Chapter 30

Hull and White model

Consider

$$dr(t) = (\alpha(t) - \beta(t)r(t))\ dt + \sigma(t)\ dW(t),$$

where $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ are nonrandom functions of t.

We can solve the stochastic differential equation. Set

$$K(t) = \int_0^t \beta(u)\ du.$$

Then

$$\begin{aligned} d\left(e^{K(t)}r(t)\right) &= e^{K(t)}\Big(\beta(t)r(t)\ dt + dr(t)\Big) \\ &= e^{K(t)}\left(\alpha(t)\ dt + \sigma(t)\ dW(t)\right). \end{aligned}$$

Integrating, we get

$$e^{K(t)}r(t) = r(0) + \int_0^t e^{K(u)}\alpha(u)\ du + \int_0^t e^{K(u)}\sigma(u)\ dW(u),$$

so

$$r(t) = e^{-K(t)}\left[r(0) + \int_0^t e^{K(u)}\alpha(u)\ du + \int_0^t e^{K(u)}\sigma(u)\ dW(u)\right].$$

From Theorem 1.69 in Chapter 29, we see that $r(t)$ is a Gaussian process with mean function

$$m_r(t) = e^{-K(t)}\left[r(0) + \int_0^t e^{K(u)}\alpha(u)\ du\right] \qquad (0.1)$$

and covariance function

$$\rho_r(s,t) = e^{-K(s)-K(t)}\int_0^{s\wedge t} e^{2K(u)}\sigma^2(u)\ du. \qquad (0.2)$$

The process $r(t)$ is also Markov.

We want to study $\int_0^T r(t)\ dt$. To do this, we define

$$X(t) = \int_0^t e^{K(u)}\sigma(u)\ dW(u), \quad Y(T) = \int_0^T e^{-K(t)}X(t)\ dt.$$

Then

$$\begin{aligned} r(t) &= e^{-K(t)}\left[r(0) + \int_0^t e^{K(u)}\alpha(u)\ du\right] + e^{-K(t)}X(t), \\ \int_0^T r(t)\ dt &= \int_0^T e^{-K(t)}\left[r(0) + \int_0^t e^{K(u)}\alpha(u)\ du\right]\ dt + Y(T). \end{aligned}$$

According to Theorem 1.70 in Chapter 29, $\int_0^T r(t)\ dt$ is normal. Its mean is

$$I\!\!E \int_0^T r(t)\ dt = \int_0^T e^{-K(t)}\left[r(0) + \int_0^t e^{K(u)}\alpha(u)\ du\right]\ dt, \tag{0.3}$$

and its variance is

$$\begin{aligned} \text{var}\left(\int_0^T r(t)\ dt\right) &= I\!\!E Y^2(T) \\ &= \int_0^T e^{2K(v)}\sigma^2(v)\left(\int_v^T e^{-K(y)}\ dy\right)^2\ dv. \end{aligned}$$

The price at time 0 of a zero-coupon bond paying \$1 at time T is

$$\begin{aligned} B(0,T) &= I\!\!E \exp\left\{-\int_0^T r(t)\ dt\right\} \\ &= \exp\left\{(-1)I\!\!E\int_0^T r(t)\ dt + \tfrac{1}{2}(-1)^2 \text{var}\left(\int_0^T r(t)\ dt\right)\right\} \\ &= \exp\Bigg\{-r(0)\int_0^T e^{-K(t)}\ dt - \int_0^T\int_0^t e^{-K(t)+K(u)}\alpha(u)\ du\ dt \\ &\qquad + \tfrac{1}{2}\int_0^T e^{2K(v)}\sigma^2(v)\left(\int_v^T e^{-K(y)}\ dy\right)^2\ dv\Bigg\} \\ &= \exp\{-r(0)C(0,T) - A(0,T)\}, \end{aligned}$$

where

$$\begin{aligned} C(0,T) &= \int_0^T e^{-K(t)}\ dt, \\ A(0,T) &= \int_0^T\int_0^t e^{-K(t)+K(u)}\alpha(u)\ du\ dt - \tfrac{1}{2}\int_0^T e^{2K(v)}\sigma^2(v)\left(\int_v^T e^{-K(y)}\ dy\right)^2\ dv. \end{aligned}$$

