Chapter 33

Change of numéraire

Consider a Brownian motion driven market model with time horizon T^*. For now, we will have one asset, which we call a "stock" even though in applications it will usually be an interest rate dependent claim. The price of the stock is modeled by

$$dS(t) = r(t)\ S(t)\ dt + \sigma(t)S(t)\ dW(t), \tag{0.1}$$

where the interest rate process $r(t)$ and the volatility process $\sigma(t)$ are adapted to some filtration $\{\mathcal{F}(t);\ 0 \le t \le T^*\}$. W is a Brownian motion relative to this filtration, but $\{\mathcal{F}(t);\ 0 \le t \le T^*\}$ may be larger than the filtration generated by W.

This is *not* a geometric Brownian motion model. We are particularly interested in the case that the interest rate is stochastic, given by a term structure model we have not yet specified.

We shall work only under the risk-neutral measure, which is reflected by the fact that the mean rate of return for the stock is $r(t)$.

We define the *accumulation factor*

$$\beta(t) = \exp\left\{\int_0^t r(u)\ du\right\},$$

so that the discounted stock price $\frac{S(t)}{\beta(t)}$ is a martingale. Indeed,

$$d\left(\frac{S(t)}{\beta(t)}\right) = \frac{S(t)}{\beta(t)}\sigma(t)\ dW(t).$$

The zero-coupon bond prices are given by

$$\begin{aligned} B(t,T) &= I\!\!E\left[\exp\left\{-\int_t^T r(u)\ du\right\}\Big|\mathcal{F}(t)\right] \\ &= I\!\!E\left[\frac{\beta(t)}{\beta(T)}\Big|\mathcal{F}(t)\right], \end{aligned}$$

so

$$\frac{B(t,T)}{\beta(t)} = I\!E\left[\frac{1}{\beta(T)}\bigg|\mathcal{F}(t)\right]$$

is also a martingale (tower property).

The T-*forward price* $F(t,T)$ of the stock is the price set at time t for delivery of one share of stock at time T with payment at time T. The value of the forward contract at time t is zero, so

$$\begin{aligned} 0 &= I\!E\left[\frac{\beta(t)}{\beta(T)}\left(S(T) - F(t,T)\right)\bigg|\mathcal{F}(t)\right] \\ &= \beta(t) I\!E\left[\frac{S(T)}{\beta(T)}\bigg|Ft\right] - F(t,T) I\!E\left[\frac{\beta(t)}{\beta(T)}\bigg|\mathcal{F}(t)\right] \\ &= \beta(t)\frac{S(t)}{\beta(t)} - F(t,T)B(t,T) \\ &= S(t) - F(t,T)B(t,T) \end{aligned}$$

Therefore,

$$F(t,T) = \frac{S(t)}{B(t,T)}.$$

Definition 33.1 (Numéraire) Any asset in the model whose price is always strictly positive can be taken as the numéraire. We then denominate all other assets in units of this numéraire.

Example 33.1 (Money market as numéraire) The money market could be the numéraire. At time t, the stock is worth $\frac{S(t)}{\beta(t)}$ units of money market and the T-maturity bond is worth $\frac{B(t,T)}{\beta(t)}$ units of money market. ■

Example 33.2 (Bond as numéraire) The T-maturity bond could be the numéraire. At time $t \le T$, the stock is worth $F(t,T)$ units of T-maturity bond and the T-maturity bond is worth 1 unit. ■

We will say that a probability measure $I\!P_N$ is *risk-neutral for the numéraire* N if every asset price, divided by N, is a martingale under $I\!P_N$. The original probability measure $I\!P$ is risk-neutral for the numéraire β (Example 33.1).

Theorem 0.71 *Let N be a numéraire, i.e., the price process for some asset whose price is always strictly positive. Then $I\!P_N$ defined by*

$$I\!P_N(A) = \frac{1}{N(0)}\int_A \frac{N(T^*)}{\beta(T^*)}\, dI\!P, \quad \forall A \in \mathcal{F}(T^*),$$

is risk-neutral for N.

