Chapter 34

Brace-Gatarek-Musiela model

34.1 Review of HJM under risk-neutral $I\!P$

$$
\begin{aligned}
f(t,T) = &\quad \text{Forward rate at time } t \text{ for borrowing at time } T. \\
df(t,T) = &\quad \sigma(t,T)\sigma^*(t,T)\,dt + \sigma(t,T)\,dW(t),
\end{aligned}
$$

where

$$\sigma^*(t,T) = \int_t^T \sigma(t,u)\,du$$

The interest rate is $r(t) = f(t,t)$. The bond prices

$$
\begin{aligned}
B(t,T) &= I\!E\left[\exp\left\{-\int_t^T r(u)\,du\right\}\Big|\mathcal{F}(t)\right] \\
&= \exp\left\{-\int_t^T f(t,u)\,du\right\}
\end{aligned}
$$

satisfy

$$dB(t,T) = r(t)\,B(t,T)\,dt - \underbrace{\sigma^*(t,T)}_{\text{volatility of } T\text{-maturity bond.}}\,B(t,T)\,dW(t).$$

To implement HJM, you specify a function

$$\sigma(t,T), \quad 0 \le t \le T.$$

A simple choice we would like to use is

$$\sigma(t,T) = \sigma f(t,T)$$

where $\sigma > 0$ is the constant "volatility of the forward rate". This is not possible because it leads to

$$
\begin{aligned}
\sigma^*(t,T) &= \sigma \int_t^T f(t,u)\,du, \\
df(t,T) &= \sigma^2 f(t,T)\left(\int_t^T f(t,u)\,du\right)\,dt + \sigma f(t,T)\,dW(t),
\end{aligned}
$$

and Heath, Jarrow and Morton show that solutions to this equation explode before T.

The problem with the above equation is that the dt term grows like the square of the forward rate. To see what problem this causes, consider the similar deterministic ordinary differential equation

$$f'(t) = f^2(t),$$

where $f(0) = c > 0$. We have

$$\begin{aligned}
\frac{f'(t)}{f^2(t)} &= 1, \\
-\frac{d}{dt}\frac{1}{f(t)} &= 1, \\
-\frac{1}{f(t)} + \frac{1}{f(0)} &= \int_0^t 1\,du = t \\
-\frac{1}{f(t)} &= t - \frac{1}{f(0)} = t - 1/c = \frac{ct-1}{c}, \\
f(t) &= \frac{c}{1-ct}.
\end{aligned}$$

This solution explodes at $t = 1/c$.

34.2 Brace-Gatarek-Musiela model

New variables:

Current time t

Time to maturity $\tau = T - t$.

Forward rates:

$$r(t,\tau) = f(t, t+\tau), \quad r(t,0) = f(t,t) = r(t), \tag{2.1}$$

$$\frac{\partial}{\partial\tau} r(t,\tau) = \frac{\partial}{\partial T} f(t, t+\tau) \tag{2.2}$$

Bond prices:

$$\begin{aligned}
D(t,\tau) &= B(t, t+\tau) \qquad (2.3)\\
&= \exp\left\{-\int_t^{t+\tau} f(t,v)\,dv\right\} \\
(u = v - t;\ du = dv): \quad &= \exp\left\{-\int_0^{\tau} f(t,t+u)\,du\right\} \\
&= \exp\left\{-\int_0^{\tau} r(t,u)\,du\right\}
\end{aligned}$$

$$\frac{\partial}{\partial\tau} D(t,\tau) = \frac{\partial}{\partial T} B(t,t+\tau) = -r(t,\tau) D(t,\tau). \tag{2.4}$$

We will now write $\sigma(t,\tau)=\sigma(t,T-t)$ rather than $\sigma(t,T)$. In this notation, the HJM model is

$$df(t,T)=\sigma(t,\tau)\sigma^*(t,\tau)\;dt+\sigma(t,\tau)\;dW(t), \tag{2.5}$$
$$dB(t,T)=r(t)B(t,T)\;dt-\sigma^*(t,\tau)B(t,T)\;dW(t), \tag{2.6}$$

where

$$\sigma^*(t,\tau)=\int_0^\tau \sigma(t,u)\;du, \tag{2.7}$$
$$\frac{\partial}{\partial\tau}\sigma^*(t,\tau)=\sigma(t,\tau). \tag{2.8}$$

