Chapter 32

A two-factor model (Duffie & Kan)

Let us define:

$$X_1(t) = \text{Interest rate at time } t$$
$$X_2(t) = \text{Yield at time } t \text{ on a bond maturing at time } t + \tau_0$$

Let $X_1(0) > 0$, $X_2(0) > 0$ be given, and let $X_1(t)$ and $X_2(t)$ be given by the coupled stochastic differential equations

$$dX_1(t) = (a_{11}X_1(t) + a_{12}X_2(t) + b_1)\ dt + \sigma_1\sqrt{\beta_1 X_1(t) + \beta_2 X_2(t) + \alpha}\ \ dW_1(t), \tag{SDE1}$$
$$dX_2(t) = (a_{21}X_1(t) + a_{22}X_2(t) + b_2)\ dt + \sigma_2\sqrt{\beta_1 X_1(t) + \beta_2 X_2(t) + \alpha}\ (\rho\ dW_1(t) + \sqrt{1-\rho^2}\ dW_2(t)), \tag{SDE2}$$

where W_1 and W_2 are independent Brownian motions. To simplify notation, we define

$$Y(t) \stackrel{\triangle}{=} \beta_1 X_1(t) + \beta_2 X_2(t) + \alpha,$$
$$W_3(t) \stackrel{\triangle}{=} \rho W_1(t) + \sqrt{1-\rho^2}W_2(t).$$

Then W_3 is a Brownian motion with

$$dW_1(t)\ dW_3(t) = \rho\ dt,$$

and

$$dX_1\ dX_1 = \sigma_1^2 Y\ dt, \quad dX_2\ dX_2 = \sigma_2^2 Y\ dt, \quad dX_1\ dX_2 = \rho\sigma_1\sigma_2 Y\ dt.$$

32.1 Non-negativity of Y

$$\begin{aligned}
dY &= \beta_1\ dX_1 + \beta_2\ dX_2 \\
&= (\beta_1 a_{11} X_1 + \beta_1 a_{12} X_2 + \beta_1 b_1)\ dt + (\beta_2 a_{21} X_1 + \beta_2 a_{22} X_2 + \beta_2 b_2)\ dt \\
&\qquad + \sqrt{Y}(\beta_1 \sigma_1\ dW_1 + \beta_2 \rho \sigma_2\ dW_1 + \beta_2 \sqrt{1-\rho^2}\sigma_2\ dW_2) \\
&= [(\beta_1 a_{11} + \beta_2 a_{21}) X_1 + (\beta_1 a_{12} + \beta_2 a_{22}) X_2]\ dt + (\beta_1 b_1 + \beta_2 b_2)\ dt \\
&\qquad + (\beta_1^2 \sigma_1^2 + 2\beta_1\beta_2\rho\sigma_1\sigma_2 + \beta_2^2\sigma_2^2)^{\frac{1}{2}} \sqrt{Y(t)}\ dW_4(t)
\end{aligned}$$

where

$$W_4(t) = \frac{(\beta_1\sigma_1 + \beta_2\rho\sigma_2)W_1(t) + \beta_2\sqrt{1-\rho^2}\sigma_2 W_2(t)}{\sqrt{\beta_1^2\sigma_1^2 + 2\beta_1\beta_2\rho\sigma_1\sigma_2 + \beta_2^2\sigma_2^2}}$$

is a Brownian motion. We shall choose the parameters so that:

Assumption 1: For some γ, $\quad \beta_1 a_{11} + \beta_2 a_{21} = \gamma\beta_1, \quad \beta_1 a_{12} + \beta_2 a_{22} = \gamma\beta_2.$

