Chapter 31

Cox-Ingersoll-Ross model

In the Hull & White model, $r(t)$ is a Gaussian process. Since, for each t, $r(t)$ is normally distributed, there is a positive probability that $r(t) < 0$. The Cox-Ingersoll-Ross model is the simplest one which avoids negative interest rates.

We begin with a d-dimensional Brownian motion $(W_1, W_2, \ldots, W_d)$. Let $\beta > 0$ and $\sigma > 0$ be constants. For $j = 1, \ldots, d$, let $X_j(0) \in I\!\!R$ be given so that

$$X_1^2(0) + X_2^2(0) + \ldots + X_d^2(0) \geq 0,$$

and let X_j be the solution to the stochastic differential equation

$$dX_j(t) = -\tfrac{1}{2}\beta X_j(t)\ dt + \tfrac{1}{2}\sigma\ dW_j(t).$$

X_j is called the *Orstein-Uhlenbeck* process. It always has a drift toward the origin. The solution to this stochastic differential equation is

$$X_j(t) = e^{-\frac{1}{2}\beta t}\left[X_j(0) + \tfrac{1}{2}\sigma \int_0^t e^{\frac{1}{2}\beta u}\ dW_j(u)\right].$$

This solution is a Gaussian process with mean function

$$m_j(t) = e^{-\frac{1}{2}\beta t} X_j(0)$$

and covariance function

$$\rho(s,t) = \frac{1}{4}\sigma^2 e^{-\frac{1}{2}\beta(s+t)} \int_0^{s\wedge t} e^{\beta u}\ du.$$

Define

$$r(t) \stackrel{\triangle}{=} X_1^2(t) + X_2^2(t) + \ldots + X_d^2(t).$$

If $d = 1$, we have $r(t) = X_1^2(t)$ and for each t, $I\!\!P\{r(t) > 0\} = 1$, but (see Fig. 31.1)

$$I\!\!P\left\{\text{There are infinitely many values of } t > 0 \text{ for which } r(t) = 0\right\} = 1$$

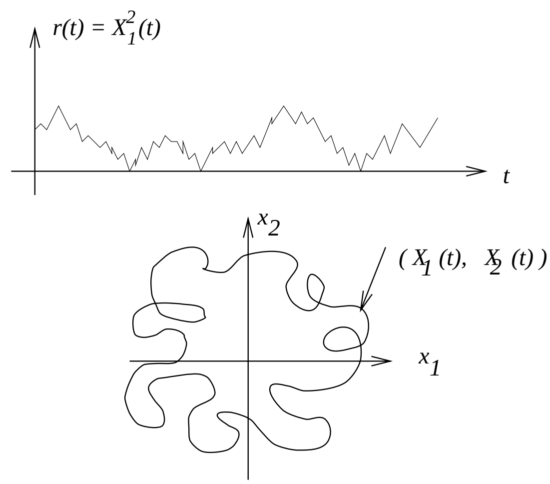

Figure 31.1: $r(t)$ *can be zero.*

If $d \geq 2$, (see Fig. 31.1)

$$\mathbb{P}\{\text{There is at least one value of } t > 0 \text{ for which } r(t) = 0\} \;=\; 0.$$

Let $f(x_1, x_2, \ldots, x_d) = x_1^2 + x_2^2 + \ldots + x_d^2$. Then

$$f_{x_i} = 2x_i, \quad f_{x_i x_j} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Itô's formula implies

$$\begin{aligned} dr(t) &= \sum_{i=1}^{d} f_{x_i}\, dX_i + \tfrac{1}{2} \sum_{i=1}^{d} f_{x_i x_i}\, dX_i\, dX_i \\ &= \sum_{i=1}^{d} 2X_i \left(-\tfrac{1}{2}\beta X_i\, dt + \tfrac{1}{2}\sigma\, dW_i(t)\right) + \sum_{i=1}^{d} \frac{1}{4}\sigma^2\, dW_i\, dW_i \\ &= -\beta r(t)\, dt + \sigma \sum_{i=1}^{d} X_i\, dW_i + \frac{d\sigma^2}{4}\, dt \\ &= \left(\frac{d\sigma^2}{4} - \beta r(t)\right) dt + \sigma\sqrt{r(t)} \sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}}\, dW_i(t). \end{aligned}$$

Define

$$W(t) = \sum_{i=1}^{d} \int_0^t \frac{X_i(u)}{\sqrt{r(u)}}\, dW_i(u).$$

Then W is a martingale,

$$dW = \sum_{i=1}^{d} \frac{X_i}{\sqrt{r}}\, dW_i,$$
$$dW\, dW = \sum_{i=1}^{d} \frac{X_i^2}{r}\, dt = dt,$$

so W is a Brownian motion. We have

$$dr(t) = \left(\frac{d\sigma^2}{4} - \beta r(t) \right)\, dt + \sigma\sqrt{r(t)}\, dW(t).$$

The *Cox-Ingersoll-Ross (CIR) process* is given by

$$dr(t) = (\alpha - \beta r(t))\, dt + \sigma\sqrt{r(t)}\, dW(t),$$

We define

$$d = \frac{4\alpha}{\sigma^2} > 0.$$

If d happens to be an integer, then we have the representation

$$r(t) = \sum_{i=1}^{d} X_i^2(t),$$

but we do not require d to be an integer. If $d < 2$ (i.e., $\alpha < \frac{1}{2}\sigma^2$), then

$$I\!P\{\text{There are infinitely many values of } t > 0 \text{ for which } r(t) = 0\} \;=\; 1.$$

This is not a good parameter choice.