We now derive the differentials of $r(t,\tau)$ and $D(t,\tau)$, analogous to (2.5) and (2.6) We have

$$\begin{aligned} dr(t,\tau) &= \underbrace{df(t,t+\tau)}_{\text{differential applies only to first argument}} + \frac{\partial}{\partial T}f(t,t+\tau)\;dt \\ &\overset{(2.5),(2.2)}{=} \sigma(t,\tau)\sigma^*(t,\tau)\;dt+\sigma(t,\tau)\;dW(t)+\frac{\partial}{\partial\tau}r(t,\tau)\;dt \\ &\overset{(2.8)}{=} \frac{\partial}{\partial\tau}\left[r(t,\tau)+\tfrac{1}{2}(\sigma^*(t,\tau))^2\right]\;dt+\sigma(t,\tau)\;dW(t). \end{aligned} \tag{2.9}$$

Also,

$$\begin{aligned} dD(t,\tau) &= \underbrace{dB(t,t+\tau)}_{\text{differential applies only to first argument}} + \frac{\partial}{\partial T}B(t,t+\tau)\;dt \\ &\overset{(2.6),(2.4)}{=} r(t)\;B(t,t+\tau)\;dt-\sigma^*(t,\tau)B(t,t+\tau)\;dW(t)-r(t,\tau)D(t,\tau)\;dt \\ &\overset{(2.1)}{=} [r(t,0)-r(t,\tau)]\,D(t,\tau)\;dt-\sigma^*(t,\tau)D(t,\tau)\;dW(t). \end{aligned} \tag{2.10}$$

34.3 LIBOR

Fix $\delta>0$ (say, $\delta=\frac{1}{4}$ year). \$ $D(t,\delta)$ invested at time t in a $(t+\delta)$-maturity bond grows to \$ 1 at time $t+\delta$. $L(t,0)$ is defined to be the corresponding rate of simple interest:

$$D(t,\delta)(1+\delta L(t,0))=1,$$
$$1+\delta L(t,0)=\frac{1}{D(t,\delta)}=\exp\left\{\int_0^\partial r(t,u)\;du\right\},$$
$$L(t,0)=\frac{\exp\left\{\int_0^\partial r(t,u)\;du\right\}-1}{\delta}.$$

34.4 Forward LIBOR

$\delta > 0$ is still fixed. At time t, agree to invest \$ $\frac{D(t,\tau+\delta)}{D(t,\tau)}$ at time $t+\tau$, with payback of \$1 at time $t+\tau+\delta$. Can do this at time t by shorting $\frac{D(t,\tau+\delta)}{D(t,\tau)}$ bonds maturing at time $t+\tau$ and going long one bond maturing at time $t+\tau+\delta$. The value of this portfolio at time t is

$$-\frac{D(t,\tau+\delta)}{D(t,\tau)} D(t,\tau) + D(t,\tau+\delta) = 0.$$

The *forward LIBOR* $L(t,\tau)$ is defined to be the simple (forward) interest rate for this investment:

$$\begin{aligned}
\frac{D(t,\tau+\delta)}{D(t,\tau)}\left(1+\delta L(t,\tau)\right) &= 1, \\
1+\delta L(t,\tau) = \frac{D(t,\tau)}{D(t,\tau+\delta)} &= \frac{\exp\left\{-\int_0^\tau r(t,u)\,du\right\}}{\exp\left\{-\int_0^{\tau+\delta} r(t,u)\,du\right\}} \\
&= \exp\left\{\int_\tau^{\tau+\delta} r(t,u)\,du\right\}, \\
L(t,\tau) &= \frac{\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\,du\right\}-1}{\delta}. \qquad (4.1)
\end{aligned}$$