Note: $I\!\!P$ and $I\!\!P_N$ are equivalent, i.e., have the same probability zero sets, and

$$I\!\!P(A) = N(0) \int_A \frac{\beta(T^*)}{N(T^*)}\, dI\!\!P_N, \quad \forall A \in \mathcal{F}(T^*).$$

Proof: Because N is the price process for some asset, N/β is a martingale under $I\!\!P$. Therefore,

$$\begin{aligned}
I\!\!P_N(\Omega) &= \frac{1}{N(0)} \int_\Omega \frac{N(T^*)}{\beta(T^*)}\, dI\!\!P \\
&= \frac{1}{N(0)}.I\!\!E\left[\frac{N(T^*)}{\beta(T^*)}\right] \\
&= \frac{1}{N(0)} \frac{N(0)}{\beta(0)} \\
&= 1,
\end{aligned}$$

and we see that $I\!\!P_N$ is a probability measure.

Let Y be an asset price. Under $I\!\!P$, Y/β is a martingale. We must show that under $I\!\!P_N$, Y/N is a martingale. For this, we need to recall how to combine conditional expectations with change of measure (Lemma 1.54). If $0 \le t \le T \le T^*$ and X is $\mathcal{F}(T)$-measurable, then

$$\begin{aligned}
I\!\!E_N\left[X \middle| \mathcal{F}(t)\right] &= \frac{N(0)\beta(t)}{N(t)} I\!\!E\left[\frac{N(T)}{N(0)\beta(T)} X \middle| \mathcal{F}(t)\right] \\
&= \frac{\beta(t)}{N(t)} I\!\!E\left[\frac{N(T)}{\beta(T)} X \middle| \mathcal{F}(t)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
I\!\!E_N\left[\frac{Y(T)}{N(T)} \middle| \mathcal{F}(t)\right] &= \frac{\beta(t)}{N(t)} I\!\!E\left[\frac{N(T)}{\beta(T)} \frac{Y(T)}{N(T)} \middle| \mathcal{F}(t)\right] \\
&= \frac{\beta(t)}{N(t)} \frac{Y(t)}{\beta(t)} \\
&= \frac{Y(t)}{N(t)},
\end{aligned}$$

which is the martingale property for Y/N under $I\!\!P_N$. ■

33.1 Bond price as numéraire

Fix $T \in (0, T^*]$ and let $B(t,T)$ be the numéraire. The risk-neutral measure for this numéraire is

$$\begin{aligned}
I\!\!P_T(A) &= \frac{1}{B(0,T)} \int_A \frac{B(T,T)}{\beta(T)}\, dI\!\!P \\
&= \frac{1}{B(0,T)} \int_A \frac{1}{\beta(T)}\, dI\!\!P \quad \forall A \in \mathcal{F}(T).
\end{aligned}$$

Because this bond is not defined after time T, we change the measure only "up to time T", i.e., using $\frac{1}{B(0,T)} \frac{B(T,T)}{\beta(T)}$ and only for $A \in \mathcal{F}(T)$.

$I\!\!P_T$ is called the T*-forward measure.* Denominated in units of T-maturity bond, the value of the stock is

$$F(t,T) = \frac{S(t)}{B(t,T)}, \quad 0 \le t \le T.$$

This is a martingale under $I\!\!P_T$, and so has a differential of the form

$$dF(t,T) = \sigma_F(t,T)F(t,T)\, dW_T(t), \quad 0 \le t \le T, \tag{1.1}$$

i.e., a differential without a dt term. The process $\{W_T;\ 0 \le t \le T\}$ is a Brownian motion under $I\!\!P_T$. We may assume without loss of generality that $\sigma_F(t,T) \ge 0$.

We write $F(t)$ rather than $F(t,T)$ from now on.

33.2 Stock price as numéraire

Let $S(t)$ be the numéraire. In terms of this numéraire, the stock price is identically 1. The risk-neutral measure under this numéraire is

$$I\!\!P_S(A) = \frac{1}{S(0)} \int_A \frac{S(T^*)}{\beta(T^*)}\, dI\!\!P, \quad \forall A \in \mathcal{F}(T^*).$$

Denominated in shares of stock, the value of the T-maturity bond is

$$\frac{B(t,T)}{S(t)} = \frac{1}{F(t)}.$$

This is a martingale under $I\!\!P_S$, and so has a differential of the form

$$d\left(\frac{1}{F(t)}\right) = \gamma(t,T)\left(\frac{1}{F(t)}\right)\, dW_S(t), \tag{2.1}$$

where $\{W_S(t);\ 0 \le t \le T^*\}$ is a Brownian motion under $I\!\!P_S$. We may assume without loss of generality that $\gamma(t,T) \ge 0$.