If $d \geq 2$ (i.e., $\alpha \geq \frac{1}{2}\sigma^2$), then

$$I\!P\{\text{There is at least one value of } t > 0 \text{ for which } r(t) = 0\} \;=\; 0.$$

With the CIR process, one can derive formulas under the assumption that $d = \frac{4\alpha}{\sigma^2}$ is a positive integer, and they are still correct even when d is not an integer.

For example, here is the distribution of $r(t)$ for fixed $t > 0$. Let $r(0) \geq 0$ be given. Take

$$X_1(0) = 0,\ X_2(0) = 0,\ \ldots,\ X_{d-1}(0) = 0,\ X_d(0) = \sqrt{r(0)}.$$

For $i = 1, 2, \ldots, d-1$, $X_i(t)$ is normal with mean zero and variance

$$\rho(t,t) = \frac{\sigma^2}{4\beta}(1 - e^{-\beta t}).$$

$X_d(t)$ is normal with mean

$$m_d(t) = e^{-\frac{1}{2}\beta t}\sqrt{r(0)}$$

and variance $\rho(t,t)$. Then

$$r(t) = \underbrace{\rho(t,t)\sum_{i=1}^{d-1}\left(\frac{X_i(t)}{\sqrt{\rho(t,t)}}\right)^2}_{\text{Chi-square with } d-1=\frac{4\alpha-\sigma^2}{\sigma^2} \text{ degrees of freedom}} + \underbrace{X_d^2(t)}_{\text{Normal squared and independent of the other term}} \tag{0.1}$$

Thus $r(t)$ has a *non-central chi-square distribution.*

31.1 Equilibrium distribution of $r(t)$

As $t\to\infty$, $m_d(t)\to 0$. We have

$$r(t) = \rho(t,t)\sum_{i=1}^{d}\left(\frac{X_i(t)}{\sqrt{\rho(t,t)}}\right)^2.$$

As $t\to\infty$, we have $\rho(t,t) = \frac{\sigma^2}{4\beta}$, and so the limiting distribution of $r(t)$ is $\frac{\sigma^2}{4\beta}$ times a chi-square with $d = \frac{4\alpha}{\sigma^2}$ degrees of freedom. The chi-square density with $\frac{4\alpha}{\sigma^2}$ degrees of freedom is

$$f(y) = \frac{1}{2^{2\alpha/\sigma^2}\Gamma\left(\frac{2\alpha}{\sigma^2}\right)} y^{\frac{2\alpha-\sigma^2}{\sigma^2}} e^{-y/2}.$$

We make the change of variable $r = \frac{\sigma^2}{4\beta}y$. The limiting density for $r(t)$ is

$$\begin{aligned} p(r) &= \frac{4\beta}{\sigma^2}.\frac{1}{2^{2\alpha/\sigma^2}\Gamma\left(\frac{2\alpha}{\sigma^2}\right)}\left(\frac{4\beta}{\sigma^2}r\right)^{\frac{2\alpha-\sigma^2}{\sigma^2}} e^{-\frac{2\beta}{\sigma^2}r} \\ &= \left(\frac{2\beta}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}} \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2}\right)} r^{\frac{2\alpha-\sigma^2}{\sigma^2}} e^{-\frac{2\beta}{\sigma^2}r}. \end{aligned}$$

We computed the mean and variance of $r(t)$ in Section 15.7.

31.2 Kolmogorov forward equation

Consider a Markov process governed by the stochastic differential equation

$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t).$$

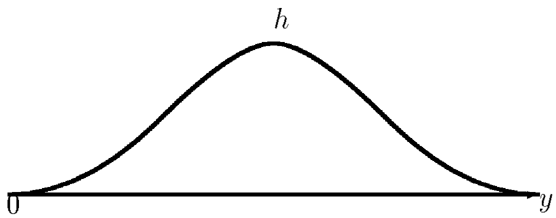

Figure 31.2: *The function* $h(y)$

Because we are going to apply the following analysis to the case $X(t) = r(t)$, we assume that $X(t) \geq 0$ for all t.

We start at $X(0) = x \geq 0$ at time 0. Then $X(t)$ is random with density $p(0,t,x,y)$ (in the y variable). Since 0 and x will not change during the following, we omit them and write $p(t,y)$ rather than $p(0,t,x,y)$. We have

$$I\!\!E h(X(t)) = \int_0^\infty h(y)p(t,y)\;dy$$

for any function h.

The Kolmogorov forward equation (KFE) is a partial differential equation in the "forward" variables t and y. We derive it below.

