{aligned}$$

Therefore, the value of the American derivative security is

$$\max_\tau \widetilde{IE}\left[(1+r)^{-\tau}g(S_\tau)\right] \le \widetilde{IE}\left[(1+r)^{-n}g(S_n)\right],$$

and this last expression is the value of the European derivative security. Of course, the LHS cannot be strictly less than the RHS above, since stopping at time n is always allowed, and we conclude that

$$\max_\tau \widetilde{IE}\left[(1+r)^{-\tau}g(S_\tau)\right] = \widetilde{IE}\left[(1+r)^{-n}g(S_n)\right].$$

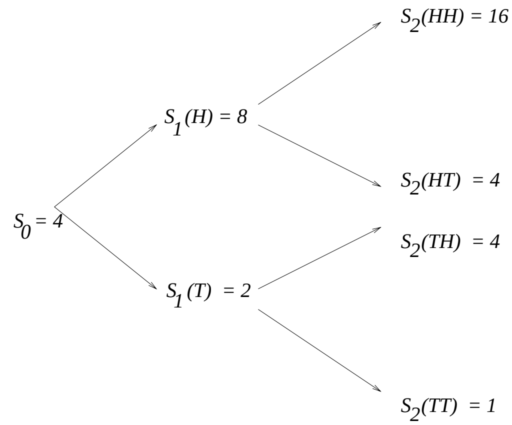

Figure 7.3: *A three period binomial model.*

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7.3 Stopped Martingales

Let $\{Y_k\}_{k=0}^n$ be a stochastic process and let τ be a stopping time. We denote by $\{Y_{k\wedge\tau}\}_{k=0}^n$ the *stopped process*

$$Y_{k\wedge\tau(\omega)}(\omega),\ k=0,1,\ldots,n.$$

Example 7.1 (Stopped Process) Figure 7.3 shows our familiar 3-period binomial example.

Define

$$\tau(\omega)=\begin{cases}1 & \text{if } \omega_1=T,\\ 2 & \text{if } \omega_1=H.\end{cases}$$

Then

$$S_{2\wedge\tau(\omega)}(\omega)=\begin{cases}S_2(HH)=16 & \text{if } \omega=HH,\\ S_2(HT)=4 & \text{if } \omega=HT,\\ S_1(T)=2 & \text{if } \omega=TH,\\ S_1(T)=2 & \text{if } \omega=TT.\end{cases}$$

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Theorem 3.26 *A stopped martingale (or submartingale, or supermartingale) is still a martingale (or submartingale, or supermartingale respectively).*

Proof: Let $\{Y_k\}_{k=0}^n$ be a martingale, and τ be a stopping time. Choose some $k\in\{0,1,\ldots,n\}$. The set $\{\tau\le k\}$ is in $\mathcal{F}_k$, so the set $\{\tau\ge k+1\}=\{\tau\le k\}^c$ is also in $\mathcal{F}_k$. We compute

$$\begin{aligned}I\!E\left[Y_{(k+1)\wedge\tau}|\mathcal{F}_k\right] &= I\!E\left[I_{\{\tau\le k\}}Y_\tau+I_{\{\tau\ge k+1\}}Y_{k+1}|\mathcal{F}_k\right]\\ &= I_{\{\tau\le k\}}Y_\tau+I_{\{\tau\ge k+1\}}I\!E[Y_{k+1}|\mathcal{F}_k]\\ &= I_{\{\tau\le k\}}Y_\tau+I_{\{\tau\ge k+1\}}Y_k\\ &= Y_{k\wedge\tau}.\end{aligned}$$