Connection with forward rates:

$$\begin{aligned}
\frac{\partial}{\partial\delta}\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\,du\right\}\Bigg|_{\delta=0} &= r(t,\tau+\delta)\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\,du\right\}\Bigg|_{\delta=0} \\
&= r(t,\tau),
\end{aligned}$$

so

$$\begin{aligned}
f(t,t+\tau) = r(t,\tau) &= \lim_{\delta\downarrow 0}\frac{\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\,du\right\}-1}{\delta} \\
L(t,\tau) &= \frac{\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\,du\right\}-1}{\delta}, \quad \delta > 0 \text{ fixed.} \qquad (4.2)
\end{aligned}$$

$r(t,\tau)$ is the continuously compounded rate. $L(t,\tau)$ is the simple rate over a period of duration δ.

We cannot have a log-normal model for $r(t,\tau)$ because solutions explode as we saw in Section 34.1. For fixed positive δ, we *can* have a log-normal model for $L(t,\tau)$.

34.5 The dynamics of $L(t,\tau)$

We want to choose $\sigma(t,\tau),\ t \geq 0,\ \tau \geq 0$, appearing in (2.5) so that

$$dL(t,\tau) = (\ldots)\,dt + L(t,\tau)\,\gamma(t,\tau)\,dW(t)$$

for some $\gamma(t,\tau),\ t \geq 0, \tau \geq 0$. This is the BGM model, and is a subclass of HJM models, corresponding to particular choices of $\sigma(t,\tau)$.

Recall (2.9):

$$dr(t,\tau) = \frac{\partial}{\partial u}\left[r(t,u) + \tfrac{1}{2}(\sigma^*(t,u))^2\right]\ dt + \sigma(t,u)\ dW(t).$$

Therefore,

$$\begin{aligned} d\left(\int_\tau^{\tau+\delta} r(t,u)\ du\right) &= \int_\tau^{\tau+\delta} dr(t,u)\ du \\ &= \int_\tau^{\tau+\delta} \frac{\partial}{\partial u}\left[r(t,u) + \tfrac{1}{2}(\sigma^*(t,u))^2\right]\ du\ dt + \int_\tau^{\tau+\delta} \sigma(t,u)\ du\ dW(t) \\ &= \left[r(t,\tau+\delta) - r(t,\tau) + \tfrac{1}{2}(\sigma^*(t,\tau+\delta))^2 - \tfrac{1}{2}(\sigma^*(t,\tau))^2\right]\ dt \\ &\qquad + [\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau)]\ dW(t) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} dL(t,\tau) &\overset{(4.1)}{=} d\left[\frac{\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\ du\right\} - 1}{\delta}\right] \\ &= \frac{1}{\delta}\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\ du\right\}\ d\int_\tau^{\tau+\delta} r(t,u)\ du \\ &\qquad + \frac{1}{2\delta}\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\ du\right\}\ \left(d\int_\tau^{\tau+\delta} r(t,u)\ du\right)^2 \\ &\overset{(4.1),(5.1)}{=} \frac{1}{\delta}[1 + \delta L(t,\tau)] \times \\ &\qquad \times \Big\{[r(t,\tau+\delta) - r(t,\delta) + \tfrac{1}{2}(\sigma^*(t,\tau+\delta))^2 - \tfrac{1}{2}(\sigma^*(t,\tau))^2]\ dt \\ &\qquad\qquad + [\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau)]\ dW(t) \\ &\qquad\qquad + \tfrac{1}{2}[\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau)]^2\ dt\Big\} \\ &= \frac{1}{\delta}[1 + \delta L(t,\tau)]\Big\{[r(t,\tau+\delta) - r(t,\delta)]\ dt \\ &\qquad\qquad + \sigma^*(t,\tau+\delta)[\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau)]\ dt \\ &= \qquad\qquad + [\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau)]\ dW(t)\Big\}. \end{aligned} \tag{5.2}$$