Then

$$\begin{aligned}
dY &= [\gamma\beta_1 X_1 + \gamma\beta_2 X_2 + \alpha\gamma]\ dt + (\beta_1 b_1 + \beta_2 b_2 - \alpha\gamma)\ dt \\
&\qquad + (\beta_1^2\sigma_1^2 + 2\beta_1\beta_2\rho\sigma_1\sigma_2 + \beta_2^2\sigma_2^2)^{\frac{1}{2}}\sqrt{Y}\ dW_4 \\
&= \gamma Y\ dt + (\beta_1 b_1 + \beta_2 b_2 - \alpha\gamma)\ dt + (\beta_1^2\sigma_1^2 + 2\beta_1\beta_2\rho\sigma_1\sigma_2 + \beta_2^2\sigma_2^2)^{\frac{1}{2}}\sqrt{Y}\ dW_4.
\end{aligned}$$

From our discussion of the CIR process, we recall that Y will stay strictly positive provided that:

Assumption 2: $\quad Y(0) = \beta_1 X_1(0) + \beta_2 X_2(0) + \alpha > 0,$

and

Assumption 3: $\quad \beta_1 b_1 + \beta_2 b_2 - \gamma\alpha \geq \frac{1}{2}(\beta_1^2\sigma_1^2 + 2\beta_1\beta_2\rho\sigma_1\sigma_2 + \beta_2^2\sigma_2^2).$

Under Assumptions 1,2, and 3,

$$Y(t) > 0, \quad 0 \leq t < \infty, \text{ almost surely,}$$

and (SDE1) and (SDE2) make sense. These can be rewritten as

$$dX_1(t) = (a_{11}X_1(t) + a_{12}X_2(t) + b_1)\ dt + \sigma_1\sqrt{Y(t)}\ dW_1(t), \qquad \text{(SDE1')}$$

$$dX_2(t) = (a_{21}X_1(t) + a_{22}X_2(t) + b_2)\ dt + \sigma_2\sqrt{Y(t)}\ dW_3(t). \qquad \text{(SDE2')}$$

32.2 Zero-coupon bond prices

The value at time $t \leq T$ of a zero-coupon bond paying \$1 at time T is

$$B(t,T) = I\!\!E\left[\exp\left\{-\int_t^T X_1(u)\,du\right\}\Big|\mathcal{F}(t)\right].$$

Since the pair (X_1, X_2) of processes is Markov, this is random only through a dependence on $X_1(t), X_2(t)$. Since the coefficients in (SDE1) and (SDE2) do not depend on time, the bond price depends on t and T only through their difference $\tau = T - t$. Thus, there is a function $B(x_1, x_2, \tau)$ of the dummy variables x_1, x_2 and τ, so that

$$B(X_1(t), X_2(t), T-t) = I\!\!E\left[\exp\left\{-\int_t^T X_1(u)\,du\right\}\Big|\mathcal{F}(t)\right].$$

The usual tower property argument shows that

$$\exp\left\{-\int_0^t X_1(u)\,du\right\} B(X_1(t), X_2(t), T-t)$$

is a martingale. We compute its stochastic differential and set the dt term equal to zero.

$$\begin{aligned}
&d\left(\exp\left\{-\int_0^t X_1(u)\,du\right\} B(X_1(t), X_2(t), T-t)\right)\\
&= \exp\left\{-\int_0^t X_1(u)\,du\right\}\Big[-X_1 B\,dt + B_{x_1}\,dX_1 + B_{x_2}\,dX_2 - B_\tau\,dt\\
&\qquad + \tfrac{1}{2}B_{x_1x_1}\,dX_1\,dX_1 + B_{x_1x_2}\,dX_1\,dX_2 + \tfrac{1}{2}B_{x_2x_2}\,dX_2\,dX_2\Big]\\
&= \exp\left\{-\int_0^t X_1(u)\,du\right\}\Big[\Big(-X_1 B + (a_{11}X_1 + a_{12}X_2 + b_1)B_{x_1} + (a_{21}X_1 + a_{22}X_2 + b_2)B_{x_2} - B_\tau\\
&\qquad + \tfrac{1}{2}\sigma_1^2 Y B_{x_1x_1} + \rho\sigma_1\sigma_2 Y B_{x_1x_2} + \tfrac{1}{2}\sigma_2^2 Y B_{x_2x_2}\Big)\,dt\\
&\qquad + \sigma_1\sqrt{Y}B_{x_1}\,dW_1 + \sigma_2\sqrt{Y}B_{x_2}\,dW_3\Big]
\end{aligned}$$