Let $h(y)$ be a smooth function of $y \geq 0$ which vanishes near $y = 0$ and for all large values of y (see Fig. 31.2). Itô's formula implies

$$dh(X(t)) = \left[h'(X(t))b(X(t)) + \tfrac{1}{2}h''(X(t))\sigma^2(X(t))\right]\;dt + h'(X(t))\sigma(X(t))\;dW(t),$$

so

$$\begin{aligned} h(X(t)) &= h(X(0)) + \int_0^t \left[h'(X(s))b(X(s)) + \tfrac{1}{2}h''(X(s))\sigma^2(X(s))\right]\;ds + \\ &\quad \int_0^t h'(X(s))\sigma(X(s))\;dW(s), \\ I\!\!E h(X(t)) &= h(X(0)) + I\!\!E\int_0^t \left[h'(X(s))b(X(s))\;dt + \tfrac{1}{2}h''(X(s))\sigma^2(X(s))\right]\;ds, \end{aligned}$$

or equivalently,

$$\int_0^\infty h(y)p(t,y)\,dy = h(X(0)) + \int_0^t \int_0^\infty h'(y)b(y)p(s,y)\,dy\,ds + \frac{1}{2}\int_0^t \int_0^\infty h''(y)\sigma^2(y)p(s,y)\,dy\,ds.$$

Differentiate with respect to t to get

$$\int_0^\infty h(y)p_t(t,y)\,dy = \int_0^\infty h'(y)b(y)p(t,y)\,dy + \frac{1}{2}\int_0^\infty h''(y)\sigma^2(y)p(t,y)\,dy.$$

Integration by parts yields

$$\int_0^\infty h'(y)b(y)p(t,y)\,dy = \underbrace{h(y)b(y)p(t,y)\Big|_{y=0}^{y=\infty}}_{=0} - \int_0^\infty h(y)\frac{\partial}{\partial y}\left(b(y)p(t,y)\right)\,dy,$$

$$\begin{aligned}\int_0^\infty h''(y)\sigma^2(y)p(t,y)\,dy &= \underbrace{h'(y)\sigma^2(y)p(t,y)\Big|_{y=0}^{y=\infty}}_{=0} - \int_0^\infty h'(y)\frac{\partial}{\partial y}\left(\sigma^2(y)p(t,y)\right)\,dy \\ &= \underbrace{-h(y)\frac{\partial}{\partial y}\left(\sigma^2(y)p(t,y)\right)\Big|_{y=0}^{y=\infty}}_{=0} + \int_0^\infty h(y)\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y)\right)\,dy.\end{aligned}$$

Therefore,

$$\int_0^\infty h(y)p_t(t,y)\,dy = -\int_0^\infty h(y)\frac{\partial}{\partial y}\left(b(y)p(t,y)\right)\,dy + \frac{1}{2}\int_0^\infty h(y)\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y)\right)\,dy,$$

or equivalently,

$$\int_0^\infty h(y)\left[p_t(t,y) + \frac{\partial}{\partial y}\left(b(y)p(t,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y)\right)\right]\,dy = 0.$$

This last equation holds for every function h of the form in Figure 31.2. It implies that

$$p_t(t,y) + \frac{\partial}{\partial y}\left((b(y)p(t,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y)\right) = 0. \qquad \text{(KFE)}$$

If there were a place where (KFE) did not hold, then we could take $h(y) > 0$ at that and nearby points, but take h to be zero elsewhere, and we would obtain

$$\int_0^\infty h\left[p_t + \frac{\partial}{\partial y}(bp) - \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2 p)\right]\,dy \neq 0.$$

If the process $X(t)$ has an equilibrium density, it will be

$$p(y) = \lim_{t \to \infty} p(t, y).$$

In order for this limit to exist, we must have

$$0 = \lim_{t \to \infty} p_t(t, y).$$

Letting $t \to \infty$ in (KFE), we obtain the equilibrium Kolmogorov forward equation

$$\frac{\partial}{\partial y}\left(b(y)p(y)\right) - \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(y)\right) = 0.$$

When an equilibrium density exists, it is the unique solution to this equation satisfying

$$p(y) \geq 0 \quad \forall y \geq 0,$$
$$\int_0^\infty p(y)\, dy = 1.$$

31.3 Cox-Ingersoll-Ross equilibrium density

We computed this to be

$$p(r) = Cr^{\frac{2\alpha - \sigma^2}{\sigma^2}} e^{-\frac{2\beta}{\sigma^2} r},$$

where

$$C = \left(\frac{2\beta}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}} \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2}\right)}.$$

We compute

$$\begin{aligned}
p'(r) &= \frac{2\alpha - \sigma^2}{\sigma^2} . \frac{p(r)}{r} - \frac{2\beta}{\sigma^2} p(r) \\
&= \frac{2}{\sigma^2 r}\left(\alpha - \tfrac{1}{2}\sigma^2 - \beta r\right) p(r), \\
p''(r) &= -\frac{2}{\sigma^2 r^2}\left(\alpha - \tfrac{1}{2}\sigma^2 - \beta r\right) p(r) + \frac{2}{\sigma^2 r}(-\beta) p(r) + \frac{2}{\sigma^2 r}\left(\alpha - \tfrac{1}{2}\sigma^2 - \beta r\right) p'(r) \\
&= \frac{2}{\sigma^2 r}\left(-\frac{1}{r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r) - \beta + \frac{2}{\sigma^2 r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r)^2\right)\ p(r)
\end{aligned}$$