But

$$\frac{\partial}{\partial\tau}L(t,\tau) = \frac{\partial}{\partial\tau}\left[\frac{\exp\left\{\int_\tau^{\tau+\delta} r(t,u)\,du\right\} - 1}{\delta}\right]$$
$$= \exp\left\{\int_\tau^{\tau+\delta} r(t,u)\,du\right\}.[r(t,\tau+\delta) - r(t,\delta)]$$
$$= \frac{1}{\delta}[1+\delta L(t,\tau)][r(t,\tau+\delta) - r(t,\delta)].$$

Therefore,

$$dL(t,\tau) = \frac{\partial}{\partial\tau}L(t,\tau)\,dt + \frac{1}{\delta}[1+\delta L(t,\tau)][\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau)].[\sigma^*(t,\tau+\delta)\,dt + dW(t)].$$

Take $\gamma(t,\tau)$ to be given by

$$\gamma(t,\tau)L(t,\tau) = \frac{1}{\delta}[1+\delta L(t,\tau)][\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau)]. \tag{5.3}$$

Then

$$dL(t,\tau) = [\frac{\partial}{\partial\tau}L(t,\tau) + \gamma(t,\tau)L(t,\tau)\sigma^*(t,\tau+\delta)]\,dt + \gamma(t,\tau)L(t,\tau)\,dW(t). \tag{5.4}$$

Note that (5.3) is equivalent to

$$\sigma^*(t,\tau+\delta) = \sigma^*(t,\tau) + \frac{\delta L(t,\tau)\gamma(t,\tau)}{1+\delta L(t,\tau)}. \tag{5.3'}$$

Plugging this into (5.4) yields

$$dL(t,\tau) = \left[\frac{\partial}{\partial\tau}L(t,\tau) + \gamma(t,\tau)L(t,\tau)\sigma^*(t,\tau) + \frac{\delta L^2(t,\tau)\gamma^2(t,\tau)}{1+\delta L(t,\tau)}\right]\,dt$$
$$+ \gamma(t,\tau)L(t,\tau)\,dW(t). \tag{5.4'}$$

34.6 Implementation of BGM

Obtain the initial *forward LIBOR curve*

$$L(0,\tau), \quad \tau \geq 0,$$

from market data. Choose a *forward LIBOR volatility function* (usually nonrandom)

$$\gamma(t,\tau), \quad t \geq 0,\ \tau \geq 0.$$

Because LIBOR gives no rate information on time periods smaller than δ, we must also choose a *partial bond volatility function*

$$\sigma^*(t,\tau), \quad t \geq 0,\ 0 \leq \tau < \delta$$

for maturities less than δ from the current time variable t.

With these functions, we can for each $\tau \in [0,\delta)$ solve (5.4') to obtain

$$L(t,\tau), \quad t \geq 0,\ 0 \leq \tau < \delta.$$

Plugging the solution into (5.3'), we obtain $\sigma^*(t,\tau)$ for $\delta \leq \tau < 2\delta$. We then solve (5.4') to obtain

$$L(t,\tau), \quad t \geq 0,\ \delta \leq \tau < 2\delta,$$

and we continue recursively.

Remark 34.1 BGM is a special case of HJM with HJM's $\sigma^*(t,\tau)$ generated recursively by (5.3'). In BGM, $\gamma(t,\tau)$ is usually taken to be nonrandom; the resulting $\sigma^*(t,\tau)$ is random.

Remark 34.2 (5.4) (equivalently, (5.4')) is a stochastic *partial* differential equation because of the $\frac{\partial}{\partial\tau}L(t,\tau)$ term. This is not as terrible as it first appears. Returning to the HJM variables t and T, set

$$K(t,T) = L(t,T-t).$$

Then

$$dK(t,T) = dL(t,T-t) - \frac{\partial}{\partial\tau}L(t,T-t)\ dt$$

and (5.4) and (5.4') become

$$\begin{aligned} dK(t,T) &= \gamma(t,T-t)K(t,T)\left[\sigma^*(t,T-t+\delta)\ dt + dW(t)\right] \\ &= \gamma(t,T-t)K(t,T)\left[\sigma^*(t,T-t)\ dt + \frac{\delta K(t,T)\gamma(t,T-t)}{1+\delta K(t,T)}\ dt + dW(t)\right]. \end{aligned} \tag{6.1}$$

