Chapter 5

Stopping Times and American Options

5.1 American Pricing

Let us first review the **European pricing formula in a Markov model**. Consider the Binomial model with n periods. Let $V_n = g(S_n)$ be the payoff of a derivative security. Define by backward recursion:

$$\begin{aligned} v_n(x) &= g(x) \\ v_k(x) &= \frac{1}{1+r}[\tilde{p}v_{k+1}(ux) + \tilde{q}v_{k+1}(dx)]. \end{aligned}$$

Then $v_k(S_k)$ is the value of the option at time k, and the hedging portfolio is given by

$$\Delta_k = \frac{v_{k+1}(uS_k) - v_{k+1}(dS_k)}{(u-d)S_k}, \quad k = 0, 1, 2, \ldots, n-1.$$

Now consider an American option. Again a function g is specified. In any period k, the holder of the derivative security can "exercise" and receive payment $g(S_k)$. Thus, the hedging portfolio should create a wealth process which satisfies

$$X_k \geq g(S_k), \forall k, \text{ almost surely.}$$

This is because the value of the derivative security at time k is at least $g(S_k)$, and the wealth process value at that time must equal the value of the derivative security.

American algorithm.

$$\begin{aligned} v_n(x) &= g(x) \\ v_k(x) &= \max\left\{\frac{1}{1+r}(\tilde{p}v_{k+1}(ux) + \tilde{q}v_{k+1}(dx)),\ g(x)\right\} \end{aligned}$$

Then $v_k(S_k)$ is the value of the option at time k.

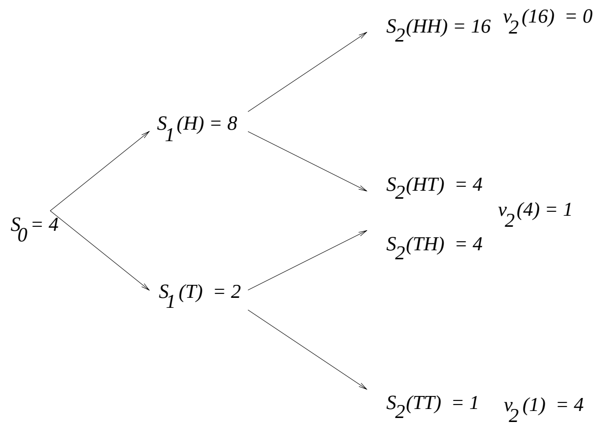

Figure 5.1: *Stock price and final value of an American put option with strike price 5.*

Example 5.1 See Fig. 5.1. $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, \tilde{p} = \tilde{q} = \frac{1}{2}, n = 2$. Set $v_2(x) = g(x) = (5 - x)^+$. Then

$$
\begin{aligned}
v_1(8) &= \max\left\{\frac{4}{5}\left[\tfrac{1}{2}.0 + \tfrac{1}{2}.1\right], (5-8)^+\right\} \\
&= \max\left\{\frac{2}{5}, 0\right\} \\
&= 0.40 \\
v_1(2) &= \max\left\{\frac{4}{5}\left[\tfrac{1}{2}.1 + \tfrac{1}{2}.4\right], (5-2)^+\right\} \\
&= \max\{2, 3\} \\
&= 3.00 \\
v_0(4) &= \max\left\{\frac{4}{5}\left[\tfrac{1}{2}.(0.4) + \tfrac{1}{2}.(3.0)\right], (5-4)^+\right\} \\
&= \max\{1.36, 1\} \\
&= 1.36
\end{aligned}
$$

Let us now construct the hedging portfolio for this option. Begin with initial wealth $X_0 = 1.36$. Compute Δ_0 as follows:

$$
\begin{aligned}
0.40 &= v_1(S_1(H)) \\
&= S_1(H)\Delta_0 + (1+r)(X_0 - \Delta_0 S_0) \\
&= 8\Delta_0 + \frac{5}{4}(1.36 - 4\Delta_0) \\
&= 3\Delta_0 + 1.70 \Longrightarrow \Delta_0 = -0.43 \\
3.00 &= v_1(S_1(T)) \\
&= S_1(T)\Delta_0 + (1+r)(X_0 - \Delta_0 S_0) \\
&= 2\Delta_0 + \frac{5}{4}(1.36 - 4\Delta_0) \\
&= -3\Delta_0 + 1.70 \Longrightarrow \Delta_0 = -0.43
\end{aligned}
$$