We want to verify the equilibrium Kolmogorov forward equation for the CIR process:

$$\frac{\partial}{\partial r}\left((\alpha - \beta r)p(r)\right) - \tfrac{1}{2}\frac{\partial^2}{\partial r^2}(\sigma^2 r p(r)) = 0. \qquad \text{(EKFE)}$$

Now

$$\frac{\partial}{\partial r}\left((\alpha-\beta r)p(r)\right) = -\beta p(r) + (\alpha-\beta r)p'(r),$$
$$\frac{\partial^2}{\partial r^2}(\sigma^2 r p(r)) = \frac{\partial}{\partial r}(\sigma^2 p(r) + \sigma^2 r p'(r))$$
$$= 2\sigma^2 p'(r) + \sigma^2 r p''(r).$$

The LHS of (EKFE) becomes

$$\begin{aligned}
&-\beta p(r) + (\alpha-\beta r)p'(r) - \sigma^2 p'(r) - \tfrac{1}{2}\sigma^2 r p''(r)\\
&= p(r)\Big[-\beta + (\alpha-\beta r-\sigma^2)\frac{2}{\sigma^2 r}(\alpha-\tfrac{1}{2}\sigma^2-\beta r)\\
&\qquad + \frac{1}{r}(\alpha-\tfrac{1}{2}\sigma^2-\beta r) + \beta - \frac{2}{\sigma^2 r}(\alpha-\tfrac{1}{2}\sigma^2-\beta r)^2\Big]\\
&= p(r)\Big[(\alpha-\tfrac{1}{2}\sigma^2-\beta r)\frac{2}{\sigma^2 r}(\alpha-\tfrac{1}{2}\sigma^2-\beta r)\\
&\qquad - \tfrac{1}{2}\sigma^2\frac{2}{\sigma^2 r}(\alpha-\tfrac{1}{2}\sigma^2-\beta r)\\
&\qquad + \frac{1}{r}(\alpha-\tfrac{1}{2}\sigma^2-\beta r) - \frac{2}{\sigma^2 r}(\alpha-\tfrac{1}{2}\sigma^2-\beta r)^2\Big]\\
&= 0,
\end{aligned}$$

as expected.

31.4 Bond prices in the CIR model

The interest rate process $r(t)$ is given by

$$dr(t) = (\alpha - \beta r(t))\, dt + \sigma\sqrt{r(t)}\, dW(t),$$

where $r(0)$ is given. The bond price process is

$$B(t,T) = I\!\!E\left[\exp\left\{-\int_t^T r(u)\, du\right\}\Big|\mathcal{F}(t)\right].$$

Because

$$\exp\left\{-\int_0^t r(u)\, du\right\} B(t,T) = I\!\!E\left[\exp\left\{-\int_0^T r(u)\, du\right\}\Big|\mathcal{F}(t)\right],$$

the tower property implies that this is a martingale. The Markov property implies that $B(t,T)$ is random only through a dependence on $r(t)$. Thus, there is a function $B(r,t,T)$ of the three dummy variables r, t, T such that the *process* $B(t,T)$ is the *function* $B(r,t,T)$ evaluated at $r(t), t, T$, i.e.,

$$B(t,T) = B(r(t),t,T).$$

Because $\exp\left\{-\int_0^t r(u)\,du\right\} B(r(t),t,T)$ is a martingale, its differential has no dt term. We compute

$$\begin{aligned} &d\left(\exp\left\{-\int_0^t r(u)\,du\right\} B(r(t),t,T)\right)\\ &= \exp\left\{-\int_0^t r(u)\,du\right\}\Big[-r(t)B(r(t),t,T)\,dt + B_r(r(t),t,T)\,dr(t) + \\ &\qquad \tfrac{1}{2}B_{rr}(r(t),t,T)\,dr(t)\,dr(t) + B_t(r(t),t,T)\,dt\Big]. \end{aligned}$$

The expression in $[\dots]$ equals

$$\begin{aligned} &= -rB\,dt + B_r(\alpha-\beta r)\,dt + B_r\sigma\sqrt{r}\,dW \\ &\qquad + \tfrac{1}{2}B_{rr}\sigma^2 r\,dt + B_t\,dt. \end{aligned}$$

Setting the dt term to zero, we obtain the partial differential equation

$$\begin{aligned} -\,rB(r,t,T) + B_t(r,t,T) + (\alpha-\beta r)B_r(r,t,T) + \tfrac{1}{2}\sigma^2 r B_{rr}(r,t,T) = 0,\\ 0 \le t < T, \quad r \ge 0. \end{aligned} \tag{4.1}$$

The terminal condition is

$$B(r,T,T) = 1, \quad r \ge 0.$$

Surprisingly, this equation has a closed form solution. Using the Hull & White model as a guide, we look for a solution of the form

$$B(r,t,T) = e^{-rC(t,T)-A(t,T)},$$

where $C(T,T)=0,\ A(T,T)=0$. Then we have

$$\begin{aligned} B_t &= (-rC_t - A_t)B,\\ B_r &= -CB, \quad B_{rr} = C^2 B, \end{aligned}$$

and the partial differential equation becomes

$$\begin{aligned} 0 &= -rB + (-rC_t - A_t)B - (\alpha-\beta r)CB + \tfrac{1}{2}\sigma^2 r C^2 B\\ &= rB(-1 - C_t + \beta C + \tfrac{1}{2}\sigma^2 C^2) - B(A_t + \alpha C) \end{aligned}$$

