Chapter 35

Notes and References

35.1 Probability theory and martingales.

Probability theory is usually learned in two stages. In the first stage, one learns that a discrete random variable has a probability mass function and a continuous random variable has a density. These can be used to compute expectations and variances, and even conditional expectations. Furthermore, one learns how transformations of continuous random variables cause changes in their densities. A well-written book which contains all these things is DeGroot (1986).

The second stage of probability theory is measure theoretic. In this stage one views a random variable as a function from a sample space Ω to the set of real numbers $\mathbb{R}$. Certain subsets of Ω are called *events*, and the collection of all events forms a σ-algebra $\mathcal{F}$. Each set A in $\mathcal{F}$ has a probability $\mathbb{P}(A)$. This point of view handles both discrete and continuous random variables within the same unifying framework. A conditional expectation is itself a random variable, measurable with respect to the conditioning σ-algebra. This point of view is indispensible for treating the rather complicated conditional expectations which arise in martingale theory. A well-written book on measure-theoretic probability is Billingsley (1986). A succinct book on measure-theoretic probability and martingales in discrete time is Williams (1991). A more detailed book is Chung (1968).

The measure-theoretic view of probability theory was begun by Kolmogorov (1933). The term *martingale* was apparently first used by Ville (1939), although the concept dates back to 1934 work of Lévy. The first complete account of martingale theory is Doob (1953).

35.2 Binomial asset pricing model.

The binomial asset pricing model was developed by Cox, Ross & Rubinstein (1979). Accounts of this model can be found in several places, including Cox & Rubinstein (1985), Dothan (1990) and Ritchken (1987). Many models are first developed and understood in continuous time, and then binomial versions are developed for purposes of implementation.

35.3 Brownian motion.

In 1828 Robert Brown observed irregular movement of pollen suspended in water. This motion is now known to be caused by the buffeting of the pollen by water molecules, as explained by Einstein (1905). Bachelier (1900) used Brownian motion (not geometric Brownian motion) as a model of stock prices, even though Brownian motion can take negative values. Lévy (1939, 1948) discovered many of the nonintuitive properties of Brownian motion. The first mathematically rigorous construction of Brownian motion was carried out by Wiener (1923, 1924).

Brownian motion and its properties are presented in a numerous texts, including Billingsley (1986). The development in this course is a summary of that found in Karatzas & Shreve (1991).

35.4 Stochastic integrals.

The integral with respect to Brownian motion was developed by Itô (1944). It was introduced to finance by Merton (1969). A mathematical construction of this integral, with a minimum of fuss, is given by Øksendal (1995).

The quadratic variation of martingales was introduced by Fisk (1966) and developed into the form used in this course by Kunita & Watanabe (1967).

35.5 Stochastic calculus and financial markets.

Stochastic calculus begins with Itô (1944). Many finance books, including (in order of increasing mathematical difficulty) Hull (1993), Dothan (1990) and Duffie (1992), include sections on Itô's integral and formula. Some other books on dynamic models in finance are Cox & Rubinstein (1985), Huang & Litzenberger (1988), Ingersoll (1987), and Jarrow (1988). An excellent reference for practitioners, now in preprint form, is Musiela & Rutkowski (1996). Some mathematics texts on stochastic calculus are Øksendal (1995), Chung & Williams (1983), Protter (1990) and Karatzas & Shreve (1991).

Samuelson (1965, 1973) presents the argument that geometric Brownian motion is a good model for stock prices. This is often confused with the *efficient market hypothesis*, which asserts that all information which can be learned from technical analysis of stock prices is already reflected in those prices. According to this hypothesis, past stock prices may be useful to estimate the parameters of the distribution of future returns, but they do not provide information which permits an investor to outperform the market. The mathematical formulation of the efficient market hypothesis is that there is a probability measure under which all discounted stock prices are martingales, a much weaker condition than the claim that stock prices follow a geometric Brownian motion. Some empirical studies supporting the efficient market hypothesis are Kendall (1953), Osborne (1959), Sprenkle (1961), Boness (1964), Alexander (1961) and Fama (1965). The last of these papers discusses other distributions which fit stock prices better than geometric Brownian motion. A criticism of the efficient market hypothesis is provided by LeRoy (1989). A provocative article on the source of stock price movements is Black (1986).