The partial differential equation for $B(x_1, x_2, \tau)$ is

$$\begin{aligned}
-x_1 B - B_\tau + (a_{11}x_1 + a_{12}x_2 + b_1)B_{x_1} + (a_{21}x_1 + a_{22}x_2 + b_2)B_{x_2} + \tfrac{1}{2}\sigma_1^2(\beta_1 x_1 + \beta_2 x_2 + \alpha)B_{x_1x_1}\\
+ \rho\sigma_1\sigma_2(\beta_1 x_1 + \beta_2 x_2 + \alpha)B_{x_1x_2} + \tfrac{1}{2}\sigma_2^2(\beta_1 x_1 + \beta_2 x_2 + \alpha)B_{x_2x_2} = 0. \quad \text{(PDE)}
\end{aligned}$$

We seek a solution of the form

$$B(x_1, x_2, \tau) = \exp\{-x_1 C_1(\tau) - x_2 C_2(\tau) - A(\tau)\},$$

valid for all $\tau \geq 0$ and all x_1, x_2 satisfying

$$\beta_1 x_1 + \beta_2 x_2 + \alpha > 0. \quad (*)$$

We must have

$$B(x_1, x_2, 0) = 1, \quad \forall x_1, x_2 \quad \text{satisfying (*)},$$

because $\tau = 0$ corresponds to $t = T$. This implies the initial conditions

$$C_1(0) = C_2(0) = A(0) = 0. \tag{IC}$$

We want to find $C_1(\tau), C_2(\tau), A(\tau)$ for $\tau > 0$. We have

$$\begin{aligned}
B_\tau(x_1, x_2, \tau) &= \left[-x_1 C_1'(\tau) - x_2 C_2'(\tau) - A'(\tau)\right] B(x_1, x_2, \tau), \\
B_{x_1}(x_1, x_2, \tau) &= -C_1(\tau) B(x_1, x_2, \tau), \\
B_{x_2}(x_1, x_2, \tau) &= -C_2(\tau) B(x_1, x_2, \tau), \\
B_{x_1 x_1}(x_1, x_2, \tau) &= C_1^2(\tau) B(x_1, x_2, \tau), \\
B_{x_1 x_2}(x_1, x_2, \tau) &= C_1(\tau) C_2(\tau) B(x_1, x_2, \tau), \\
B_{x_2 x_2}(x_1, x_2, \tau) &= C_2^2(\tau) B(x_1, x_2, \tau).
\end{aligned}$$