We first solve the ordinary differential equation

$$-1 - C_t(t,T) + \beta C(t,T) + \tfrac{1}{2}\sigma^2 C^2(t,T) = 0; \quad C(T,T) = 0,$$

and then set

$$A(t,T) = \alpha\int_t^T C(u,T)\,du,$$

so $A(T,T)=0$ and

$$A_t(t,T) = -\alpha C(t,T).$$

It is tedious but straightforward to check that the solutions are given by

$$C(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}\beta\sinh(\gamma(T-t))},$$

$$A(t,T) = -\frac{2\alpha}{\sigma^2}\log\left[\frac{\gamma e^{\frac{1}{2}\beta(T-t)}}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}\beta\sinh(\gamma(T-t))}\right],$$

where

$$\gamma = \tfrac{1}{2}\sqrt{\beta^2 + 2\sigma^2}, \quad \sinh u = \frac{e^u - e^{-u}}{2}, \quad \cosh u = \frac{e^u + e^{-u}}{2}.$$

Thus in the CIR model, we have

$$I\!\!E\left[\exp\left\{-\int_t^T r(u)\,du\right\}\middle|\mathcal{F}(t)\right] = B(r(t),t,T),$$

where

$$B(r,t,T) = \exp\left\{-rC(t,T) - A(t,T)\right\}, \quad 0 \le t < T, \;\; r \ge 0,$$

and $C(t,T)$ and $A(t,T)$ are given by the formulas above. Because the coefficients in

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t)$$

do not depend on t, the function $B(r,t,T)$ depends on t and T only through their difference $\tau = T-t$. Similarly, $C(t,T)$ and $A(t,T)$ are functions of $\tau = T-t$. We write $B(r,\tau)$ instead of $B(r,t,T)$, and we have

$$B(r,\tau) = \exp\left\{-rC(\tau) - A(\tau)\right\}, \quad \tau \ge 0, \;\; r \ge 0,$$

where

$$C(\tau) = \frac{\sinh(\gamma\tau)}{\gamma\cosh(\gamma\tau) + \frac{1}{2}\beta\sinh(\gamma\tau)},$$

$$A(\tau) = -\frac{2\alpha}{\sigma^2}\log\left[\frac{\gamma e^{\frac{1}{2}\beta\tau}}{\gamma\cosh(\gamma\tau) + \frac{1}{2}\beta\sinh(\gamma\tau)}\right],$$

$$\gamma = \tfrac{1}{2}\sqrt{\beta^2 + 2\sigma^2}.$$

We have

$$B(r(0),T) = I\!\!E\exp\left\{-\int_0^T r(u)\,du\right\}.$$

Now $r(u) > 0$ for each u, almost surely, so $B(r(0),T)$ is strictly decreasing in T. Moreover,

$$B(r(0),0) = 1,$$

$$\lim_{T\to\infty} B(r(0),T) = I\!E \exp\left\{-\int_0^\infty r(u)\,du\right\} = 0.$$

But also,

$$B(r(0),T) = \exp\left\{-r(0)C(T) - A(T)\right\},$$

so

$$r(0)C(0) + A(0) = 0,$$
$$\lim_{T\to\infty} [r(0)C(T) + A(T)] = \infty,$$

and

$$r(0)C(T) + A(T)$$

is strictly inreasing in T.

31.5 Option on a bond

The value at time t of an option on a bond in the CIR model is

$$v(t,r(t)) = I\!E\left[\exp\left\{-\int_t^{T_1} r(u)\,du\right\}(B(T_1,T_2) - K)^+ \middle| \mathcal{F}(t)\right],$$

where T_1 is the expiration time of the option, T_2 is the maturity time of the bond, and $0 \le t \le T_1 \le T_2$. As usual, $\exp\left\{-\int_0^t r(u)\,du\right\} v(t,r(t))$ is a martingale, and this leads to the partial differential equation

$$-rv + v_t + (\alpha - \beta r)v_r + \tfrac{1}{2}\sigma^2 r v_{rr} = 0, \quad 0 \le t < T_1, \quad r \ge 0.$$

(where $v = v(t,r)$.) The terminal condition is

$$v(T_1,r) = (B(r,T_1,T_2) - K)^+, \quad r \ge 0.$$

Other European derivative securities on the bond are priced using the same partial differential equation with the terminal condition appropriate for the particular security.