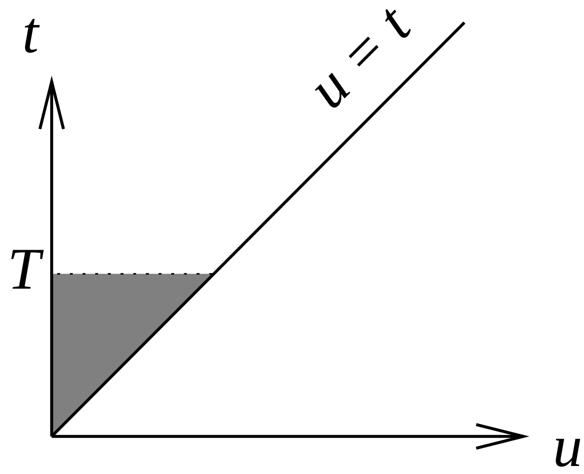

Figure 30.1: *Range of values of* u, t *for the integral.*

30.1 Fiddling with the formulas

Note that (see Fig 30.1)

$$\begin{aligned}
&\int_0^T \int_0^t e^{-K(t)+K(u)} \alpha(u)\, du\, dt \\
&= \int_0^T \int_u^T e^{-K(t)+K(u)} \alpha(u)\, dt\, du \\
(y = t;\; v = u) &= \int_0^T e^{K(v)} \alpha(v) \left(\int_v^T e^{-K(y)}\, dy \right) dv.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A(0,T) &= \int_0^T \left[e^{K(v)} \alpha(v) \left(\int_v^T e^{-K(y)}\, dy \right) - \tfrac{1}{2} e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{-K(y)}\, dy \right)^2 \right] dv, \\
C(0,T) &= \int_0^T e^{-K(y)}\, dy, \\
B(0,T) &= \exp\left\{ -r(0)C(0,T) - A(0,T) \right\}.
\end{aligned}$$

Consider the price at time $t \in [0,T]$ of the zero-coupon bond:

$$B(t,T) = I\!E \left[\exp\left\{ -\int_t^T r(u)\, du \right\} \middle| \mathcal{F}(t) \right].$$

Because r is a Markov process, this should be random only through a dependence on $r(t)$. In fact,

$$B(t,T) = \exp\left\{ -r(t)C(t,T) - A(t,T) \right\},$$

where

$$A(t,T) = \int_t^T \left[e^{K(v)} \alpha(v) \left(\int_v^T e^{-K(y)}\, dy \right) - \tfrac{1}{2} e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{-K(y)}\, dy \right)^2 \right] dv,$$

$$C(t,T) = e^{K(t)} \int_t^T e^{-K(y)}\, dy.$$

The reason for these changes is the following. We are now taking the initial time to be t rather than zero, so it is plausible that $\int_0^T \ldots\, dv$ should be replaced by $\int_t^T \ldots\, dv$. Recall that

$$K(v) = \int_0^v \beta(u)\, du,$$

and this should be replaced by

$$K(v) - K(t) = \int_t^v \beta(u)\, du.$$

Similarly, $K(y)$ should be replaced by $K(y) - K(t)$. Making these replacements in $A(0,T)$, we see that the $K(t)$ terms cancel. In $C(0,T)$, however, the $K(t)$ term does not cancel.

30.2 Dynamics of the bond price

Let $C_t(t,T)$ and $A_t(t,T)$ denote the partial derivatives with respect to t. From the formula

$$B(t,T) = \exp\left\{ -r(t)C(t,T) - A(t,T) \right\},$$

we have

$$\begin{aligned} dB(t,T) &= B(t,T) \left[-C(t,T)\, dr(t) - \tfrac{1}{2} C^2(t,T)\, dr(t)\, dr(t) - r(t) C_t(t,T)\, dt - A_t(t,T)\, dt \right] \\ &= B(t,T) \Big[-C(t,T) \left(\alpha(t) - \beta(t) r(t) \right) dt \\ &\qquad - C(t,T)\sigma(t)\, dW(t) - \tfrac{1}{2} C^2(t,T) \sigma^2(t)\, dt \\ &\qquad - r(t) C_t(t,T)\, dt - A_t(t,T)\, dt \Big]. \end{aligned}$$

