Chapter 4

The Markov Property

4.1 Binomial Model Pricing and Hedging

Recall that V_m is the given simple European derivative security, and the value and portfolio processes are given by:

$$V_k = (1+r)^k \widetilde{\mathbb{E}}[(1+r)^{-m} V_m | \mathcal{F}_k], \quad k = 0, 1, \ldots, m-1.$$

$$\Delta_k(\omega_1, \ldots, \omega_k) = \frac{V_{k+1}(\omega_1, \ldots, \omega_k, H) - V_{k+1}(\omega_1, \ldots, \omega_k, T)}{S_{k+1}(\omega_1, \ldots, \omega_k, H) - S_{k+1}(\omega_1, \ldots, \omega_k, T)}, \quad k = 0, 1, \ldots, m-1.$$

Example 4.1 (Lookback Option) $u = 2, d = 0.5, r = 0.25, S_0 = 4, \tilde{p} = \frac{1+r-d}{u-d} = 0.5, \tilde{q} = 1 - \tilde{p} = 0.5.$ Consider a simple European derivative security with expiration 2, with payoff given by (See Fig. 4.1):

$$V_2 = \max_{0 \le k \le 2} (S_k - 5)^+.$$

Notice that

$$V_2(HH) = 11, \ \ V_2(HT) = 3 \neq V_2(TH) = 0, \ \ V_2(TT) = 0.$$

The payoff is thus "path dependent". Working backward in time, we have:

$$V_1(H) = \frac{1}{1+r}[\tilde{p}V_2(HH) + \tilde{q}V_2(HT)] = \frac{4}{5}[0.5 \times 11 + 0.5 \times 3] = 5.60,$$

$$V_1(T) = \frac{4}{5}[0.5 \times 0 + 0.5 \times 0] = 0,$$

$$V_0 = \frac{4}{5}[0.5 \times 5.60 + 0.5 \times 0] = 2.24.$$

Using these values, we can now compute:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = 0.93,$$

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = 0.67,$$

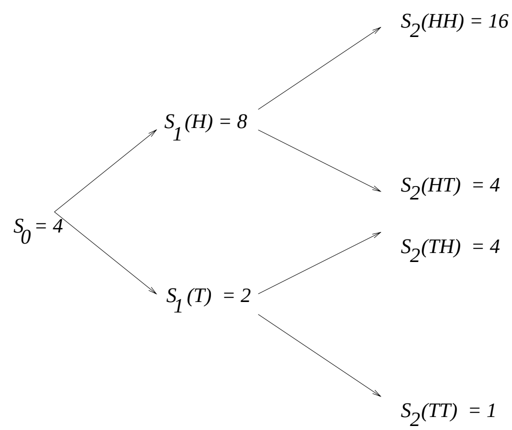

Figure 4.1: *Stock price underlying the lookback option.*

$$\Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = 0.$$

Working forward in time, we can check that

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = 5.59;\;\; V_1(H) = 5.60,$$

$$X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = 0.01;\;\; V_1(T) = 0,$$

$$X_1(HH) = \Delta_1(H) S_1(HH) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)) = 11.01;\;\; V_1(HH) = 11,$$

etc.

■

Example 4.2 (European Call) Let $u = 2, d = \frac{1}{2}, r = \frac{1}{4}, S_0 = 4, \tilde{p} = \tilde{q} = \frac{1}{2}$, and consider a European call with expiration time 2 and payoff function

