Chapter 6

Properties of American Derivative Securities

6.1 The properties

Definition 6.1 An *American derivative security* is a sequence of non-negative random variables $\{G_k\}_{k=0}^n$ such that each G_k is $\mathcal{F}_k$-measurable. The owner of an American derivative security can exercise at any time k, and if he does, he receives the payment G_k.

(a) The value V_k of the security at time k is

$$V_k = \max_{\tau} (1+r)^k \widetilde{I\!\!E}[(1+r)^{-\tau} G_\tau | \mathcal{F}_k],$$

where the maximum is over all stopping times τ satisfying $\tau \geq k$ almost surely.

(b) The discounted value process $\{(1+r)^{-k} V_k\}_{k=0}^n$ is the smallest supermartingale which satisfies

$$V_k \geq G_k, \forall k, \quad \text{almost surely.}$$

(c) Any stopping time τ which satisfies

$$V_0 = \widetilde{I\!\!E}[(1+r)^{-\tau} G_\tau]$$

is an optimal exercise time. In particular

$$\tau \stackrel{\triangle}{=} \min\{k; V_k = G_k\}$$

is an optimal exercise time.

(d) The hedging portfolio is given by

$$\Delta_k(\omega_1, \ldots, \omega_k) = \frac{V_{k+1}(\omega_1, \ldots, \omega_k, H) - V_{k+1}(\omega_1, \ldots, \omega_k, T)}{S_{k+1}(\omega_1, \ldots, \omega_k, H) - S_{k+1}(\omega_1, \ldots, \omega_k, T)}, k = 0, 1, \ldots, n-1.$$

(e) Suppose for some k and ω, we have $V_k(\omega) = G_k(\omega)$. Then the owner of the derivative security should exercise it. If he does not, then the seller of the security can immediately consume

$$V_k(\omega) - \frac{1}{1+r}\widetilde{I\!E}[V_{k+1}|\mathcal{F}_k](\omega)$$

and still maintain the hedge.

6.2 Proofs of the Properties

Let $\{G_k\}_{k=0}^n$ be a sequence of non-negative random variables such that each G_k is $\mathcal{F}_k$-measurable. Define T_k to be the set of all stopping times τ satisfying $k \le \tau \le n$ almost surely. Define also

$$V_k \stackrel{\triangle}{=} (1+r)^k \max_{\tau \in T_k} \widetilde{I\!E}\left[(1+r)^{-\tau} G_\tau | \mathcal{F}_k\right].$$

Lemma 2.18 $V_k \ge G_k$ *for every* k.

Proof: Take $\tau \in T_k$ to be the constant k. ■

Lemma 2.19 *The process* $\{(1+r)^{-k} V_k\}_{k=0}^n$ *is a supermartingale.*

Proof: Let τ^* attain the maximum in the definition of V_{k+1}, i.e.,

$$(1+r)^{-(k+1)} V_{k+1} = \widetilde{I\!E}\left[(1+r)^{-\tau^*} G_{\tau^*} | \mathcal{F}_{k+1}\right].$$

Because τ^* is also in T_k, we have

$$\begin{aligned}
\widetilde{I\!E}[(1+r)^{-(k+1)} V_{k+1}|\mathcal{F}_k] &= \widetilde{I\!E}\left[\widetilde{I\!E}[(1+r)^{-\tau^*} G_{\tau^*}|\mathcal{F}_{k+1}]|\mathcal{F}_k\right] \\
&= \widetilde{I\!E}[(1+r)^{-\tau^*} G_{\tau^*}|\mathcal{F}_k] \\
&\le \max_{\tau \in T_k} \widetilde{I\!E}\left[(1+r)^{-\tau} G_\tau|\mathcal{F}_k\right] \\
&= (1+r)^{-k} V_k.
\end{aligned}$$