Theorem 2.72 *The volatility $\gamma(t,T)$ in (2.1) is equal to the volatility $\sigma_F(t,T)$ in (1.1). In other words, (2.1) can be rewritten as*

$$d\left(\frac{1}{F(t)}\right) = \sigma_F(t,T)\left(\frac{1}{F(t)}\right)\, dW_S(t), \tag{2.1'}$$

Proof: Let $g(x) = 1/x$, so $g'(x) = -1/x^2,\ g''(x) = 2/x^3$. Then

$$\begin{aligned}
d\left(\frac{1}{F(t)}\right) &= dg(F(t)) \\
&= g'(F(t))\ dF(t) + \tfrac{1}{2}g''(F(t))\ dF(t)\ dF(t) \\
&= -\frac{1}{F^2(t)}\sigma_F(t,T)F(t,T)\ dW_T(t) + \frac{1}{F^3(t)}\sigma_F^2(t,T)F^2(t,T)\ dt \\
&= \frac{1}{F(t)}\left[-\sigma_F(t,T)\ dW_T(t) + \sigma_F^2(t,T)\ dt\right] \\
&= \sigma_F(t,T)\left(\frac{1}{F(t)}\right)[-dW_T(t) + \sigma_F(t,T)\ dt].
\end{aligned}$$

Under $I\!P_T$, $-W_T$ is a Brownian motion. Under this measure, $\frac{1}{F(t)}$ has volatility $\sigma_F(t,T)$ and mean rate of return $\sigma_F^2(t,T)$. The change of measure from $I\!P_T$ to $I\!P_S$ makes $\frac{1}{F(t)}$ a martingale, i.e., it changes the mean return to zero, but the change of measure does not affect the volatility. Therefore, $\gamma(t,T)$ in (2.1) must be $\sigma_F(t,T)$ and W_S must be

$$W_S(t) = -W_T(t) + \int_0^t \sigma_F(u,T)\ du.$$

■

33.3 Merton option pricing formula

The price at time zero of a European call is

$$\begin{aligned}
V(0) &= I\!E\left[\frac{1}{\beta(T)}(S(T)-K)^+\right] \\
&= I\!E\left[\frac{S(T)}{\beta(T)}\mathbf{1}_{\{S(T)>K\}}\right] - K I\!E\left[\frac{1}{\beta(T)}\mathbf{1}_{\{S(T)>K\}}\right] \\
&= S(0)\int_{\{S(T)>K\}}\frac{S(T)}{S(0)\beta(T)}\ dI\!P - KB(0,T)\int_{\{S(T)>K\}}\frac{1}{B(0,T)\beta(T)}\ dI\!P \\
&= S(0)I\!P_S\{S(T)>K\} - KB(0,T)I\!P_T\{S(T)>K\} \\
&= S(0)I\!P_S\{F(T)>K\} - KB(0,T)I\!P_T\{F(T)>K\} \\
&= S(0)I\!P_S\left\{\frac{1}{F(T)} < \frac{1}{K}\right\} - KB(0,T)I\!P_T\{F(T)>K\}.
\end{aligned}$$

This is a completely general formula which permits computation as soon as we specify $\sigma_F(t,T)$. If we assume that $\sigma_F(t,T)$ is a constant σ_F, we have the following:

$$\begin{aligned}
\frac{1}{F(T)} &= \frac{B(0,T)}{S(0)} \exp\left\{\sigma_F W_S(T) - \tfrac{1}{2}\sigma_F^2 T\right\}, \\
I\!\!P_S\left(\frac{1}{F(T)} < \frac{1}{K}\right) &= I\!\!P_S\left\{\sigma_F W_S(T) - \tfrac{1}{2}\sigma_F^2 T < \log\frac{S(0)}{KB(0,T)}\right\} \\
&= I\!\!P_S\left\{\frac{W_S(T)}{\sqrt{T}} < \frac{1}{\sigma_F\sqrt{T}} \log\frac{S(0)}{KB(0,T)} + \tfrac{1}{2}\sigma_F\sqrt{T}\right\} \\
&= N(\rho_1),
\end{aligned}$$

where

$$\rho_1 = \frac{1}{\sigma_F\sqrt{T}}\left[\log\frac{S(0)}{KB(0,T)} + \tfrac{1}{2}\sigma_F^2 T\right].$$