Remark 34.3 From (5.3) we have

$$\gamma(t,\tau)L(t,\tau) = [1+\delta L(t,\tau)]\frac{\sigma^*(t,\tau+\delta)-\sigma^*(t,\tau)}{\delta}.$$

If we let $\delta\downarrow 0$, then

$$\gamma(t,\tau)L(t,\tau) \rightarrow \frac{\partial}{\partial\delta}\sigma^*(t,\tau+\delta)\bigg|_{\delta=0} = \sigma(t,\tau),$$

and so

$$\gamma(t,T-t)K(t,T) \rightarrow \sigma(t,T-t).$$

We saw before (eq. 4.2) that as $\delta\downarrow 0$,

$$L(t,\tau) \rightarrow r(t,\tau) = f(t,t+\tau),$$

so

$$K(t,T) \to f(t,T).$$

Therefore, the limit as $\delta \downarrow 0$ of (6.1) is given by equation (2.5):

$$df(t,T) = \sigma(t, T-t)\left[\sigma^*(t, T-t)\ dt + dW(t)\right].$$

Remark 34.4 Although the dt term in (6.1) has the term $\frac{\delta\gamma^2(t,T-t)K^2(t,T)}{1+K(t,T)}$ involving K^2, solutions to this equation do not explode because

$$\begin{aligned} \frac{\delta\gamma^2(t,T-t)K^2(t,T)}{1+\delta K(t,T)} &\le \frac{\delta\gamma^2(t,T-t)K^2(t,T)}{\delta K(t,T)} \\ &\le \gamma^2(t,T-t)K(t,T). \end{aligned}$$

34.7 Bond prices

Let $\beta(t) = \exp\left\{\int_0^t r(u)\ du\right\}$. From (2.6) we have

$$\begin{aligned} d\left(\frac{B(t,T)}{\beta(t)}\right) &= \frac{1}{\beta(t)}[-r(t)B(t,T)\ dt + dB(t,T)] \\ &= -\frac{B(t,T)}{\beta(t)}\sigma^*(t,T-t)\ dW(t). \end{aligned}$$

The solution $\frac{B(t,T)}{\beta(t)}$ to this stochastic differential equation is given by

$$\frac{B(t,T)}{\beta(t)B(0,T)} = \exp\left\{-\int_0^t \sigma^*(u,T-u)\ dW(u) - \tfrac{1}{2}\int_0^t (\sigma^*(u,T-u))^2\ du\right\}.$$

This is a martingale, and we can use it to switch to the *forward measure*

$$\begin{aligned} I\!\!P_T(A) &= \frac{1}{B(0,T)}\int_A \frac{1}{\beta(T)}\ dI\!\!P \\ &= \int_A \frac{B(T,T)}{\beta(T)B(0,T)}\ dI\!\!P \quad \forall A \in \mathcal{F}(T). \end{aligned}$$

Girsanov's Theorem implies that

$$W_T(t) = W(t) + \int_0^t \sigma^*(u,T-u)\ du, \quad 0 \le t \le T,$$

is a Brownian motion under $I\!\!P_T$.

34.8 Forward LIBOR under more forward measure

From (6.1) we have

$$\begin{aligned} dK(t,T) &= \gamma(t,T-t)K(t,T)\left[\sigma^*(t,T-t+\delta)\,dt + dW(t)\right] \\ &= \gamma(t,T-t)K(t,T)\,dW_{T+\delta}(t), \end{aligned}$$

so

$$K(t,T) = K(0,T)\exp\left\{\int_0^t \gamma(u,T-u)\,dW_{T+\delta}(u) - \tfrac{1}{2}\int_0^t \gamma^2(u,T-u)\,du\right\}$$

and

$$\begin{aligned} K(T,T) &= K(0,T)\exp\left\{\int_0^T \gamma(u,T-u)\,dW_{T+\delta}(u) - \tfrac{1}{2}\int_0^T \gamma^2(u,T-u)\,du\right\} \\ &= K(t,T)\exp\left\{\int_t^T \gamma(u,T-u)\,dW_{T+\delta}(u) - \tfrac{1}{2}\int_t^T \gamma^2(u,T-u)\,du\right\}. \end{aligned} \tag{8.1}$$