■

Lemma 2.20 *If* $\{Y_k\}_{k=0}^n$ *is another process satisfying*

$$Y_k \ge G_k, k = 0, 1, \ldots, n, \text{ a.s.},$$

and $\{(1+r)^{-k} Y_k\}_{k=0}^n$ *is a supermartingale, then*

$$Y_k \ge V_k, k = 0, 1, \ldots, n, \text{ a.s.}$$

Proof: The optional sampling theorem for the supermartingale $\{(1+r)^{-k}Y_k\}_{k=0}^n$ implies

$$\widetilde{I\!E}[(1+r)^{-\tau}Y_\tau|\mathcal{F}_k] \le (1+r)^{-k}Y_k, \forall \tau \in T_k.$$

Therefore,

$$\begin{aligned} V_k &= (1+r)^k \max_{\tau \in T_k} \widetilde{I\!E}[(1+r)^{-\tau}G_\tau|\mathcal{F}_k] \\ &\le (1+r)^k \max_{\tau \in T_k} \widetilde{I\!E}[(1+r)^{-\tau}Y_\tau|\mathcal{F}_k] \\ &\le (1+r)^{-k}(1+r)^k Y_k \\ &= Y_k. \end{aligned}$$

■

Lemma 2.21 *Define*

$$\begin{aligned} C_k &= V_k - \frac{1}{1+r}\widetilde{I\!E}[V_{k+1}|\mathcal{F}_k] \\ &= (1+r)^k \left\{ (1+r)^{-k}V_k - \widetilde{I\!E}[(1+r)^{-(k+1)}V_{k+1}|\mathcal{F}_k] \right\}. \end{aligned}$$

Since $\{(1+r)^{-k}V_k\}_{k=0}^n$ is a supermartingale, C_k must be non-negative almost surely. Define

$$\Delta_k(\omega_1, \ldots, \omega_k) = \frac{V_{k+1}(\omega_1, \ldots, \omega_k, H) - V_{k+1}(\omega_1, \ldots, \omega_k, T)}{S_{k+1}(\omega_1, \ldots, \omega_k, H) - S_{k+1}(\omega_1, \ldots, \omega_k, T)}.$$

Set $X_0 = V_0$ and define recursively

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - C_k - \Delta_k S_k).$$

Then

$$X_k = V_k \;\; \forall k.$$

Proof: We proceed by induction on k. The induction hypothesis is that $X_k = V_k$ for some $k \in \{0, 1, \ldots, n-1\}$, i.e., for each fixed $(\omega_1, \ldots, \omega_k)$ we have

$$X_k(\omega_1, \ldots, \omega_k) = V_k(\omega_1, \ldots, \omega_k).$$

We need to show that

$$X_{k+1}(\omega_1, \ldots, \omega_k, H) = V_{k+1}(\omega_1, \ldots, \omega_k, H),$$

$$X_{k+1}(\omega_1, \ldots, \omega_k, T) = V_{k+1}(\omega_1, \ldots, \omega_k, T).$$

We prove the first equality; the proof of the second is similar. Note first that

$$\begin{aligned} &V_k(\omega_1, \ldots, \omega_k) - C_k(\omega_1, \ldots, \omega_k) \\ &= \frac{1}{1+r}\widetilde{I\!E}[V_{k+1}|\mathcal{F}_k](\omega_1, \ldots, \omega_k) \\ &= \frac{1}{1+r}\left(\tilde{p}V_{k+1}(\omega_1, \ldots, \omega_k, H) + \tilde{q}V_{k+1}(\omega_1, \ldots, \omega_k, T)\right). \end{aligned}$$

Since $(\omega_1, \ldots, \omega_k)$ will be fixed for the rest of the proof, we will suppress these symbols. For example, the last equation can be written simply as

$$V_k - C_k = \frac{1}{1+r}\left(\tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T)\right).$$

