Chapter 7

Jensen’s Inequality

7.1 Jensen’s Inequality for Conditional Expectations

Lemma 1.23 If \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is convex and \( \mathbb{E}[|\varphi(X)|] < \infty \), then

\[
\mathbb{E}[\varphi(X) | G] \geq \varphi(\mathbb{E}[X | G]).
\]

For instance, if \( G = \{ \phi, \Omega \}, \varphi(x) = x^2 \):

\[
\mathbb{E}X^2 \geq (\mathbb{E}X)^2.
\]

Proof: Since \( \varphi \) is convex we can express it as follows (See Fig. 7.1):

\[
\varphi(x) = \max_{h \in \omega} h(x).
\]

Now let \( h(x) = ax + b \) lie below \( \varphi \). Then,

\[
\begin{align*}
\mathbb{E}[\varphi(X) | G] & \geq \mathbb{E}[aX + b | G] \\
& = a \mathbb{E}[X | G] + b \\
& = h(\mathbb{E}[X | G])
\end{align*}
\]

This implies

\[
\begin{align*}
\mathbb{E}[\varphi(X) | G] & \geq \max_{h \in \omega} h(\mathbb{E}[X | G]) \\
& = \varphi(\mathbb{E}[X | G]).
\end{align*}
\]
Theorem 1.24 If \( \{Y_k\}_{k=0}^n \) is a martingale and \( \phi \) is convex then \( \{\phi(Y_k)\}_{k=0}^n \) is a submartingale.

Proof:

\[
\mathbb{E}[\phi(Y_{k+1}) | \mathcal{F}_k] \geq \phi(\mathbb{E}[Y_{k+1} | \mathcal{F}_k]) = \phi(Y_k).
\]

\[
\blacksquare
\]

7.2 Optimal Exercise of an American Call

This follows from Jensen’s inequality.

Corollary 2.25 Given a convex function \( g : [0, \infty) \to \mathbb{R} \) where \( g(0) = 0 \). For instance, \( g(x) = (x - K)^+ \) is the payoff function for an American call. Assume that \( r \geq 0 \). Consider the American derivative security with payoff \( g(S_k) \) in period \( k \). The value of this security is the same as the value of the simple European derivative security with final payoff \( g(S_n) \), i.e.,

\[
\mathbb{E}[(1 + r)^{-n} g(S_n)] = \max_\tau \mathbb{E}[(1 + r)^{-\tau} g(S_{\tau})],
\]

where the LHS is the European value and the RHS is the American value. In particular \( \tau = n \) is an optimal exercise time.

Proof: Because \( g \) is convex, for all \( \lambda \in [0, 1] \) we have (see Fig. 7.2):

\[
g(\lambda x) = g(\lambda x + (1 - \lambda)0) \\
\leq \lambda g(x) + (1 - \lambda)g(0) \\
= \lambda g(x).
\]
Therefore, \[ g \left( \frac{1}{1 + r} S_{k+1} \right) \leq \frac{1}{1 + r} g(S_{k+1}) \]
and
\[
\mathbb{E}^\tau \left[ (1 + r)^{-k} g(S_{k+1}) \bigg| \mathcal{F}_k \right] 
= (1 + r)^{-k} \mathbb{E} \left[ \frac{1}{1 + r} g(S_{k+1}) \bigg| \mathcal{F}_k \right] 
\geq (1 + r)^{-k} \mathbb{E} \left[ g \left( \frac{1}{1 + r} S_{k+1} \right) \bigg| \mathcal{F}_k \right] 
\geq (1 + r)^{-k} g \left( \mathbb{E} \left[ \frac{1}{1 + r} S_{k+1} \bigg| \mathcal{F}_k \right] \right) 
= (1 + r)^{-k} g(S_k),
\]
So \( \{(1 + r)^{-k} g(S_k)\}_{k=0}^n \) is a submartingale. Let \( \tau \) be a stopping time satisfying \( 0 \leq \tau \leq n \). The optional sampling theorem implies
\[
(1 + r)^{-\tau} g(S_\tau) \leq \mathbb{E} \left[ (1 + r)^{-n} g(S_n) \bigg| \mathcal{F}_\tau \right].
\]
Taking expectations, we obtain
\[
\mathbb{E} \left[ (1 + r)^{-\tau} g(S_\tau) \right] \leq \mathbb{E} \left( \mathbb{E} \left[ (1 + r)^{-n} g(S_n) \bigg| \mathcal{F}_\tau \right] \right) 
= \mathbb{E} \left[ (1 + r)^{-n} g(S_n) \right].
\]
Therefore, the value of the American derivative security is
\[
\max_\tau \mathbb{E} \left[ (1 + r)^{-\tau} g(S_\tau) \right] \leq \mathbb{E} \left[ (1 + r)^{-n} g(S_n) \right],
\]
and this last expression is the value of the European derivative security. Of course, the LHS cannot be strictly less than the RHS above, since stopping at time \( n \) is always allowed, and we conclude that
\[
\max_\tau \mathbb{E} \left[ (1 + r)^{-\tau} g(S_\tau) \right] = \mathbb{E} \left[ (1 + r)^{-n} g(S_n) \right].
\]
Let \( \{Y_k\}_{k=0}^n \) be a stochastic process and let \( \tau \) be a stopping time. We denote by \( \{Y_{k\wedge \tau}\}_{k=0}^n \) the stopped process

\[
Y_{k\wedge \tau}(\omega), \quad k = 0, 1, \ldots, n.
\]

**Example 7.1 (Stopped Process)**  Figure 7.3 shows our familiar 3-period binomial example.

Define

\[
\tau(\omega) = \begin{cases} 
1 & \text{if } \omega_1 = T, \\
2 & \text{if } \omega_1 = H.
\end{cases}
\]

Then

\[
S_{2\wedge \tau(\omega)}(\omega) = \begin{cases} 
S_2(HH) = 16 & \text{if } \omega = HH, \\
S_2(HT) = 4 & \text{if } \omega = HT, \\
S_1(T) = 2 & \text{if } \omega = TH, \\
S_1(T) = 2 & \text{if } \omega = TT.
\end{cases}
\]

**Theorem 3.26** A stopped martingale (or submartingale, or supermartingale) is still a martingale (or submartingale, or supermartingale respectively).

**Proof:** Let \( \{Y_k\}_{k=0}^n \) be a martingale, and \( \tau \) be a stopping time. Choose some \( k \in \{0, 1, \ldots, n\} \). The set \( \{\tau \leq k\} \) is in \( \mathcal{F}_k \), so the set \( \{\tau \geq k + 1\} = \{\tau \leq k\}^c \) is also in \( \mathcal{F}_k \). We compute

\[
\mathbb{E} \left[ Y_{(k+1)\wedge \tau} | \mathcal{F}_k \right] = \mathbb{E} \left[ I_{\{\tau \leq k\}} Y_\tau + I_{\{\tau \geq k+1\}} Y_{k+1} | \mathcal{F}_k \right] = I_{\{\tau \leq k\}} Y_\tau + I_{\{\tau \geq k+1\}} \mathbb{E}[Y_{k+1} | \mathcal{F}_k] = I_{\{\tau \leq k\}} Y_\tau + I_{\{\tau \geq k+1\}} Y_k = Y_{k\wedge \tau}.
\]