We assume that γ is nonrandom. Then

$$X(t) = \int_t^T \gamma(u,T-u)\,dW_{T+\delta}(u) - \tfrac{1}{2}\int_t^T \gamma^2(u,T-u)\,du \tag{8.2}$$

is normal with variance

$$\rho^2(t) = \int_t^T \gamma^2(u,T-u)\,du$$

and mean $-\frac{1}{2}\rho^2(t)$.

34.9 Pricing an interest rate caplet

Consider a floating rate interest payment settled in arrears. At time $T+\delta$, the floating rate interest payment due is $\delta L(T,0) = \delta K(T,T)$, the LIBOR at time T. A caplet protects its owner by requiring him to pay only the cap δc if $\delta K(T,T) > \delta c$. Thus, the value of the caplet at time $T+\delta$ is $\delta(K(T,T)-c)^+$. We determine its value at times $0 \le t \le T+\delta$.

Case I: $T \le t \le T+\delta$.

$$\begin{aligned} C_{T+\delta}(t) &= I\!E\left[\frac{\beta(t)}{\beta(T+\delta)}\delta(K(T,T)-c)^+\middle|\mathcal{F}(t)\right] \\ &= \delta(K(T,T)-c)^+ I\!E\left[\frac{\beta(t)}{\beta(T+\delta)}\middle|\mathcal{F}(t)\right] \\ &= \delta(K(T,T)-c)^+ B(t,T+\delta). \end{aligned} \tag{9.1}$$

Case II: $0 \le t \le T$.
Recall that

$$I\!P_{T+\delta}(A) = \int_A Z(T+\delta)\, dI\!P, \quad \forall A \in \mathcal{F}(T+\delta),$$

where

$$Z(t) = \frac{B(t,T+\delta)}{\beta(t)B(0,T+\delta)}.$$

We have

$$\begin{aligned}
C_{T+\delta}(t) &= I\!E\left[\frac{\beta(t)}{\beta(T+\delta)}\delta(K(T,T)-c)^+ \Big| \mathcal{F}(t)\right] \\
&= \delta B(t,T+\delta)\underbrace{\frac{\beta(t)B(0,T+\delta)}{B(t,T+\delta)}}_{\frac{1}{Z(t)}} I\!E\left[\underbrace{\frac{B(T+\delta,T+\delta)}{\beta(T+\delta)B(0,T+\delta)}}_{Z(T+\delta)}(K(T,T)-c)^+ \Big| \mathcal{F}(t)\right] \\
&= \delta B(t,T+\delta) I\!E_{T+\delta}\left[(K(T,T)-c)^+ \Big| \mathcal{F}(t)\right]
\end{aligned}$$

From (8.1) and (8.2) we have

$$K(T,T) = K(t,T)\exp\{X(t)\},$$

where $X(t)$ is normal under $I\!P_{T+\delta}$ with variance $\rho^2(t) = \int_t^T \gamma^2(u, T-u)\, du$ and mean $-\frac{1}{2}\rho^2(t)$. Furthermore, $X(t)$ is independent of $\mathcal{F}(t)$.

$$C_{T+\delta}(t) = \delta B(t,T+\delta) I\!E_{T+\delta}\left[(K(t,T)\exp\{X(t)\}-c)^+ \Big| \mathcal{F}(t)\right].$$

Set

$$\begin{aligned}
g(y) &= I\!E_{T+\delta}\left[(y\exp\{X(t)\}-c)^+\right] \\
&= y\ N\left(\frac{1}{\rho(t)}\log\frac{y}{c} + \tfrac{1}{2}\rho(t)\right) - c\ N\left(\frac{1}{\rho(t)}\log\frac{y}{c} - \tfrac{1}{2}\rho(t)\right).
\end{aligned}$$

Then

$$C_{T+\delta}(t) = \delta\ B(t,T+\delta)\ g(K(t,T)), \quad 0 \le t \le T-\delta. \tag{9.2}$$

In the case of constant γ, we have

$$\rho(t) = \gamma\sqrt{T-t},$$

and (9.2) is called the *Black caplet formula*.