Using $\Delta_0 = -0.43$ results in

$$X_1(H) = v_1(S_1(H)) = 0.40, \quad X_1(T) = v_1(S_1(T)) = 3.00$$

Now let us compute Δ_1 (Recall that $S_1(T) = 2$):

$$\begin{aligned}
1 &= v_2(4) \\
&= S_2(TH)\Delta_1(T) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) \\
&= 4\Delta_1(T) + \frac{5}{4}(3 - 2\Delta_1(T)) \\
&= 1.5\Delta_1(T) + 3.75 \Longrightarrow \Delta_1(T) = -1.83 \\
4 &= v_2(1) \\
&= S_2(TT)\Delta_1(T) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) \\
&= \Delta_1(T) + \frac{5}{4}(3 - 2\Delta_1(T)) \\
&= -1.5\Delta_1(T) + 3.75 \Longrightarrow \Delta_1(T) = -0.16
\end{aligned}$$

We get different answers for $\Delta_1(T)$! If we had $X_1(T) = 2$, the value of the *European* put, we would have

$$1 = 1.5\Delta_1(T) + 2.5 \Longrightarrow \Delta_1(T) = -1,$$

$$4 = -1.5\Delta_1(T) + 2.5 \Longrightarrow \Delta_1(T) = -1,$$

■

5.2 Value of Portfolio Hedging an American Option

$$\begin{aligned}
X_{k+1} &= \Delta_k S_{k+1} + (1+r)(X_k - C_k - \Delta_k S_k) \\
&= (1+r)X_k + \Delta_k(S_{k+1} - (1+r)S_k) - (1+r)C_k
\end{aligned}$$

Here, C_k is the amount "consumed" at time k.

- The discounted value of the portfolio is a *supermartingale*.
- The value satisfies $X_k \geq g(S_k)$, $k = 0, 1, \ldots, n$.
- The value process is the smallest process with these properties.

When do you consume? If

$$\widetilde{I\!E}((1+r)^{-(k+1)} v_{k+1}(S_{k+1}) | \mathcal{F}_k] < (1+r)^{-k} v_k(S_k),$$

or, equivalently,

$$\widetilde{I\!E}(\frac{1}{1+r} v_{k+1}(S_{k+1}) | \mathcal{F}_k] < v_k(S_k)$$

and the holder of the American option does not exercise, then the seller of the option can consume to close the gap. By doing this, he can ensure that $X_k = v_k(S_k)$ for all k, where v_k is the value defined by the American algorithm in Section 5.1.

In the previous example, $v_1(S_1(T)) = 3$, $v_2(S_2(TH)) = 1$ and $v_2(S_2(TT)) = 4$. Therefore,

$$\begin{aligned}\widetilde{I\!\!E}[\frac{1}{1+r}v_2(S_2)|\mathcal{F}_1](T) &= \frac{4}{5}\left[\tfrac{1}{2}.1 + \tfrac{1}{2}.4\right]\\ &= \frac{4}{5}\left[\frac{5}{2}\right]\\ &= 2,\\ v_1(S_1(T)) &= 3,\end{aligned}$$

so there is a gap of size 1. If the owner of the option does not exercise it at time one in the state $\omega_1 = T$, then the seller can consume 1 at time 1. Thereafter, he uses the usual hedging portfolio

$$\Delta_k = \frac{v_{k+1}(uS_k) - v_{k+1}(dS_k)}{(u-d)S_k}$$

In the example, we have $v_1(S_1(T)) = g(S_1(T))$. It is optimal for the owner of the American option to exercise whenever its value $v_k(S_k)$ agrees with its intrinsic value $g(S_k)$.