$$V_2 = (S_2 - 5)^+.$$

Note that

$$V_2(HH) = 11,\; V_2(HT) = V_2(TH) = 0,\; V_2(TT) = 0,$$

$$V_1(H) = \frac{4}{5}[\tfrac{1}{2}.11 + \tfrac{1}{2}.0] = 4.40$$

$$V_1(T) = \frac{4}{5}[\tfrac{1}{2}.0 + \tfrac{1}{2}.0] = 0$$

$$V_0 = \frac{4}{5}[\tfrac{1}{2} \times 4.40 + \tfrac{1}{2} \times 0] = 1.76.$$

Define $v_k(x)$ to be the value of the call at time k when $S_k = x$. Then

$$v_2(x) = (x-5)^+$$

$$v_1(x) = \frac{4}{5}[\tfrac{1}{2} v_2(2x) + \tfrac{1}{2} v_2(x/2)],$$

$$v_0(x) = \frac{4}{5}[\tfrac{1}{2} v_1(2x) + \tfrac{1}{2} v_1(x/2)].$$

In particular,

$$v_2(16) = 11,\ v_2(4) = 0,\ v_2(1) = 0,$$

$$v_1(8) = \frac{4}{5}[\tfrac{1}{2}.11 + \tfrac{1}{2}.0] = 4.40,$$

$$v_1(2) = \frac{4}{5}[\tfrac{1}{2}.0 + \tfrac{1}{2}.0] = 0,$$

$$v_0 = \frac{4}{5}[\tfrac{1}{2} \times 4.40 + \tfrac{1}{2} \times 0] = 1.76.$$

Let $\delta_k(x)$ be the number of shares in the hedging portfolio at time k when $S_k = x$. Then

$$\delta_k(x) = \frac{v_{k+1}(2x) - v_{k+1}(x/2)}{2x - x/2}, \quad k = 0, 1.$$

■

4.2 Computational Issues

For a model with n periods (coin tosses), Ω has 2^n elements. For period k, we must solve 2^k equations of the form

$$V_k(\omega_1, \ldots, \omega_k) = \frac{1}{1+r}[\tilde{p}V_{k+1}(\omega_1, \ldots, \omega_k, H) + \tilde{q}V_{k+1}(\omega_1, \ldots, \omega_k, T)].$$

For example, a three-month option has 66 trading days. If each day is taken to be one period, then $n = 66$ and $2^{66} \sim 7 \times 10^{19}$.

There are three possible ways to deal with this problem:

1. Simulation. We have, for example, that

   $$V_0 = (1+r)^{-n}\widetilde{I\!E}V_n,$$

   and so we could compute V_0 by simulation. More specifically, we could simulate n coin tosses $\omega = (\omega_1, \ldots, \omega_n)$ under the risk-neutral probability measure. We could store the value of $V_n(\omega)$. We could repeat this several times and take the average value of V_n as an approximation to $\widetilde{I\!E}V_n$.

2. Approximate a many-period model by a continuous-time model. Then we can use calculus and partial differential equations. We'll get to that.

3. Look for Markov structure. Example 4.2 has this. In period 2, the option in Example 4.2 has three possible values $v_2(16), v_2(4), v_2(1)$, rather than four possible values $V_2(HH), V_2(HT), V_2(TH), V_2(TT)$. If there were 66 periods, then in period 66 there would be 67 possible stock price values (since the final price depends only on the *number* of up-ticks of the stock price – i.e., heads – so far) and hence only 67 possible option values, rather than $2^{66} \sim 7 \times 10^{19}$.

4.3 Markov Processes

Technical condition always present: We consider only functions on R and subsets of R which are Borel-measurable, i.e., we only consider subsets A of R that are in $\mathcal{B}$ and functions $g : I\!\!R \to I\!\!R$ such that g^{-1} is a function $\mathcal{B} \to \mathcal{B}$.

Definition 4.1 () Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{\mathcal{F}_k\}_{k=0}^n$ be a filtration under $\mathcal{F}$. Let $\{X_k\}_{k=0}^n$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. This process is said to be *Markov* if:

- The stochastic process $\{X_k\}$ is adapted to the filtration $\{\mathcal{F}_k\}$, and
- *(The Markov Property).* For each $k = 0, 1, \ldots, n-1$, the distribution of X_{k+1} conditioned on $\mathcal{F}_k$ is the same as the distribution of X_{k+1} conditioned on X_k.

4.3.1 Different ways to write the Markov property

(a) (Agreement of distributions). For every $A \in \mathcal{B} \stackrel{\triangle}{=} \mathcal{B}(I\!\!R)$, we have

$$\begin{aligned} I\!\!P(X_{k+1} \in A | \mathcal{F}_k) &= I\!\!E[I_A(X_{k+1}) | \mathcal{F}_k] \\ &= I\!\!E[I_A(X_{k+1}) | X_k] \\ &= I\!\!P[X_{k+1} \in A | X_k]. \end{aligned}$$