Because we have used the risk-neutral pricing formula

$$B(t,T) = I\!E \left[\exp\left\{ -\int_t^T r(u)\, du \right\} \Big| \mathcal{F}(t) \right]$$

to obtain the bond price, its differential must be of the form

$$dB(t,T) = r(t) B(t,T)\, dt + (\ldots)\, dW(t).$$

Therefore, we must have

$$-C(t,T)\left(\alpha(t)-\beta(t)r(t)\right)-\tfrac{1}{2}C^2(t,T)\sigma^2(t)-r(t)C_t(t,T)-A_t(t,T) \;=\; r(t).$$

We leave the verification of this equation to the homework. After this verification, we have the formula

$$dB(t,T)=r(t)B(t,T)\;dt-\sigma(t)C(t,T)B(t,T)\;dW(t).$$

In particular, the volatility of the bond price is $\sigma(t)C(t,T)$.

30.3 Calibration of the Hull & White model

Recall:

$$\begin{aligned}
dr(t) &= (\alpha(t)-\beta(t)r(t))\;dt+\sigma(t)\;dB(t),\\
K(t) &= \int_0^t \beta(u)\;du,\\
A(t,T) &= \int_t^T \left[e^{K(v)}\alpha(v)\left(\int_v^T e^{-K(y)}\;dy\right)-\tfrac{1}{2}e^{2K(v)}\sigma^2(v)\left(\int_v^T e^{-K(y)}\;dy\right)^2\right]\;dv,\\
C(t,T) &= e^{K(t)}\int_t^T e^{-K(y)}\;dy,\\
B(t,T) &= \exp\left\{-r(t)C(t,T)-A(t,T)\right\}.
\end{aligned}$$

Suppose we obtain $B(0,T)$ for all $T\in[0,T^*]$ from market data (with some interpolation). Can we determine the functions $\alpha(t)$, $\beta(t)$, and $\sigma(t)$ for all $t\in[0,T^*]$? Not quite. Here is what we can do.

We take the following input data for the calibration:

1. $B(0,T),\ 0\le T\le T^*$;
2. $r(0)$;
3. $\alpha(0)$;
4. $\sigma(t),\ 0\le t\le T^*$ (usually assumed to be constant);
5. $\sigma(0)C(0,T),\ 0\le T\le T^*$, i.e., the volatility at time zero of bonds of all maturities.

Step 1. From 4 and 5 we solve for

$$C(0,T)=\int_0^T e^{-K(y)}\;dy.$$

We can then compute

$$\begin{aligned} \frac{\partial}{\partial T} C(0,T) &= e^{-K(T)} \\ \implies K(T) &= -\log \frac{\partial}{\partial T} C(0,T), \\ \frac{\partial}{\partial T} K(T) &= \frac{\partial}{\partial T} \int_0^T \beta(u)\, du = \beta(T). \end{aligned}$$

We now have $\beta(T)$ for all $T \in [0, T^*]$.

Step 2. From the formula

$$B(0,T) = \exp\{-r(0)C(0,T) - A(0,T)\},$$

we can solve for $A(0,T)$ for all $T \in [0, T^*]$. Recall that

$$A(0,T) = \int_0^T \left[e^{K(v)} \alpha(v) \left(\int_v^T e^{-K(y)}\, dy \right) - \tfrac{1}{2} e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{-K(y)}\, dy \right)^2 \right] dv.$$

We can use this formula to determine $\alpha(T),\ 0 \le T \le T^*$ as follows:

$$\begin{aligned} \frac{\partial}{\partial T} A(0,T) &= \int_0^T \left[e^{K(v)} \alpha(v) e^{-K(T)} - e^{2K(v)} \sigma^2(v) e^{-K(T)} \left(\int_v^T e^{-K(y)}\, dy \right) \right] dv, \\ e^{K(T)} \frac{\partial}{\partial T} A(0,T) &= \int_0^T \left[e^{K(v)} \alpha(v) - e^{2K(v)} \sigma^2(v) \left(\int_v^T e^{-K(y)}\, dy \right) \right] dv, \\ \frac{\partial}{\partial T} \left[e^{K(T)} \frac{\partial}{\partial T} A(0,T) \right] &= e^{K(T)} \alpha(T) - \int_0^T e^{2K(v)} \sigma^2(v)\, e^{-K(T)}\, dv, \\ e^{K(T)} \frac{\partial}{\partial T} \left[e^{K(T)} \frac{\partial}{\partial T} A(0,T) \right] &= e^{2K(T)} \alpha(T) - \int_0^T e^{2K(v)} \sigma^2(v)\, dv, \\ \frac{\partial}{\partial T} \left[e^{K(T)} \frac{\partial}{\partial T} \left[e^{K(T)} \frac{\partial}{\partial T} A(0,T) \right] \right] &= \alpha'(T) e^{2K(T)} + 2\alpha(T)\beta(T) e^{2K(T)} - e^{2K(T)} \sigma^2(T), \quad 0 \le T \le T^*. \end{aligned}$$