31.6 Deterministic time change of CIR model

Process time scale: In this time scale, the interest rate $r(t)$ is given by the constant coefficient CIR equation

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t).$$

Real time scale: In this time scale, the interest rate $\hat{r}(\hat{t})$ is given by a time-dependent CIR equation

$$d\hat{r}(\hat{t}) = (\hat{\alpha}(\hat{t}) - \hat{\beta}(\hat{t})\hat{r}(\hat{t}))\,d\hat{t} + \hat{\sigma}(\hat{t})\sqrt{\hat{r}(\hat{t})}\,d\hat{W}(\hat{t}).$$

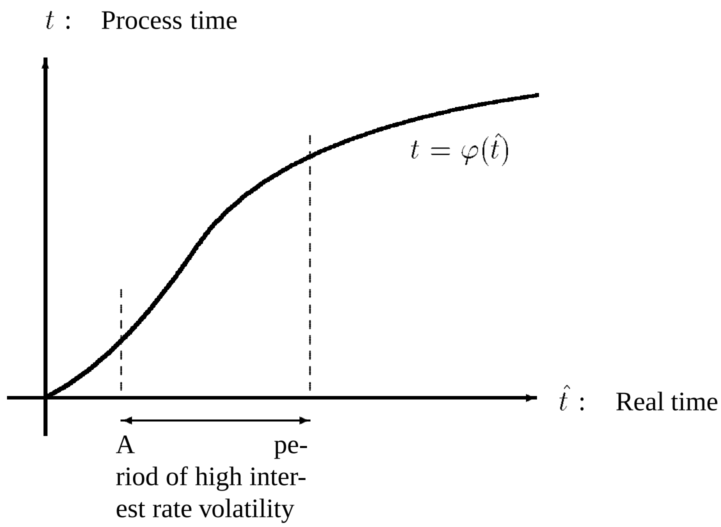

Figure 31.3: *Time change function.*

There is a strictly increasing time change function $t = \varphi(\hat{t})$ which relates the two time scales (See Fig. 31.3).

Let $\hat{B}(\hat{r}, \hat{t}, \hat{T})$ denote the price at real time $\hat{t}$ of a bond with maturity $\hat{T}$ when the interest rate at time $\hat{t}$ is $\hat{r}$. We want to set things up so

$$\hat{B}(\hat{r}, \hat{t}, \hat{T}) = B(r, t, T) = e^{-rC(t,T)-A(t,T)},$$

where $t = \varphi(\hat{t}), T = \varphi(\hat{T})$, and $C(t,T)$ and $A(t,T)$ are as defined previously.

We need to determine the relationship between $\hat{r}$ and r. We have

$$B(r(0), 0, T) = I\!E \exp\left\{ - \int_0^T r(t)\ dt \right\},$$

$$B(\hat{r}(0), 0, \hat{T}) = I\!E \exp\left\{ - \int_0^{\hat{T}} \hat{r}(\hat{t})\ d\hat{t} \right\}.$$

With $T = \varphi(\hat{T})$, make the change of variable $t = \varphi(\hat{t})$, $dt = \varphi'(\hat{t})\ d\hat{t}$ in the first integral to get

$$B(r(0), 0, T) = I\!E \exp\left\{ - \int_0^{\hat{T}} r(\varphi(\hat{t}))\varphi'(\hat{t})\ d\hat{t} \right\},$$

and this will be $B(\hat{r}(0), 0, \hat{T})$ if we set

$$\boxed{\hat{r}(\hat{t}) = r(\varphi(\hat{t}))\ \varphi'(\hat{t}).}$$

31.7 Calibration

$$\begin{aligned}\hat{B}(\hat{r}(\hat{t}),\hat{t},\hat{T}) &= B\left(\frac{\hat{r}(\hat{t})}{\varphi'(\hat{t})},\varphi(\hat{t}),\varphi(\hat{T})\right)\\ &= \exp\left\{-\hat{r}(\hat{t})\frac{C(\varphi(\hat{t}),\varphi(\hat{T}))}{\varphi'(\hat{t})}-A(\varphi(\hat{t}),\varphi(\hat{T}))\right\}\\ &= \exp\left\{-\hat{r}(\hat{t})\hat{C}(\hat{t},\hat{T})-\hat{A}(\hat{t},\hat{T})\right\},\end{aligned}$$

where

$$\begin{aligned}\hat{C}(\hat{t},\hat{T}) &= \frac{C(\varphi(\hat{t}),\varphi(\hat{T}))}{\varphi'(\hat{t})}\\ \hat{A}(\hat{t},\hat{T}) &= A(\varphi(\hat{t}),\varphi(\hat{T}))\end{aligned}$$

do *not* depend on $\hat{t}$ and $\hat{T}$ only through $\hat{T}-\hat{t}$, since, in the real time scale, the model coefficients are time dependent.

Suppose we know $\hat{r}(0)$ and $\hat{B}(\hat{r}(0),0,\hat{T})$ for all $\hat{T}\in[0,\hat{T}^*]$. We calibrate by writing the equation

$$\hat{B}(\hat{r}(0),0,\hat{T}) = \exp\left\{-\hat{r}(0)\hat{C}(0,\hat{T})-\hat{A}(0,\hat{T})\right\},$$

or equivalently,

$$-\log\hat{B}(\hat{r}(0),0,\hat{T}) = \frac{\hat{r}(0)}{\varphi'(0)}C(\varphi(0),\varphi(\hat{T}))+A(\varphi(0),\varphi(\hat{T})).$$

Take α,β and σ so the equilibrium distribution of $r(t)$ seems reasonable. These values determine the functions C,A. Take $\varphi'(0)=1$ (we justify this in the next section). For each $\hat{T}$, solve the equation for $\varphi(\hat{T})$:

$$-\log\hat{B}(\hat{r}(0),0,\hat{T}) = \hat{r}(0)C(0,\varphi(\hat{T}))+A(0,\varphi(\hat{T})). \quad (*)$$