(b) (Agreement of expectations of all functions). For every (Borel-measurable) function $h : I\!\!R \to I\!\!R$ for which $I\!\!E|h(X_{k+1})| < \infty$, we have

$$I\!\!E[h(X_{k+1}) | \mathcal{F}_k] = I\!\!E[h(X_{k+1}) | X_k].$$

(c) (Agreement of Laplace transforms.) For every $u \in I\!\!R$ for which $I\!\!E e^{uX_{k+1}} < \infty$, we have

$$I\!\!E\left[e^{uX_{k+1}} \middle| \mathcal{F}_k \right] = I\!\!E\left[e^{uX_{k+1}} \middle| X_k \right].$$

(If we fix u and define $h(x) = e^{ux}$, then the equations in (b) and (c) are the same. However in (b) we have a condition which holds for *every* function h, and in (c) we assume this condition only for functions h of the form $h(x) = e^{ux}$. A main result in the theory of Laplace transforms is that if the equation holds for every h of this special form, then it holds for every h, i.e., (c) implies (b).)

(d) (Agreement of characteristic functions) For every $u \in I\!\!R$, we have

$$I\!\!E\left[e^{iuX_{k+1}} | \mathcal{F}_k \right] = I\!\!E\left[e^{iuX_{k+1}} | X_k \right],$$

where $i = \sqrt{-1}$. (Since $|e^{iux}| = |\cos x + \sin x| \le 1$ we don't need to assume that $I\!\!E|e^{iux}| < \infty$.)

Remark 4.1 In every case of the Markov properties where $\mathbb{E}[\dots|X_k]$ appears, we could just as well write $g(X_k)$ for some function g. For example, form (a) of the Markov property can be restated as:

> For every $A \in \mathcal{B}$, we have
>
> $$\mathbb{P}(X_{k+1} \in A|\mathcal{F}_k) = g(X_k),$$
>
> where g is a function that depends on the set A.

Conditions (a)-(d) are equivalent. The Markov property as stated in (a)-(d) involves the process at a "current" time k and one future time $k+1$. Conditions (a)-(d) are also equivalent to conditions involving the process at time k and multiple future times. We write these apparently stronger but actually equivalent conditions below.

Consequences of the Markov property. Let j be a positive integer.

(A) For every $A_{k+1} \subset \mathbb{R}, \dots, A_{k+j} \subset \mathbb{R}$,

$$\mathbb{P}[X_{k+1} \in A_{k+1}, \dots, X_{k+j} \in A_{k+j}|\mathcal{F}_k] = \mathbb{P}[X_{k+1} \in A_{k+1}, \dots, X_{k+j} \in A_{k+j}|X_k].$$

(A') For every $A \in \mathbb{R}^j$,

$$\mathbb{P}[(X_{k+1}, \dots, X_{k+j}) \in A|\mathcal{F}_k] = \mathbb{P}[(X_{k+1}, \dots, X_{k+j}) \in A|X_k].$$

(B) For every function $h : \mathbb{R}^j \to \mathbb{R}$ for which $\mathbb{E}|h(X_{k+1}, \dots, X_{k+j})| < \infty$, we have

$$\mathbb{E}[h(X_{k+1}, \dots, X_{k+j})|\mathcal{F}_k] = \mathbb{E}[h(X_{k+1}, \dots, X_{k+j})|X_k].$$

(C) For every $u = (u_{k+1}, \dots, u_{k+j}) \in \mathbb{R}^j$ for which $\mathbb{E}|e^{u_{k+1}X_{k+1}+\dots+u_{k+j}X_{k+j}}| < \infty$, we have

$$\mathbb{E}[e^{u_{k+1}X_{k+1}+\dots+u_{k+j}X_{k+j}}|\mathcal{F}_k] = \mathbb{E}[e^{u_{k+1}X_{k+1}+\dots+u_{k+j}X_{k+j}}|X_k].$$

(D) For every $u = (u_{k+1}, \dots, u_{k+j}) \in \mathbb{R}^j$ we have

$$\mathbb{E}[e^{i(u_{k+1}X_{k+1}+\dots+u_{k+j}X_{k+j})}|\mathcal{F}_k] = \mathbb{E}[e^{i(u_{k+1}X_{k+1}+\dots+u_{k+j}X_{k+j})}|X_k].$$

Once again, every expression of the form $\mathbb{E}(\dots|X_k)$ can also be written as $g(X_k)$, where the function g depends on the random variable represented by $\dots$ in this expression.

Remark. All these Markov properties have analogues for vector-valued processes.