Definition 5.1 (Stopping Time) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{\mathcal{F}_k\}_{k=0}^n$ be a filtration. A *stopping time* is a random variable $\tau : \Omega \rightarrow \{0, 1, 2, \ldots, n\} \cup \{\infty\}$ with the property that:

$$\{\omega \in \Omega; \tau(\omega) = k\} \in \mathcal{F}_k, \ \forall k = 0, 1, \ldots, n, \infty.$$

Example 5.2 Consider the binomial model with $n = 2, S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}$, so $\tilde{p} = \tilde{q} = \frac{1}{2}$. Let v_0, v_1, v_2 be the value functions defined for the American put with strike price 5. Define

$$\tau(\omega) = \min\{k; v_k(S_k) = (5 - S_k)^+\}.$$

The stopping time τ corresponds to "stopping the first time the value of the option agrees with its intrinsic value". It is an optimal exercise time. We note that

$$\tau(\omega) = \begin{cases} 1 & \text{if } \omega \in A_T \\ 2 & \text{if } \omega \in A_H \end{cases}$$

We verify that τ is indeed a stopping time:

$$\begin{aligned}\{\omega; \tau(\omega) = 0\} &= \phi \in \mathcal{F}_0\\ \{\omega; \tau(\omega) = 1\} &= A_T \in \mathcal{F}_1\\ \{\omega; \tau(\omega) = 2\} &= A_H \in \mathcal{F}_2\end{aligned}$$

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Example 5.3 (A random time which is not a stopping time) In the same binomial model as in the previous example, define

$$\rho(\omega) = \min\{k; S_k(\omega) = m_2(\omega)\},$$

where $m_2 \stackrel{\triangle}{=} \min_{0 \le j \le 2} S_j$. In other words, ρ stops when the stock price reaches its minimum value. This random variable is given by

$$\rho(\omega) = \begin{cases} 0 & \text{if } \omega \in A_H, \\ 1 & \text{if } \omega = TH, \\ 2 & \text{if } \omega = TT \end{cases}$$

We verify that ρ is *not* a stopping time:

$$\begin{aligned} \{\omega; \rho(\omega) = 0\} &= A_H \notin \mathcal{F}_0 \\ \{\omega; \rho(\omega) = 1\} &= \{TH\} \notin \mathcal{F}_1 \\ \{\omega; \rho(\omega) = 2\} &= \{TT\} \in \mathcal{F}_2 \end{aligned}$$

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5.3 Information up to a Stopping Time

Definition 5.2 Let τ be a stopping time. We say that a set $A \subset \Omega$ is *determined by time* τ provided that

$$A \cap \{\omega; \tau(\omega) = k\} \in \mathcal{F}_k, \forall k.$$

The collection of sets determined by τ is a σ-algebra, which we denote by $\mathcal{F}_\tau$.

Example 5.4 In the binomial model considered earlier, let

$$\tau = \min\{k; v_k(S_k) = (5 - S_k)^+\},$$

i.e.,

$$\tau(\omega) = \begin{cases} 1 & \text{if } \omega \in A_T \\ 2 & \text{if } \omega \in A_H \end{cases}$$

The set $\{HT\}$ is determined by time τ, but the set $\{TH\}$ is not. Indeed,

$$\begin{aligned} \{HT\} \cap \{\omega; \tau(\omega) = 0\} &= \phi \in \mathcal{F}_0 \\ \{HT\} \cap \{\omega; \tau(\omega) = 1\} &= \phi \in \mathcal{F}_1 \\ \{HT\} \cap \{\omega; \tau(\omega) = 2\} &= \{HT\} \in \mathcal{F}_2 \end{aligned}$$

but

$$\{TH\} \cap \{\omega; \tau(\omega) = 1\} = \{TH\} \notin \mathcal{F}_1.$$

The atoms of $\mathcal{F}_\tau$ are

$$\{HT\},\ \{HH\},\ A_T = \{TH, TT\}.$$

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Notation 5.1 (Value of Stochastic Process at a Stopping Time) If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\{\mathcal{F}_k\}_{k=0}^n$ is a filtration under $\mathcal{F}$, $\{X_k\}_{k=0}^n$ is a stochastic process adapted to this filtration, and τ is a stopping time with respect to the same filtration, then X_τ is an $\mathcal{F}_\tau$-measurable random variable whose value at ω is given by