The first derivation of the Black-Scholes formula given in this course, using only Itô's formula, is similar to that originally given by Black & Scholes (1973). An important companion paper is Merton (1973), which makes good reading even today. (This and many other papers by Merton are collected in Merton (1990).) Even though geometric Brownian motion is a less than perfect model for stock prices, the Black-Scholes option hedging formula seems not to be very sensitive to deficiencies in the model.

35.6 Markov processes.

Markov processes which are solutions to stochastic differential equations are called *diffusion processes*. A good introduction to this topic, including discussions of the Kolmogorov forward and backward equations, is Chapter 15 of Karlin & Taylor (1981). The other books cited previously, Øksendal (1995), Protter (1990), Chung & Williams (1983), and Karatzas & Shreve (1991), all treat this subject. Kloeden & Platen (1992) is a thorough study of the numerical solution of stochastic differential equations.

The constant elasticity of variance model for option pricing appears in Cox & Ross (1976). Another alternative model for the stock price underlying options, due to Föllmer & Schweizer (1993), has the geometric Ornstein-Uhlenbeck process as a special case.

The Feynman-Kac Theorem, connecting stochastic differential equations to partial differential equations, is due to Feyman (1948) and Kac (1951). A numerical treatment of the partial differential equations arising in finance is contained in Wilmott, Dewynne and Howison (1993, 1995) and also Duffie (1992).

35.7 Girsanov's theorem, the martingale representation theorem, and risk-neutral measures.

Girsanov's Theorem in the generality stated here is due to Girsanov (1960), although the result for constant θ was established much earlier by Cameron & Martin (1944). The theorem requires a technical condition to ensure that $\mathbb{E}Z(T) = 1$, so that $\widetilde{\mathbb{P}}$ is a probability measure; see Karatzas & Shreve (1991), page 198.

The form of the martingale representation theorem presented here is from Kunita & Watanabe (1967). It can also be found in Karatzas & Shreve (1991), page 182.

The application of the Girsanov Theorem and the martingale representation theorem to risk-neutral pricing is due to Harrison & Pliska (1981). This methodology frees the Brownian-motion driven model from the assumption of constant interest rate and volatility; these parameters can be random through dependence on the path of the underlying asset, or even through dependence on the paths of other assets. When both the interest rate and volatility of an asset are allowed to be stochastic, the Brownian-motion driven model is mathematically the most general possible for asset prices without jumps.

When asset processes have jumps, risk-free hedging is generally not possible. Some works on hedging and/or optimization in models which allow for jumps are Aase (1993), Back (1991), Bates (1988,1992), Beinert & Trautman (1991), Elliott & Kopp (1990), Jarrow & Madan (1991b,c), Jones (1984), Madan & Seneta (1990), Madan & Milne (1991), Mercurio & Runggaldier (1993), Merton (1976), Naik & Lee (1990), Schweizer (1992a,b), Shirakawa (1990,1991) and Xue (1992).

The Fundamental Theorem of Asset Pricing, as stated here, can be found in Harrison & Pliska (1981, 1983). It is tempting to believe the converse of Part I, i.e., that the absence of arbitrage implies the existence of a risk-neutral measure. This is true in discrete-time models, but in continuous-time models, a slightly stronger condition is needed to guarantee existence of a risk-neutral measure. For the continuous-time case, results have been obtained by many authors, including Stricker (1990), Delbaen (1992), Lakner (1993), Delbaen & Schachermayer (1994a,b), and Fritelli & Lakner (1994, 1995).

In addition to the fundamental papers of Harrison & Kreps (1979), and Harrison & Pliska (1981, 1983), some other works on the relationship between market completeness and uniqueness of the risk-neutral measure are Artzner & Heath (1990), Delbaen (1992), Jacka (1992), Jarrow & Madan (1991a), Müller (1989) and Taqqu & Willinger (1987).