We compute

$$\begin{aligned}
X_{k+1}(H) &= \Delta_k S_{k+1}(H) + (1+r)(X_k - C_k - \Delta_k S_k) \\
&= \frac{V_{k+1}(H) - V_{k+1}(T)}{S_{k+1}(H) - S_{k+1}(T)}\left(S_{k+1}(H) - (1+r)S_k\right) \\
&\quad +(1+r)(V_k - C_k) \\
&= \frac{V_{k+1}(H) - V_{k+1}(T)}{(u-d)S_k}\left(uS_k - (1+r)S_k\right) \\
&\quad +\tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T) \\
&= (V_{k+1}(H) - V_{k+1}(T))\tilde{q} + \tilde{p}V_{k+1}(H) + \tilde{q}V_{k+1}(T) \\
&= V_{k+1}(H).
\end{aligned}$$

6.3 Compound European Derivative Securities

In order to derive the optimal stopping time for an American derivative security, it will be useful to study compound European derivative securities, which are also interesting in their own right.

A compound European derivative security consists of $n+1$ different simple European derivative securities (with the same underlying stock) expiring at times $0, 1, \ldots, n$; the security that expires at time j has payoff C_j. Thus a compound European derivative security is specified by the process $\{C_j\}_{j=0}^n$, where each C_j is $\mathcal{F}_j$-measurable, i.e., the process $\{C_j\}_{j=0}^n$ is adapted to the filtration $\{\mathcal{F}_k\}_{k=0}^n$.

Hedging a short position (one payment). Here is how we can hedge a short position in the j'th European derivative security. The value of European derivative security j at time k is given by

$$V_k^{(j)} = (1+r)^k \widetilde{I\!E}[(1+r)^{-j}C_j|\mathcal{F}_k],\ k = 0, \ldots, j,$$

and the hedging portfolio for that security is given by

$$\Delta_k^{(j)}(\omega_1, \ldots, \omega_k) = \frac{V_{k+1}^{(j)}(\omega_1, \ldots, \omega_k, H) - V_{k+1}^{(j)}(\omega_1, \ldots, \omega_k, T)}{S_{k+1}^{(j)}(\omega_1, \ldots, \omega_k, H) - S_{k+1}^{(j)}(\omega_1, \ldots, \omega_k, T)}, k = 0, \ldots, j-1.$$

Thus, starting with wealth $V_0^{(j)}$, and using the portfolio $(\Delta_0^{(j)}, \ldots, \Delta_{j-1}^{(j)})$, we can ensure that at time j we have wealth C_j.

Hedging a short position (all payments). Superpose the hedges for the individual payments. In other words, start with wealth $V_0 = \sum_{j=0}^n V_0^{(j)}$. At each time $k \in \{0, 1, \ldots, n-1\}$, first make the payment C_k and then use the portfolio

$$\Delta_k = \Delta_k{}^{(k+1)} + \Delta_k{}^{(k+2)} + \ldots + \Delta_k{}^{(n)}$$

corresponding to all future payments. At the final time n, after making the final payment C_n, we will have exactly zero wealth.

Suppose you own a compound European derivative security$\{C_j\}_{j=0}^n$. Compute

$$V_0 = \sum_{j=0}^{n} V_0^{(j)} = \widetilde{I\!E}\left[\sum_{j=0}^{n}(1+r)^{-j}C_j\right]$$

and the hedging portfolio is $\{\Delta_k\}_{k=0}^{n-1}$. You can borrow V_0 and consume it immediately. This leaves you with wealth $X_0 = -V_0$. In each period k, *receive* the payment C_k and then use the portfolio $-\Delta_k$. At the final time n, after receiving the last payment C_n, your wealth will reach zero, i.e., you will no longer have a debt.