$$X_\tau(\omega) \stackrel{\triangle}{=} X_{\tau(\omega)}(\omega).$$

Theorem 3.17 (Optional Sampling) *Suppose that $\{Y_k, \mathcal{F}_k\}_{k=0}^{\infty}$ (or $\{Y_k, \mathcal{F}_k\}_{k=0}^{n}$) is a submartingale. Let τ and ρ be* bounded *stopping times, i.e., there is a nonrandom number n such that*

$$\tau \leq n, \ \rho \leq n, \quad \textit{almost surely.}$$

If $\tau \leq \rho$ almost surely, then

$$Y_\tau \leq I\!\!E(Y_\rho|\mathcal{F}_\tau).$$

Taking expectations, we obtain $I\!\!EY_\tau \leq I\!\!EY_\rho$, and in particular, $Y_0 = I\!\!EY_0 \leq I\!\!EY_\rho$. If $\{Y_k, \mathcal{F}_k\}_{k=0}^{\infty}$ is a supermartingale, then $\tau \leq \rho$ implies $Y_\tau \geq I\!\!E(Y_\rho|\mathcal{F}_\tau)$.
If $\{Y_k, \mathcal{F}_k\}_{k=0}^{\infty}$ is a martingale, then $\tau \leq \rho$ implies $Y_\tau = I\!\!E(Y_\rho|\mathcal{F}_\tau)$.

Example 5.5 In the example 5.4 considered earlier, we define $\rho(\omega) = 2$ for all $\omega \in \Omega$. Under the risk-neutral probability measure, the discounted stock price process $(\frac{5}{4})^{-k} S_k$ is a martingale. We compute

$$\widetilde{I\!\!E}\left[\left(\frac{4}{5}\right)^2 S_2 \bigg| \mathcal{F}_\tau\right].$$

The atoms of $\mathcal{F}_\tau$ are $\{HH\}, \{HT\}$, and A_T. Therefore,

$$\begin{aligned}
\widetilde{I\!\!E}\left[\left(\frac{4}{5}\right)^2 S_2 \bigg| \mathcal{F}_\tau\right](HH) &= \left(\frac{4}{5}\right)^2 S_2(HH), \\
\widetilde{I\!\!E}\left[\left(\frac{4}{5}\right)^2 S_2 \bigg| \mathcal{F}_\tau\right](HT) &= \left(\frac{4}{5}\right)^2 S_2(HT),
\end{aligned}$$

and for $\omega \in A_T$,

$$\begin{aligned}
\widetilde{I\!\!E}\left[\left(\frac{4}{5}\right)^2 S_2 \bigg| \mathcal{F}_\tau\right](\omega) &= \tfrac{1}{2}\left(\frac{4}{5}\right)^2 S_2(TH) + \tfrac{1}{2}\left(\frac{4}{5}\right)^2 S_2(TT) \\
&= \tfrac{1}{2} \times 2.56 + \tfrac{1}{2} \times 0.64 \\
&= 1.60
\end{aligned}$$

In every case we have gotten (see Fig. 5.2)

$$\widetilde{I\!\!E}\left[\left(\frac{4}{5}\right)^2 S_2 \bigg| \mathcal{F}_\tau\right](\omega) = \left(\frac{4}{5}\right)^{\tau(\omega)} S_{\tau(\omega)}(\omega).$$

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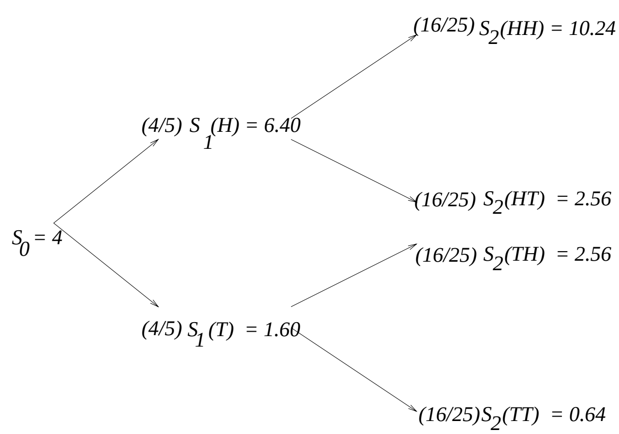

Figure 5.2: *Illustrating the optional sampling theorem.*