34.10 Pricing an interest rate cap

Let

$$T_0 = 0,\ T_1 = \delta,\ T_2 = 2\delta,\ \ldots,\ T_n = n\delta.$$

A cap is a series of payments

$$\delta(K(T_k, T_k) - c)^+ \quad \text{at time } T_{k+1},\ k = 0, 1, \ldots, n-1.$$

The value at time t of the cap is the value of all remaining caplets, i.e.,

$$C(t) = \sum_{k:t \le T_k} C_{T_k}(t).$$

34.11 Calibration of BGM

The interest rate caplet c on $L(0,T)$ at time $T+\delta$ has time-zero value

$$C_{T+\delta}(0) = \delta B(0, T+\delta)\ g(K(0,T)),$$

where g (defined in the last section) depends on

$$\int_0^T \gamma^2(u, T-u)\ du.$$

Let us suppose γ is a deterministic function of its second argument, i.e.,

$$\gamma(t, \tau) = \gamma(\tau).$$

Then g depends on

$$\int_0^T \gamma^2(T-u)\ du = \int_0^T \gamma^2(v)\ dv.$$

If we know the caplet price $C_{T+\delta}(0)$, we can "back out" the squared volatility $\int_0^T \gamma^2(v)\ dv$. If we know caplet prices

$$C_{T_0+\delta}(0), C_{T_1+\delta}(0), \ldots, C_{T_n+\delta}(0),$$

where $T_0 < T_1 < \ldots < T_n$, we can "back out"

$$\int_0^{T_0} \gamma^2(v)\ dv,\ \ \int_{T_0}^{T_1} \gamma^2(v)\ dv = \int_0^{T_1} \gamma^2(v)\ dv - \int_0^{T_0} \gamma^2(v)\ dv,$$
$$\ldots,\ \int_{T_{n-1}}^{T_n} \gamma^2(v)\ dv. \quad (11.1)$$

In this case, we may assume that γ is constant on each of the intervals

$$(0, T_0),\ (T_0, T_1),\ \ldots,\ (T_{n-1}, T_n),$$

and choose these constants to make the above integrals have the values implied by the caplet prices.

If we know caplet prices $C_{T+\delta}(0)$ for all $T \geq 0$, we can "back out" $\int_0^T \gamma^2(v)\ dv$ and then differentiate to discover $\gamma^2(\tau)$ and $\gamma(\tau) = \sqrt{\gamma^2(\tau)}$ for all $\tau \geq 0$.

To implement BGM, we need both $\gamma(\tau),\ \tau \geq 0$, and

$$\sigma^*(t,\tau), \quad t \geq 0,\ 0 \leq \tau < \delta.$$

Now $\sigma^*(t,\tau)$ is the volatility at time t of a zero coupon bond maturing at time $t+\tau$ (see (2.6)). Since δ is small (say $\frac{1}{4}$ year), and $0 \leq \tau < \delta$, it is reasonable to set

$$\sigma^*(t,\tau) = 0, \quad t \geq 0, \quad 0 \leq \tau < \delta.$$

We can now solve (or simulate) to get

$$L(t,\tau), \quad t \geq 0, \tau \geq 0,$$

or equivalently,

$$K(t,T), \quad t \geq 0, T \geq 0,$$

using the recursive procedure outlined at the start of Section 34.6.