The right-hand side of this equation is increasing in the $\varphi(\hat{T})$ variable, starting at 0 at time 0 and having limit ∞ at ∞, i.e.,

$$\begin{aligned}\hat{r}(0)C(0,0)+A(0,0)&=0,\\ \lim_{T\to\infty}\left[\hat{r}(0)C(0,T)+A(0,T)\right]&=\infty.\end{aligned}$$

Since $0\le-\log\hat{B}(\hat{r}(0),0,\hat{T})<\infty$, (*) has a unique solution for each $\hat{T}$. For $\hat{T}=0$, this solution is $\varphi(0)=0$. If $\hat{T}_1<\hat{T}_2$, then

$$-\log\hat{B}(r(0),0,\hat{T}_1) < -\log\hat{B}(r(0),0,\hat{T}_2),$$

so $\varphi(\hat{T}_1)<\varphi(\hat{T}_2)$. Thus φ is a strictly increasing time-change-function with the right properties.

31.8 Tracking down $\varphi'(0)$ in the time change of the CIR model

Result for general term structure models:

$$-\frac{\partial}{\partial T}\log B(0,T)\Big|_{T=0} = r(0).$$

Justification:

$$\begin{aligned}
B(0,T) &= I\!\!E \exp\left\{-\int_0^T r(u)\,du\right\}.\\
-\log B(0,T) &= -\log I\!\!E \exp\left\{-\int_0^T r(u)\,du\right\}\\
-\frac{\partial}{\partial T}\log B(0,T) &= \frac{I\!\!E\left[r(T)e^{-\int_0^T r(u)\,du}\right]}{I\!\!E e^{-\int_0^T r(u)\,du}}\\
-\frac{\partial}{\partial T}\log B(0,T)\Big|_{T=0} &= r(0).
\end{aligned}$$

In the real time scale associated with the calibration of CIR by time change, we write the bond price as

$$\hat{B}(\hat{r}(0),0,\hat{T}),$$

thereby indicating explicitly the initial interest rate. The above says that

$$-\frac{\partial}{\partial \hat{T}}\log \hat{B}(\hat{r}(0),0,\hat{T})\Big|_{\hat{T}=0} = \hat{r}(0).$$

The calibration of CIR by time change requires that we find a strictly increasing function φ with $\varphi(0)=0$ such that

$$-\log \hat{B}(\hat{r}(0),0,\hat{T}) = \frac{1}{\varphi'(0)}\hat{r}(0)C(\varphi(\hat{T})) + A(\varphi(\hat{T})), \quad \hat{T}\geq 0, \tag{cal}$$

where $\hat{B}(\hat{r}(0),0,\hat{T})$, determined by market data, is strictly increasing in $\hat{T}$, starts at 1 when $\hat{T}=0$, and goes to zero as $\hat{T}\to\infty$. Therefore, $-\log \hat{B}(\hat{r}(0),0,\hat{T})$ is as shown in Fig. 31.4.

Consider the function

$$\hat{r}(0)C(T) + A(T),$$

Here $C(T)$ and $A(T)$ are given by

$$\begin{aligned}
C(T) &= \frac{\sinh(\gamma T)}{\gamma\cosh(\gamma T) + \frac{1}{2}\beta\sinh(\gamma T)},\\
A(T) &= -\frac{2\alpha}{\sigma^2}\log\left[\frac{\gamma e^{\frac{1}{2}\beta T}}{\gamma\cosh(\gamma T)+\frac{1}{2}\beta\sinh(\gamma T)}\right],\\
\gamma &= \tfrac{1}{2}\sqrt{\beta^2+2\sigma^2}.
\end{aligned}$$