This gives us an ordinary differential equation for α, i.e.,

$$\alpha'(t) e^{2K(t)} + 2\alpha(t)\beta(t) e^{2K(t)} - e^{2K(t)} \sigma^2(t) = \text{known function of } t.$$

From assumption 4 and step 1, we know all the coefficients in this equation. From assumption 3, we have the initial condition $\alpha(0)$. We can solve the equation numerically to determine the function $\alpha(t),\ 0 \le t \le T^*$.

Remark 30.1 The derivation of the ordinary differential equation for $\alpha(t)$ requires three differentiations. Differentiation is an unstable procedure, i.e., functions which are close can have very different derivatives. Consider, for example,

$$\begin{aligned} f(x) &= 0 \quad \forall x \in I\!\!R, \\ g(x) &= \frac{\sin(1000x)}{100} \quad \forall x \in I\!\!R. \end{aligned}$$

Then

$$|f(x) - g(x)| \le \frac{1}{100} \quad \forall x \in I\!R,$$

but because

$$g'(x) = 10\cos(1000x),$$

we have

$$|f'(x) - g'(x)| = 10$$

for many values of x.

Assumption 5 for the calibration was that we know the volatility at time zero of bonds of all maturities. These volatilities can be implied by the prices of options on bonds. We consider now how the model prices options.

30.4 Option on a bond

Consider a European call option on a zero-coupon bond with strike price K and expiration time T_1. The bond matures at time $T_2 > T_1$. The price of the option at time 0 is

$$\begin{aligned}
&I\!E\left[e^{-\int_0^{T_1} r(u)\,du}\,(B(T_1,T_2) - K)^+\right] \\
&= I\!E e^{-\int_0^{T_1} r(u)\,du}(\exp\{-r(T_1)C(T_1,T_2) - A(T_1,T_2)\} - K)^+. \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x}\Big(\exp\{-yC(T_1,T_2) - A(T_1,T_2)\} - K\Big)^+ f(x,y)\,dx\,dy,
\end{aligned}$$

where $f(x,y)$ is the joint density of $\left(\int_0^{T_1} r(u)\,du,\ r(T_1)\right)$.

We observed at the beginning of this Chapter (equation (0.3)) that $\int_0^{T_1} r(u)\,du$ is normal with

$$\begin{aligned}
\mu_1 &\stackrel{\triangle}{=} I\!E\left[\int_0^{T_1} r(u)\,du\right] = \int_0^{T_1} I\!E r(u)\,du \\
&\qquad\qquad\qquad\qquad = \int_0^{T_1}\left[r(0)e^{-K(v)} + e^{-K(v)}\int_0^v e^{K(u)}\alpha(u)\,du\right]\,dv, \\
\sigma_1^2 &\stackrel{\triangle}{=} \text{var}\left[\int_0^{T_1} r(u)\,du\right] = \int_0^{T_1} e^{2K(v)}\sigma^2(v)\left(\int_v^{T_1} e^{-K(y)}\,dy\right)^2\,dv.
\end{aligned}$$

We also observed (equation (0.1)) that $r(T_1)$ is normal with

$$\begin{aligned}
\mu_2 &\stackrel{\triangle}{=} I\!E r(T_1) = r(0)e^{-K(T_1)} + e^{-K(T_1)}\int_0^{T_1} e^{K(u)}\alpha(u)\,du, \\
\sigma_2^2 &\stackrel{\triangle}{=} \text{var}\,(r(T_1)) = e^{-2K(T_1)}\int_0^{T_1} e^{2K(u)}\sigma^2(u)\,du.
\end{aligned}$$