34.12 Long rates

The long rate is determined by long maturity bond prices. Let n be a large fixed positive integer, so that $n\delta$ is 20 or 30 years. Then

$$\begin{aligned} \frac{1}{D(t,n\delta)} &= \exp\left\{\int_0^{n\delta} r(t,u)\ du\right\} \\ &= \prod_{k=1}^{n} \exp\left\{\int_{(k-1)\delta}^{k\delta} r(t,u)\ du\right\} \\ &= \prod_{k=1}^{n} [1+\delta L(t,(k-1)\delta)], \end{aligned}$$

where the last equality follows from (4.1). The long rate is

$$\frac{1}{n\delta} \log \frac{1}{D(t,n\delta)} = \frac{1}{n\delta} \sum_{k=1}^{n} \log[1+\delta L(t,(k-1)\delta)].$$

34.13 Pricing a swap

Let $T_0 \geq 0$ be given, and set

$$T_1 = T_0 + \delta, \quad T_2 = T_0 + 2\delta, \quad \ldots, \quad T_n = T_0 + n\delta.$$

The swap is the series of payments

$$\delta(L(T_k,0)-c) \quad \text{at time } T_{k+1}, k=0,1,\ldots,n-1.$$

For $0 \le t \le T_0$, the value of the swap is

$$\sum_{k=0}^{n-1} I\!E\left[\frac{\beta(t)}{\beta(T_{k+1})}\delta(L(T_k,0)-c)\middle|\mathcal{F}(t)\right].$$

Now

$$1+\delta L(T_k,0)=\frac{1}{B(T_k,T_{k+1})},$$

so

$$L(T_k,0)=\frac{1}{\delta}\left[\frac{1}{B(T_k,T_{k+1})}-1\right].$$

We compute

$$\begin{aligned}
&I\!E\left[\frac{\beta(t)}{\beta(T_{k+1})}\delta(L(T_k,0)-c)\middle|\mathcal{F}(t)\right]\\
&=I\!E\left[\frac{\beta(t)}{\beta(T_{k+1})}\left(\frac{1}{B(T_k,T_{k+1})}-1-\delta c\right)\middle|\mathcal{F}(t)\right]\\
&=I\!E\left[\frac{\beta(t)}{\beta(T_k)B(T_k,T_{k+1})}\underbrace{I\!E\left[\frac{\beta(T_k)}{\beta(T_{k+1})}\middle|\mathcal{F}(T_k)\right]}_{B(T_k,T_{k+1})}\middle|\mathcal{F}(t)\right]-(1+\delta c)B(t,T_{k+1})\\
&=I\!E\left[\frac{\beta(t)}{\beta(T_{k+1})}\middle|\mathcal{F}(t)\right]-(1+\delta c)B(t,T_{k+1})\\
&=B(t,T_k)-(1+\delta c)B(t,T_{k+1}).
\end{aligned}$$

The value of the swap at time t is

$$\begin{aligned}
&\sum_{k=0}^{n-1} I\!E\left[\frac{\beta(t)}{\beta(T_{k+1})}\delta(L(T_k,0)-c)\middle|\mathcal{F}(t)\right]\\
&=\sum_{k=0}^{n-1}[B(t,T_k)-(1+\delta c)B(t,T_{k+1})]\\
&=B(t,T_0)-(1+\delta c)B(t,T_1)+B(t,T_1)-(1+\delta c)B(t,T_2)+\ldots+B(t,T_{n-1})-(1+\delta c)B(t,T_n)\\
&=B(t,T_0)-\delta cB(t,T_1)-\delta cB(t,T_2)-\ldots-\delta cB(t,T_n)-B(t,T_n).
\end{aligned}$$

The forward swap rate $w_{T_0}(t)$ at time t for maturity T_0 is the value of c which makes the time-t value of the swap equal to zero:

$$w_{T_0}(t)=\frac{B(t,T_0)-B(t,T_n)}{\delta\left[B(t,T_1)+\ldots+B(t,T_n)\right]}.$$

In contrast to the cap formula, which depends on the term structure model and requires estimation of γ, the swap formula is generic.