Proof that (b) $\Longrightarrow$ (A). (with $j = 2$ in (A)) Assume (b). Then (a) also holds (take $h = I_A$). Consider

$$
\begin{aligned}
&I\!\!P[X_{k+1} \in A_{k+1}, X_{k+2} \in A_{k+2} | \mathcal{F}_k] \\
&= I\!\!E[I_{A_{k+1}}(X_{k+1}) I_{A_{k+2}}(X_{k+2}) | \mathcal{F}_k] \\
&\qquad \text{(Definition of conditional probability)} \\
&= I\!\!E[I\!\!E[I_{A_{k+1}}(X_{k+1}) I_{A_{k+2}}(X_{k+2}) | \mathcal{F}_{k+1}] | \mathcal{F}_k] \\
&\qquad \text{(Tower property)} \\
&= I\!\!E[I_{A_{k+1}}(X_{k+1}) . I\!\!E[I_{A_{k+2}}(X_{k+2}) | \mathcal{F}_{k+1}] | \mathcal{F}_k] \\
&\qquad \text{(Taking out what is known)} \\
&= I\!\!E[I_{A_{k+1}}(X_{k+1}) . I\!\!E[I_{A_{k+2}}(X_{k+2}) | X_{k+1}] | \mathcal{F}_k] \\
&\qquad \text{(Markov property, form (a).)} \\
&= I\!\!E[I_{A_{k+1}}(X_{k+1}) . g(X_{k+1}) | \mathcal{F}_k] \\
&\qquad \text{(Remark 4.1)} \\
&= I\!\!E[I_{A_{k+1}}(X_{k+1}) . g(X_{k+1}) | X_k] \\
&\qquad \text{(Markov property, form (b).)}
\end{aligned}
$$

Now take conditional expectation on both sides of the above equation, conditioned on $\sigma(X_k)$, and use the tower property on the left, to obtain

$$I\!\!P[X_{k+1} \in A_{k+1}, X_{k+2} \in A_{k+2} | X_k] = I\!\!E[I_{A_{k+1}}(X_{k+1}) . g(X_{k+1}) | X_k]. \tag{3.1}$$

Since both

$$I\!\!P[X_{k+1} \in A_{k+1}, X_{k+2} \in A_{k+2} | \mathcal{F}_k]$$

and

$$I\!\!P[X_{k+1} \in A_{k+1}, X_{k+2} \in A_{k+2} | X_k]$$

are equal to the RHS of (3.1)), they are equal to each other, and this is property (A) with $j = 2$. ■

Example 4.3 It is intuitively clear that the stock price process in the binomial model is a Markov process. We will formally prove this later. If we want to estimate the distribution of S_{k+1} based on the information in $\mathcal{F}_k$, the only relevant piece of information is the value of S_k. For example,

$$\widetilde{I\!\!E}[S_{k+1} | \mathcal{F}_k] = (\tilde{p}u + \tilde{q}d) S_k = (1+r) S_k \tag{3.2}$$

is a function of S_k. Note however that form (b) of the Markov property is stronger then (3.2); the Markov property requires that for *any* function h,

$$\widetilde{I\!\!E}[h(S_{k+1}) | \mathcal{F}_k]$$

is a function of S_k. Equation (3.2) is the case of $h(x) = x$.

Consider a model with 66 periods and a simple European derivative security whose payoff at time 66 is

$$V_{66} = \frac{1}{3}(S_{64} + S_{65} + S_{66}).$$

The value of this security at time 50 is

$$\begin{aligned} V_{50} &= (1+r)^{50}\widetilde{I\!E}[(1+r)^{-66}V_{66}|\mathcal{F}_{50}] \\ &= (1+r)^{-16}\widetilde{I\!E}[V_{66}|S_{50}], \end{aligned}$$

because the stock price process is Markov. (We are using form (B) of the Markov property here). In other words, the F_{50}-measurable random variable V_{50} can be written as

$$V_{50}(\omega_1,\ldots,\omega_{50}) = g(S_{50}(\omega_1,\ldots,\omega_{50}))$$

for some function g, which we can determine with a bit of work. ■

4.4 Showing that a process is Markov

Definition 4.2 (Independence) Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-algebras of $\mathcal{F}$. We say that $\mathcal{G}$ and $\mathcal{H}$ are *independent* if for every $A \in \mathcal{G}$ and $B \in \mathcal{H}$, we have

$$I\!P(A\cap B) = I\!P(A)I\!P(B).$$

We say that a random variable X is independent of a σ-algebra $\mathcal{G}$ if $\sigma(X)$, the σ-algebra generated by X, is independent of $\mathcal{G}$.