6.4 Optimal Exercise of American Derivative Security

In this section we derive the optimal exercise time for the owner of an American derivative security. Let $\{G_k\}_{k=0}^n$ be an American derivative security. Let τ be the stopping time the owner plans to use. (We assume that each G_k is non-negative, so we may assume without loss of generality that the owner stops at expiration – time n– if not before). Using the stopping time τ, in period j the owner will receive the payment

$$C_j = I_{\{\tau=j\}}G_j.$$

In other words, once he chooses a stopping time, the owner has effectively converted the American derivative security into a compound European derivative security, whose value is

$$\begin{aligned}
V_0^{(\tau)} &= \widetilde{I\!E}\left[\sum_{j=0}^{n}(1+r)^{-j}C_j\right]\\
&= \widetilde{I\!E}\left[\sum_{j=0}^{n}(1+r)^{-j}I_{\{\tau=j\}}G_j\right]\\
&= \widetilde{I\!E}[(1+r)^{-\tau}G_\tau].
\end{aligned}$$

The owner of the American derivative security can borrow this amount of money immediately, if he chooses, and invest in the market so as to exaclty pay off his debt as the payments $\{C_j\}_{j=0}^n$ are received. Thus, his optimal behavior is to use a stopping time τ which maximizes $V_0^{(\tau)}$.

Lemma 4.22 $V_0^{(\tau)}$ *is maximized by the stopping time*

$$\tau^* = \min\{k; V_k = G_k\}.$$

Proof: Recall the definition

$$V_0 \stackrel{\triangle}{=} \max_{\tau\in T_0}\widetilde{I\!E}\left[(1+r)^{-\tau}G_\tau\right] = \max_{\tau\in T_0} V_0^{(\tau)}$$

Let τ' be a stopping time which maximizes $V_0^{(\tau)}$, i.e., $V_0 = \widetilde{I\!E}\left[(1+r)^{-\tau'}G_{\tau'}\right]$. Because $\{(1+r)^{-k}V_k\}_{k=0}^n$ is a supermartingale, we have from the optional sampling theorem and the inequality $V_k \geq G_k$, the following:

$$\begin{aligned} V_0 &\geq \widetilde{I\!E}\left[(1+r)^{-\tau'}V_{\tau'}|\mathcal{F}_0\right] \\ &= \widetilde{I\!E}\left[(1+r)^{-\tau'}V_{\tau'}\right] \\ &\geq \widetilde{I\!E}\left[(1+r)^{-\tau'}G_{\tau'}\right] \\ &= V_0. \end{aligned}$$

Therefore,

$$V_0 = \widetilde{I\!E}\left[(1+r)^{-\tau'}V_{\tau'}\right] = \widetilde{I\!E}\left[(1+r)^{-\tau'}G_{\tau'}\right],$$

and

$$V_{\tau'} = G_{\tau'}, \quad \text{a.s.}$$

We have just shown that if τ' attains the maximum in the formula

$$V_0 = \max_{\tau \in T_0} \widetilde{I\!E}\left[(1+r)^{-\tau}G_\tau\right], \tag{4.1}$$

then

$$V_{\tau'} = G_{\tau'}, \quad \text{a.s.}$$

But we have defined

$$\tau^* = \min\{k; V_k = G_k\},$$

and so we must have $\tau^* \leq \tau' \leq n$ almost surely. The optional sampling theorem implies

$$\begin{aligned} (1+r)^{-\tau^*}G_{\tau^*} &= (1+r)^{-\tau^*}V_{\tau^*} \\ &\geq \widetilde{I\!E}\left[(1+r)^{-\tau'}V_{\tau'}|\mathcal{F}_{\tau^*}\right] \\ &= \widetilde{I\!E}\left[(1+r)^{-\tau'}G_{\tau'}|\mathcal{F}_{\tau^*}\right]. \end{aligned}$$

Taking expectations on both sides, we obtain

$$\widetilde{I\!E}\left[(1+r)^{-\tau^*}G_{\tau^*}\right] \geq \widetilde{I\!E}\left[(1+r)^{-\tau'}G_{\tau'}\right] = V_0.$$

It follows that τ^* also attains the maximum in (4.1), and is therefore an optimal exercise time for the American derivative security. ■