In fact, $\left(\int_0^{T_1} r(u)\, du,\ r(T_1)\right)$ is jointly normal, and the covariance is

$$\begin{aligned}\rho\sigma_1\sigma_2 &= \mathbb{E}\left[\int_0^{T_1} (r(u) - \mathbb{E}r(u))\ du.\ (r(T_1) - \mathbb{E}r(T_1))\right]\\ &= \int_0^{T_1} \mathbb{E}[(r(u) - \mathbb{E}r(u))\ (r(T_1) - \mathbb{E}r(T_1))]\, du\\ &= \int_0^{T_1} \rho_r(u, T_1)\ du,\end{aligned}$$

where $\rho_r(u, T_1)$ is defined in Equation 0.2.

The option on the bond has price at time zero of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x}\left(\exp\{-yC(T_1,T_2) - A(T_1,T_2)\} - K\right)^+ \cdot \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_1^2} + \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right]\right\}\ dx\ dy. \quad (4.1)$$

The price of the option at time $t \in [0, T_1]$ is

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_t^{T_1} r(u)\ du}\,(B(T_1,T_2) - K)^+ \middle| \mathcal{F}(t)\right]\\ &\quad = \mathbb{E}\left[e^{-\int_t^{T_1} r(u)\ du}(\exp\{-r(T_1)C(T_1,T_2) - A(T_1,T_2)\} - K)^+ \middle| \mathcal{F}(t)\right] \quad (4.2)\end{aligned}$$

Because of the Markov property, this is random only through a dependence on $r(t)$. To compute this option price, we need the joint distribution of $\left(\int_t^{T_1} r(u)\ du,\ r(T_1)\right)$ conditioned on $r(t)$. This pair of random variables has a jointly normal conditional distribution, and

$$\begin{aligned}
\mu_1(t) &= I\!\!E\left[\int_t^{T_1} r(u)\,du \middle| \mathcal{F}(t)\right] \\
&= \int_t^{T_1}\left[r(t)e^{-K(v)+K(t)} + e^{-K(v)}\int_t^v e^{K(u)}\alpha(u)\,du\right]dv, \\
\sigma_1^2(t) &= I\!\!E\left[\left(\int_t^{T_1} r(u)\,du - \mu_1(t)\right)^2 \middle| \mathcal{F}(t)\right] \\
&= \int_t^{T_1} e^{2K(v)}\sigma^2(v)\left(\int_v^{T_1} e^{-K(y)}\,dy\right)^2 dv, \\
\mu_2(t) &= I\!\!E\left[r(T_1)\middle| r(t)\right] \\
&= r(t)e^{-K(T_1)+K(t)} + e^{-K(T_1)}\int_t^{T_1} e^{K(u)}\alpha(u)\,du, \\
\sigma_2^2(t) &= I\!\!E\left[(r(T_1)-\mu_2(t))^2\middle|\mathcal{F}(t)\right] \\
&= e^{-2K(T_1)}\int_t^{T_1} e^{2K(u)}\sigma^2(u)\,du, \\
\rho(t)\sigma_1(t)\sigma_2(t) &= I\!\!E\left[\left(\int_t^{T_1} r(u)\,du - \mu_1(t)\right)(r(T_1)-\mu_2(t))\middle|\mathcal{F}(t)\right] \\
&= \int_t^{T_1} e^{-K(u)-K(T_1)}\int_t^u e^{2K(v)}\sigma^2(v)\,dv\,du.
\end{aligned}$$

The variances and covariances are not random. The means are random through a dependence on $r(t)$.

Advantages of the Hull & White model:

1. Leads to closed-form pricing formulas.
2. Allows calibration to fit initial yield curve exactly.

Short-comings of the Hull & White model:

1. One-factor, so only allows parallel shifts of the yield curve, i.e.,
$$B(t,T) = \exp\{-r(t)C(t,T) - A(t,T)\},$$
so bond prices of all maturities are perfectly correlated.
2. Interest rate is normally distributed, and hence can take negative values. Consequently, the bond price
$$B(t,T) = I\!\!E\left[\exp\left\{-\int_t^T r(u)\,du\right\}\middle|\mathcal{F}(t)\right]$$
can exceed 1.