Example 4.4 Consider the two-period binomial model. Recall that $\mathcal{F}_1$ is the σ-algebra of sets determined by the first toss, i.e., $\mathcal{F}_1$ contains the four sets

$$A_H \stackrel{\triangle}{=} \{HH,HT\},\ A_T \stackrel{\triangle}{=} \{TH,TT\},\ \phi,\ \Omega.$$

Let $\mathcal{H}$ be the σ-algebra of sets determined by the second toss, i.e., $\mathcal{H}$ contains the four sets

$$\{HH,TH\},\{HT,TT\},\phi,\Omega.$$

Then $\mathcal{F}_1$ and $\mathcal{H}$ are independent. For example, if we take $A=\{HH,HT\}$ from $\mathcal{F}_1$ and $B=\{HH,TH\}$ from $\mathcal{H}$, then $I\!P(A\cap B) = I\!P(HH) = p^2$ and

$$I\!P(A)I\!P(B) = (p^2+pq)(p^2+pq) = p^2(p+q)^2 = p^2.$$

Note that $\mathcal{F}_1$ and S_2 are not independent (unless $p=1$ or $p=0$). For example, one of the sets in $\sigma(S_2)$ is $\{\omega; S_2(\omega) = u^2S_0\} = \{HH\}$. If we take $A=\{HH,HT\}$ from $\mathcal{F}_1$ and $B=\{HH\}$ from $\sigma(S_2)$, then $I\!P(A\cap B) = I\!P(HH) = p^2$, but

$$I\!P(A)I\!P(B) = (p^2+pq)p^2 = p^3(p+q) = p^3.$$

■

The following lemma will be very useful in showing that a process is Markov:

Lemma 4.15 (Independence Lemma) *Let X and Y be random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Assume*

- *X is independent of $\mathcal{G}$;*
- *Y is $\mathcal{G}$-measurable.*

Let $f(x,y)$ be a function of two variables, and define

$$g(y) \stackrel{\triangle}{=} \mathbb{E}f(X,y).$$

Then

$$\mathbb{E}[f(X,Y)|\mathcal{G}] = g(Y).$$

Remark. In this lemma and the following discussion, capital letters denote random variables and lower case letters denote nonrandom variables.

Example 4.5 (Showing the stock price process is Markov) Consider an n-period binomial model. Fix a time k and define $X \stackrel{\triangle}{=} \frac{S_{k+1}}{S_k}$ and $\mathcal{G} \stackrel{\triangle}{=} \mathcal{F}_k$. Then $X = u$ if $\omega_{k+1} = H$ and $X = d$ if $\omega_{k+1} = T$. Since X depends only on the $(k+1)$st toss, X is independent of $\mathcal{G}$. Define $Y \stackrel{\triangle}{=} S_k$, so that Y is $\mathcal{G}$-measurable. Let h be any function and set $f(x,y) \stackrel{\triangle}{=} h(xy)$. Then

$$g(y) \stackrel{\triangle}{=} \mathbb{E}f(X,y) = \mathbb{E}h(Xy) = ph(uy) + qh(dy).$$

The Independence Lemma asserts that

$$\begin{aligned}
\mathbb{E}[h(S_{k+1})|\mathcal{F}_k] &= \mathbb{E}[h\left(\frac{S_{k+1}}{S_k}.S_k\right)|\mathcal{F}_k] \\
&= \mathbb{E}[f(X,Y)|\mathcal{G}] \\
&= g(Y) \\
&= ph(uS_k) + qh(dS_k).
\end{aligned}$$

This shows the stock price is Markov. Indeed, if we condition both sides of the above equation on $\sigma(S_k)$ and use the tower property on the left and the fact that the right hand side is $\sigma(S_k)$-measurable, we obtain

$$\mathbb{E}[h(S_{k+1})|S_k] = ph(uS_k) + qh(dS_k).$$

Thus $\mathbb{E}[h(S_{k+1})|\mathcal{F}_k]$ and $\mathbb{E}[h(S_{k+1})|X_k]$ are equal and form (b) of the Markov property is proved.