Similarly,

$$\begin{aligned}
F(T) &= \frac{S(0)}{B(0,T)} \exp\left\{\sigma_F W_T(T) - \tfrac{1}{2}\sigma_F^2 T\right\}, \\
I\!\!P_T\{F(T) > K\} &= I\!\!P_T\left\{\sigma_F W_T(T) - \tfrac{1}{2}\sigma_F^2 T > \log\frac{KB(0,T)}{S(0)}\right\} \\
&= I\!\!P_T\left\{\frac{W_T(T)}{\sqrt{T}} > \frac{1}{\sigma_F\sqrt{T}}\left[\log\frac{KB(0,T)}{S(0)} + \tfrac{1}{2}\sigma_F^2 T\right]\right\} \\
&= I\!\!P_T\left\{\frac{-W_T(T)}{\sqrt{T}} < \frac{1}{\sigma_F\sqrt{T}}\left[\log\frac{S(0)}{KB(0,T)} - \tfrac{1}{2}\sigma_F^2 T\right]\right\} \\
&= N(\rho_2),
\end{aligned}$$

where

$$\rho_2 = \frac{1}{\sigma_F\sqrt{T}}\left[\log\frac{S(0)}{KB(0,T)} - \tfrac{1}{2}\sigma_F^2 T\right].$$

If r is constant, then $B(0,T) = e^{-rT}$,

$$\begin{aligned}
\rho_1 &= \frac{1}{\sigma_F\sqrt{T}}\left[\log\frac{S(0)}{K} + (r + \tfrac{1}{2}\sigma_F^2)T\right], \\
\rho_2 &= \frac{1}{\sigma_F\sqrt{T}}\left[\log\frac{S(0)}{K} + (r - \tfrac{1}{2}\sigma_F^2)T\right],
\end{aligned}$$

and we have the usual Black-Scholes formula. When r is not constant, we still have the explicit formula

$$V(0) = S(0)N(\rho_1) - KB(0,T)N(\rho_2).$$

As this formula suggests, if σ_F is constant, then for $0 \le t \le T$, the value of a European call expiring at time T is

$$V(t) = S(t)N(\rho_1(t)) - KB(t,T)N(\rho_2(t)),$$

where

$$\rho_1(t) = \frac{1}{\sigma_F\sqrt{T-t}}\left[\log\frac{F(t)}{K} + \tfrac{1}{2}\sigma_F^2(T-t)\right],$$
$$\rho_2(t) = \frac{1}{\sigma_F\sqrt{T-t}}\left[\log\frac{F(t)}{K} - \tfrac{1}{2}\sigma_F^2(T-t)\right].$$

This formula also suggests a hedge: at each time t, hold $N(\rho_1(t))$ shares of stock and short $KN(\rho_2(t))$ bonds.

We want to verify that this hedge is *self-financing.* Suppose we begin with \$ $V(0)$ and at each time t hold $N(\rho_1(t))$ shares of stock. We short bonds as necessary to finance this. Will the position in the bond always be $-KN(\rho_2(t))$? If so, the value of the portfolio will always be

$$S(t)N(\rho_1(t)) - KB(t,T)N(\rho_2(t)) = V(t),$$

and we will have a hedge.

Mathematically, this question takes the following form. Let

$$\Delta(t) = N(\rho_1(t)).$$

At time t, hold $\Delta(t)$ shares of stock. If $X(t)$ is the value of the portfolio at time t, then $X(t) - \Delta(t)S(t)$ will be invested in the bond, so the number of bonds owned is $\frac{X(t)-\Delta(t)}{B(t,T)}S(t)$ and the portfolio value evolves according to

$$dX(t) = \Delta(t)\; dS(t) + \frac{X(t)-\Delta(t)}{B(t,T)}S(t)\; dB(t,T). \tag{3.1}$$

The value of the option evolves according to

$$\begin{aligned} dV(t) &= N(\rho_1(t))\; dS(t) + S(t)\; dN(\rho_1(t)) + dS(t)\; dN(\rho_1(t)) \\ &\quad - KN(\rho_2(t))\; dB(t,T) - K\; dB(t,T)\; dN(\rho_2(t)) - KB(t,T)\; dN(\rho_2(t)). \end{aligned} \tag{3.2}$$

If $X(0) = V(0)$, will $X(t) = V(t)$ for $0 \le t \le T$?

Formulas (3.1) and (3.2) are difficult to compare, so we simplify them by a change of numéraire. This change is justified by the following theorem.