35.8 Exotic options.

The reflection principle, adjusted to account for drift, is taken from Karatzas & Shreve (1991), pages 196–197.

Explicit formulas for the prices of barrier options have been obtained by Rubinstein & Reiner (1991) and Kunitomo & Ikeda (1992). Lookback options have been studied by Goldman, Sosin & Gatto (1979), Goldman, Sosin & Shepp (1979) and Conzé & Viswanathan (1991).

Because it is difficult to obtain explicit formulas for the prices of Asian options, most work has been devoted to approximations. We do not provide an explicit pricing formula here, although the partial differential equation given here by the Feynman-Kac Theorem characterizes the exact price. Bouaziz, Bryis & Crouhy (1994) provide an approximate pricing formula, Rogers & Shi (1995) provide a lower bound, and Geman & Yor (1993) obtain the Laplace transform of the price.

35.9 American options.

A general arbitrage-based theory for the pricing of American contingent claims and options begins with the articles of Bensoussan (1984) and Karatzas (1988); see Myneni (1992) for a survey and additional references. The perpetual American put problem was solved by McKean (1965).

Approximation and/or numerical solutions for the American option problem have been proposed by several authors, including Black (1975), Brennan & Schwartz (1977) (see Jaillet et al. (1990) for a treatment of the American option optimal stopping problem via variational inequalities, which leads to a justification of the Brennan-Schwartz algorithm), by Cox, Ross & Rubinstein (1979) (see Lamberton (1993) for the convergence of the associated binomial and/or finite difference schemes) and by Parkinson (1977), Johnson (1983), Geske & Johnson (1984), MacMillan (1986), Omberg (1987), Barone-Adesi & Whalley (1987), Barone-Adesi & Elliott (1991), Bunch & Johnson (1992), Broadie & Detemple (1994), and Carr & Faguet (1994).

35.10 Forward and futures contracts.

The distinction between futures contracts and daily resettled forward contracts has only recently been recognized (see Margrabe (1976), Black (1976)) and even more recently understood. Cox, Ingersoll & Ross (1981) and Jarrow & Oldfield (1981) provide a discrete-time arbitrage-based analysis of the relationship between forwards and futures, whereas Richard & Sundaresan (1981) study these claims in a continuous-time, equilibrium setting. Our presentation of this material is similar to that of Duffie & Stanton (1992), which also considers options on futures, and to Chapte 7 of Duffie (1992). For additional reading on forward and futures contracts, one may consult Duffie (1989).

35.11 Term structure models.

The Hull & White (1990) model is a generalization of the constant-coefficent Vasicek (1977) model. Implementations of the model appear in Hull & White (1994a,b). The Cox-Ingersoll-Ross model is presented in (1985a,b). The presentations of these given models here is taken from Rogers (1995). Other surveys of term structure models are Duffie & Kan (1994) and Vetzal (1994). A partial list of other term structure models is Black, Derman & Toy (1990), Brace & Musiela (1994a,b), Brennan & Schwartz (1979, 1982) (but see Hogan (1993) for discussion of a problem with this model), Duffie & Kan (1993), Ho & Lee (1986), Jamshidian (1990), and Longstaff & Schwartz (1992a,b).

The continuous-time Heath-Jarrow-Morton model appears in Heath, Jarrow & Morton (1992), and a discrete-time version is provided by Heath, Jarrow & Morton (1990). Carverhill & Pang (1995) discuss implementation. The Brace-Gatarek-Musiela variation of the HJM model is taken from Brace, et al. (1995). A summary of this model appears as Reed (1995). Related works on term structure models and swaps are Flesaker & Hughston (1995) and Jamshidian (1996).

35.12 Change of numéraire.

This material in this course is taken from Geman, El Karoui and Rochet (1995). Similar ideas were used by by Jamshidian (1989). The Merton option pricing formula appears in Merton (1973).

35.13 Foreign exchange models.

Foreign exchange options were priced by Biger & Hull (1983) and Garman & Kohlhagen (1983). The prices for differential swaps have been worked out by Jamshidian (1993a, 1993b) and Brace & Musiela (1994a).

35.14 REFERENCES

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