Not only have we shown that the stock price process is Markov, but we have also obtained a formula for $\mathbb{E}[h(S_{k+1})|\mathcal{F}_k]$ as a function of S_k. This is a special case of Remark 4.1.

■

4.5 Application to Exotic Options

Consider an n-period binomial model. Define the *running maximum* of the stock price to be

$$M_k \stackrel{\triangle}{=} \max_{1 \le j \le k} S_j.$$

Consider a simple European derivative security with payoff at time n of $v_n(S_n, M_n)$.

Examples:

- $v_n(S_n, M_n) = (M_n - K)^+$ (Lookback option);
- $v_n(S_n, M_n) = I_{M_n \geq B}(S_n - K)^+$ (Knock-in Barrier option).

Lemma 5.16 *The two-dimensional process* $\{(S_k, M_k)\}_{k=0}^n$ *is Markov. (Here we are working under the risk-neutral measure P, although that does not matter).*

Proof: Fix k. We have

$$M_{k+1} = M_k \vee S_{k+1},$$

where $\vee$ indicates the maximum of two quantities. Let $Z \triangleq \frac{S_{k+1}}{S_k}$, so

$$\widetilde{I\!P}(Z = u) = \tilde{p}, \quad \widetilde{I\!P}(Z = d) = \tilde{q},$$

and Z is independent of $\mathcal{F}_k$. Let $h(x, y)$ be a function of two variables. We have

$$\begin{aligned} h(S_{k+1}, M_{k+1}) &= h(S_{k+1}, M_k \vee S_{k+1}) \\ &= h(ZS_k, M_k \vee (ZS_k)). \end{aligned}$$

Define

$$\begin{aligned} g(x, y) &\triangleq \widetilde{I\!E} h(Zx, y \vee (Zx)) \\ &= \tilde{p} h(ux, y \vee (ux)) + \tilde{q} h(dx, y \vee (dx)). \end{aligned}$$

The Independence Lemma implies

$$\widetilde{I\!E}[h(S_{k+1}, M_{k+1}) | \mathcal{F}_k] = g(S_k, M_k) = \tilde{p} h(uS_k, M_k \vee (uS_k)) + \tilde{q} h(dS_k, M_k),$$

the second equality being a consequence of the fact that $M_k \wedge dS_k = M_k$. Since the RHS is a function of (S_k, M_k), we have proved the Markov property (form (b)) for this two-dimensional process. ■

Continuing with the exotic option of the previous Lemma... Let V_k denote the value of the derivative security at time k. Since $(1+r)^{-k} V_k$ is a martingale under $\widetilde{I\!P}$, we have

$$V_k = \frac{1}{1+r} \widetilde{I\!E}[V_{k+1} | \mathcal{F}_k], k = 0, 1, \ldots, n-1.$$

At the final time, we have

$$V_n = v_n(S_n, M_n).$$

Stepping back one step, we can compute

$$\begin{aligned} V_{n-1} &= \frac{1}{1+r} \widetilde{I\!E}[v_n(S_n, M_n) | \mathcal{F}_{n-1}] \\ &= \frac{1}{1+r} \left[\tilde{p} v_n(uS_{n-1}, uS_{n-1} \vee M_{n-1}) + \tilde{q} v_n(dS_{n-1}, M_{n-1}) \right]. \end{aligned}$$

This leads us to define

$$v_{n-1}(x,y) \stackrel{\triangle}{=} \frac{1}{1+r}\left[\tilde{p}v_n(ux, ux \vee y) + \tilde{q}v_n(dx, y)\right]$$

so that

$$V_{n-1} = v_{n-1}(S_{n-1}, M_{n-1}).$$

The general algorithm is

$$v_k(x,y) = \frac{1}{1+r}\Big[\tilde{p}v_{k+1}(ux, ux \vee y) + \tilde{q}v_{k+1}(dx, y)\Big],$$

and the value of the option at time k is $v_k(S_k, M_k)$. Since this is a simple European option, the hedging portfolio is given by the usual formula, which in this case is

$$\Delta_k = \frac{v_{k+1}(uS_k, (uS_k) \vee M_k) - v_{k+1}(dS_k, M_k)}{(u-d)S_k}$$