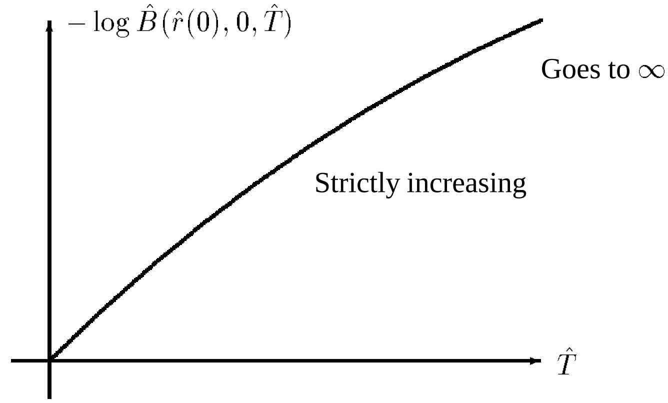

Figure 31.4: *Bond price in CIR model*

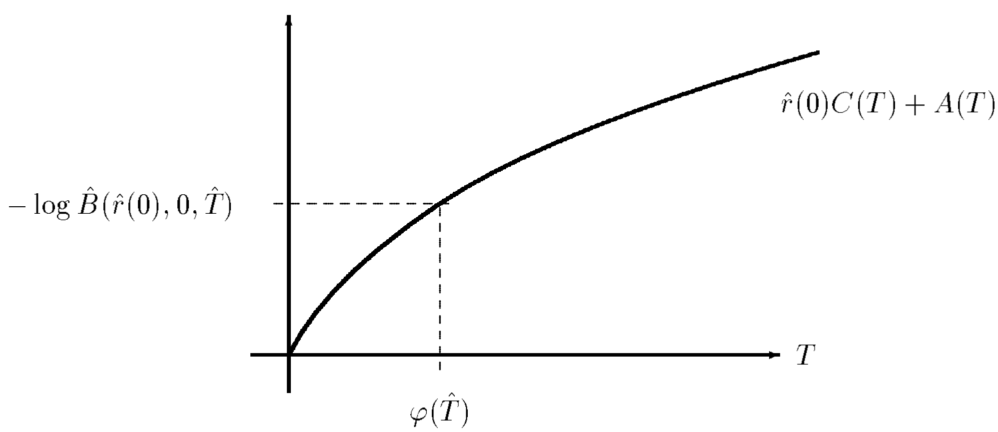

Figure 31.5: *Calibration*

The function $\hat{r}(0)C(T) + A(T)$ is zero at $T = 0$, is strictly increasing in T, and goes to ∞ as $T \to \infty$. This is because the interest rate is positive in the CIR model (see last paragraph of Section 31.4).

To solve (cal), let us first consider the related equation

$$-\log \hat{B}(\hat{r}(0), 0, \hat{T}) = \hat{r}(0)C(\varphi(\hat{T})) + A(\varphi(\hat{T})). \qquad \text{(cal')}$$

Fix $\hat{T}$ and define $\varphi(\hat{T})$ to be the unique T for which (see Fig. 31.5)

$$-\log \hat{B}(\hat{r}(0), 0, \hat{T}) = \hat{r}(0)C(T) + A(T)$$

If $\hat{T} = 0$, then $\varphi(\hat{T}) = 0$. If $\hat{T}_1 < \hat{T}_2$, then $\varphi(\hat{T}_1) < \varphi(\hat{T}_2)$. As $\hat{T} \to \infty$, $\varphi(\hat{T}) \to \infty$. We have thus defined a time-change function φ which has all the right properties, except it satisfies (cal') rather than (cal).

We conclude by showing that $\varphi'(0) = 1$ so φ also satisfies (cal). From (cal') we compute

$$\begin{aligned}\hat{r}(0) &= -\frac{\partial}{\partial \hat{T}} \log \hat{B}(\hat{r}(0), 0, \hat{T})\Big|_{\hat{T}=0} \\ &= \hat{r}(0)C'(\varphi(0))\varphi'(0) + A'(\varphi(0))\varphi'(0) \\ &= \hat{r}(0)C'(0)\varphi'(0) + A'(0)\varphi'(0).\end{aligned}$$

We show in a moment that $C'(0) = 1$, $A'(0) = 0$, so we have

$$\hat{r}(0) = \hat{r}(0)\varphi'(0).$$

Note that $\hat{r}(0)$ is the initial interest rate, observed in the market, and is striclty positive. Dividing by $\hat{r}(0)$, we obtain

$$\varphi'(0) = 1.$$

Computation of $C'(0)$:

$$\begin{aligned}C'(\tau) &= \frac{1}{\left(\gamma\cosh(\gamma\tau) + \frac{1}{2}\beta\sinh(\gamma\tau)\right)^2}\Big[\gamma\cosh(\gamma\tau)\left(\gamma\cosh(\gamma\tau) + \tfrac{1}{2}\beta\sinh(\gamma\tau)\right) \\ &\qquad - \sinh(\gamma\tau)\left(\gamma^2\sinh(\gamma\tau) + \tfrac{1}{2}\beta\gamma\cosh(\gamma\tau)\right)\Big] \\ C'(0) &= \frac{1}{\gamma^2}\left[\gamma(\gamma+0) - 0(0 + \tfrac{1}{2}\beta\gamma)\right] = 1.\end{aligned}$$

Computation of $A'(0)$:

$$\begin{aligned}A'(\tau) &= -\frac{2\alpha}{\sigma^2}\left[\frac{\gamma\cosh(\gamma\tau) + \frac{1}{2}\beta\sinh(\gamma\tau)}{\gamma e^{\beta\tau/2}}\right] \\ &\qquad \times \frac{1}{\left(\gamma\cosh(\gamma\tau) + \frac{1}{2}\beta\sinh(\gamma\tau)\right)^2}\Big[\frac{\beta\gamma}{2}e^{\beta\tau/2}\left(\gamma\cosh(\gamma\tau) + \tfrac{1}{2}\beta\sinh(\gamma\tau)\right) \\ &\qquad - \gamma e^{\beta\tau/2}\left(\gamma^2\sinh(\gamma\tau) + \tfrac{1}{2}\beta\gamma\cosh(\gamma\tau)\right)\Big], \\ A'(0) &= -\frac{2\alpha}{\sigma^2}\left[\frac{\gamma+0}{\gamma}\right]\frac{1}{(\gamma+0)^2}\left[\frac{\beta\gamma}{2}(\gamma+0) - \gamma(0 + \tfrac{1}{2}\beta\gamma)\right] \\ &= -\frac{2\alpha}{\sigma^2}\cdot\frac{1}{\gamma^2}\left[\frac{\beta\gamma^2}{2} - \tfrac{1}{2}\beta\gamma^2\right] \\ &= 0.\end{aligned}$$