Theorem 3.73 *Changes of numéraire affect portfolio values in the way you would expect.*

Proof: Suppose we have a model with k assets with prices $S_1, S_2, \ldots, S_k$. At each time t, hold $\Delta_i(t)$ shares of asset i, $i = 1, 2, \ldots, k-1$, and invest the remaining wealth in asset k. Begin with a nonrandom initial wealth $X(0)$, and let $X(t)$ be the value of the portfolio at time t. The number of shares of asset k held at time t is

$$\Delta_k(t) = \frac{\left(X(t) - \sum_{i=1}^{k-1}\Delta_i(t)S_i(t)\right)}{S_k(t)},$$

and X evolves according to the equation

$$\begin{aligned} dX &= \sum_{i=1}^{k-1} \Delta_i \; dS_i + \left(X - \sum_{i=1}^{k-1} \Delta_i S_i \right) \frac{dS_k}{S_k} \\ &= \sum_{i=1}^{k} \Delta_i \; dS_i. \end{aligned}$$

Note that

$$X_k(t) = \sum_{i=1}^{k} \Delta_i(t) S_i(t),$$

and we only get to specify $\Delta_1, \ldots, \Delta_{k-1}$, not Δ_k, in advance.

Let N be a numéraire, and define

$$\widehat{X}(t) = \frac{X(t)}{N(t)}, \quad \widehat{S_i}(t) = \frac{S_i(t)}{N(t)}, \quad i = 1, 2, \ldots, k.$$

Then

$$\begin{aligned} d\widehat{X} &= \frac{1}{N} \; dX + X \; d\left(\frac{1}{N}\right) + dX \; d\left(\frac{1}{N}\right) \\ &= \frac{1}{N} \sum_{i=1}^{k} \Delta_i \; dS_i + \left(\sum_{i=1}^{k} \Delta_i S_i \right) d\left(\frac{1}{N}\right) + \sum_{i=1}^{k} \Delta_i \; dS_i \; d\left(\frac{1}{N}\right) \\ &= \sum_{i=1}^{k} \Delta_i \left(\frac{1}{N} \; dS_i + S_i d\left(\frac{1}{N}\right) + dS_i \; d\left(\frac{1}{N}\right) \right) \\ &= \sum_{i=1}^{k} \Delta_i \; d\widehat{S_i}. \end{aligned}$$

Now

$$\begin{aligned} \Delta_k &= \frac{\left(X - \sum_{i=1}^{k-1} \Delta_i S_i \right)}{S_k} \\ &= \frac{\left(X/N - \sum_{i=1}^{k-1} \Delta_i S_i / N \right)}{S_k / N} \\ &= \frac{\widehat{X} - \sum_{i=1}^{k-1} \Delta_i \widehat{S_i}}{\widehat{S_k}}. \end{aligned}$$

Therefore,

$$d\widehat{X} = \sum_{i=1}^{k} \Delta_i \; d\widehat{S_i} + \left(\widehat{X} - \sum_{i=1}^{k-1} \Delta_i \widehat{S_i} \right) \frac{d\widehat{S_k}}{\widehat{S_k}}$$

This is the formula for the evolution of a portfolio which holds Δ_i shares of asset i, $i = 1, 2, \ldots, k-1$, and all assets and the portfolio are denominated in units of N. ■

We return to the European call hedging problem (comparison of (3.1) and (3.2)), but we now use the zero-coupon bond as numéraire. We still hold $\Delta(t) = N(\rho_1(t))$ shares of stock at each time t. In terms of the new numéraire, the asset values are

$$\text{Stock:} \quad \frac{S(t)}{B(t,T)} = F(t),$$

$$\text{Bond:} \quad \frac{B(t,T)}{B(t,T)} = 1.$$

The portfolio value evolves according to

$$d\widehat{X}(t) = \Delta(t)\ dF(t) + (\widehat{X}(t) - \Delta(t))\frac{d(1)}{1} = \Delta(t)\ dF(t). \tag{3.1'}$$

In the new numéraire, the option value formula

$$V(t) = N(\rho_1(t))S(t) - KB(t,T)N(\rho_2(t))$$

becomes

$$\widehat{V}(t) = \frac{V(t)}{B(t,T)} = N(\rho_1(t))F(t) - KN(\rho_2(t)),$$

and

$$d\widehat{V} = N(\rho_1(t))\ dF(t) + F(t)\ dN(\rho_1(t)) + dN(\rho_1(t))\ dF(t) - K\ dN(\rho_2(t)). \tag{3.2'}$$

To show that the hedge works, we must show that

$$F(t)\ dN(\rho_1(t)) + dN(\rho_1(t))\ dF(t) - K\ dN(\rho_2(t)) = 0.$$

This is a homework problem.