(PDE) becomes

$$\begin{aligned}
0 = B(x_1, x_2, \tau)\Big[&-x_1 + x_1 C_1'(\tau) + x_2 C_2'(\tau) + A'(\tau) - (a_{11}x_1 + a_{12}x_2 + b_1) C_1(\tau) \\
&- (a_{21}x_1 + a_{22}x_2 + b_2) C_2(\tau) \\
&+ \tfrac{1}{2}\sigma_1^2(\beta_1 x_1 + \beta_2 x_2 + \alpha) C_1^2(\tau) + \rho\sigma_1\sigma_2(\beta_1 x_1 + \beta_2 x_2 + \alpha) C_1(\tau) C_2(\tau) \\
&+ \tfrac{1}{2}\sigma_2^2(\beta_1 x_1 + \beta_2 x_2 + \alpha) C_2^2(\tau)\Big] \\
= x_1 B(x_1, x_2, \tau)\Big[&-1 + C_1'(\tau) - a_{11} C_1(\tau) - a_{21} C_2(\tau) \\
&+ \tfrac{1}{2}\sigma_1^2\beta_1 C_1^2(\tau) + \rho\sigma_1\sigma_2\beta_1 C_1(\tau) C_2(\tau) + \tfrac{1}{2}\sigma_2^2\beta_1 C_2^2(\tau)\Big] \\
+ x_2 B(x_1, x_2, \tau)\Big[&C_2'(\tau) - a_{12} C_1(\tau) - a_{22} C_2(\tau) \\
&+ \tfrac{1}{2}\sigma_1^2\beta_2 C_1^2(\tau) + \rho\sigma_1\sigma_2\beta_2 C_1(\tau) C_2(\tau) + \tfrac{1}{2}\sigma_2^2\beta_2 C_2^2(\tau)\Big] \\
+ B(x_1, x_2, \tau)\Big[&A'(\tau) - b_1 C_1(\tau) - b_2 C_2(\tau) \\
&+ \tfrac{1}{2}\sigma_1^2\alpha C_1^2(\tau) + \rho\sigma_1\sigma_2\alpha C_1(\tau) C_2(\tau) + \tfrac{1}{2}\sigma_2^2\alpha C_2^2(\tau)\Big]
\end{aligned}$$

We get three equations:

$$C_1'(\tau) = 1 + a_{11} C_1(\tau) + a_{21} C_2(\tau) - \tfrac{1}{2}\sigma_1^2\beta_1 C_1^2(\tau) - \rho\sigma_1\sigma_2\beta_1 C_1(\tau) C_2(\tau) - \tfrac{1}{2}\sigma_2^2\beta_1 C_2^2(\tau), \tag{1}$$

$$C_1(0) = 0;$$

$$C_2'(\tau) = a_{12} C_1(\tau) + a_{22} C_2(\tau) - \tfrac{1}{2}\sigma_1^2\beta_2 C_1^2(\tau) - \rho\sigma_1\sigma_2\beta_2 C_1(\tau) C_2(\tau) - \tfrac{1}{2}\sigma_2^2\beta_2 C_2^2(\tau), \tag{2}$$

$$C_2(0) = 0;$$

$$A'(\tau) = b_1 C_1(\tau) + b_2 C_2(\tau) - \tfrac{1}{2}\sigma_1^2\alpha C_1^2(\tau) - \rho\sigma_1\sigma_2\alpha C_1(\tau) C_2(\tau) - \tfrac{1}{2}\sigma_2^2\alpha C_2^2(\tau), \tag{3}$$

$$A(0) = 0;$$

We first solve (1) and (2) simultaneously numerically, and then integrate (3) to obtain the function $A(\tau)$.

32.3 Calibration

Let $\tau_0 > 0$ be given. The value at time t of a bond maturing at time $t + \tau_0$ is

$$B(X_1(t), X_2(t), \tau_0) = \exp\{-X_1(t)C_1(\tau_0) - X_2(t)C_2(\tau_0) - A(\tau_0)\}$$

and the yield is

$$-\frac{1}{\tau_0} \log B(X_1(t), X_2(t), \tau_0) = \frac{1}{\tau_0} \left[X_1(t)C_1(\tau_0) + X_2(t)C_2(\tau_0) + A(\tau_0)\right].$$

But we have set up the model so that $X_2(t)$ is the yield at time t of a bond maturing at time $t + \tau_0$. Thus

$$X_2(t) = \frac{1}{\tau_0} \left[X_1(t)C_1(\tau_0) + X_2(t)C_2(\tau_0) + A(\tau_0)\right].$$

This equation must hold for every value of $X_1(t)$ and $X_2(t)$, which implies that

$$C_1(\tau_0) = 0, \;\; C_2(\tau_0) = \tau_0, \;\; A(\tau) = 0.$$

We must choose the parameters

$$a_{11}, a_{12}, b_1; \;\; a_{21}, a_{22}, b_2; \;\; \beta_1, \beta_2, \alpha; \;\; \sigma_1, \rho, \sigma_2;$$

so that these three equations are satisfied.