Chapter 29

Gaussian processes

Definition 29.1 (Gaussian Process) A *Gaussian process* $X(t)$, $t \geq 0$, is a stochastic process with the property that for every set of times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, the set of random variables

$$X(t_1), X(t_2), \ldots, X(t_n)$$

is jointly normally distributed.

Remark 29.1 If X is a Gaussian process, then its distribution is determined by its *mean function*

$$m(t) = \mathbb{E}X(t)$$

and its *covariance function*

$$\rho(s,t) = \mathbb{E}[(X(s) - m(s)) \cdot (X(t) - m(t))].$$

Indeed, the joint density of $X(t_1), \ldots, X(t_n)$ is

$$\begin{aligned} &\mathbb{P}\{X(t_1) \in dx_1, \ldots, X(t_n) \in dx_n\} \\ &= \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} \exp\left\{-\tfrac{1}{2}(\mathbf{x} - m(\mathbf{t})) \cdot \Sigma^{\mathbf{-1}} \cdot (\mathbf{x} - \mathbf{m}(\mathbf{t}))^{\mathbf{T}}\right\} \, dx_1 \ldots \, dx_n, \end{aligned}$$

where Σ is the covariance matrix

$$\Sigma = \begin{bmatrix} \rho(t_1,t_1) & \rho(t_1,t_2) & \ldots & \rho(t_1,t_n) \\ \rho(t_2,t_1) & \rho(t_2,t_2) & \ldots & \rho(t_2,t_n) \\ \ldots & \ldots & \ldots & \ldots \\ \rho(t_n,t_1) & \rho(t_n,t_2) & \ldots & \rho(t_n,t_n) \end{bmatrix}$$

$\mathbf{x}$ is the row vector $[x_1, x_2, \ldots, x_n]$, $\mathbf{t}$ is the row vector $[t_1, t_2, \ldots, t_n]$, and $m(\mathbf{t}) = [m(t_1), m(t_2), \ldots, m(t_n)]$.

The moment generating function is

$$\mathbb{E} \exp\left\{\sum_{k=1}^{n} u_k X(t_k)\right\} = \exp\left\{\mathbf{u} \cdot m(\mathbf{t})^{\mathbf{T}} + \tfrac{1}{2}\mathbf{u} \cdot \Sigma \cdot \mathbf{u}^{\mathbf{T}}\right\},$$

where $\mathbf{u} = [u_1, u_2, \ldots, u_n]$.

29.1 An example: Brownian Motion

Brownian motion W is a Gaussian process with $m(t) = 0$ and $\rho(s,t) = s \wedge t$. Indeed, if $0 \le s \le t$, then

$$\begin{aligned}\rho(s,t) &= \mathbb{E}\left[W(s)W(t)\right] = \mathbb{E}\left[W(s)\left(W(t) - W(s)\right) + W^2(s)\right] \\ &= \mathbb{E}W(s).\mathbb{E}\left(W(t) - W(s)\right) + \mathbb{E}W^2(s) \\ &= \mathbb{E}W^2(s) \\ &= s \wedge t.\end{aligned}$$

To prove that a process is Gaussian, one must show that $X(t_1), \dots, X(t_n)$ has either a density or a moment generating function of the appropriate form. We shall use the m.g.f., and shall cheat a bit by considering only two times, which we usually call s and t. We will want to show that

$$\mathbb{E}\exp\left\{u_1 X(s) + u_2 X(t)\right\} = \exp\left\{u_1 m_1 + u_2 m_2 + \tfrac{1}{2}[u_1\ u_2]\begin{bmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix}\right\}.$$

Theorem 1.69 (Integral w.r.t. a Brownian) *Let $W(t)$ be a Brownian motion and $\delta(t)$ a nonrandom function. Then*

$$X(t) = \int_0^t \delta(u)\, dW(u)$$

is a Gaussian process with $m(t) = 0$ and

$$\rho(s,t) = \int_0^{s \wedge t} \delta^2(u)\, du.$$

Proof: (Sketch.) We have

$$dX = \delta\, dW.$$

Therefore,

$$\begin{aligned}de^{uX(s)} &= ue^{uX(s)}\delta(s)\, dW(s) + \tfrac{1}{2}u^2 e^{uX(s)}\delta^2(s)\, ds, \\ e^{uX(s)} &= e^{uX(0)} + \underbrace{u\int_0^s e^{uX(v)}\delta(v)\, dW(v)}_{\text{Martingale}} + \tfrac{1}{2}u^2 \int_0^s e^{uX(v)}\delta^2(v)\, dv, \\ \mathbb{E}e^{uX(s)} &= 1 + \tfrac{1}{2}u^2 \int_0^s \delta^2(v)\,\mathbb{E}e^{uX(v)}\, dv, \\ \frac{d}{ds}\mathbb{E}e^{uX(s)} &= \tfrac{1}{2}u^2\delta^2(s)\,\mathbb{E}e^{uX(s)}, \\ \mathbb{E}e^{uX(s)} &= e^{uX(0)}\exp\left\{\tfrac{1}{2}u^2\int_0^s \delta^2(v)\, dv\right\} \qquad (1.1) \\ &= \exp\left\{\tfrac{1}{2}u^2\int_0^s \delta^2(v)\, dv\right\}.\end{aligned}$$

This shows that $X(s)$ is normal with mean 0 and variance $\int_0^s \delta^2(v)\, dv$.

Now let $0 \le s < t$ be given. Just as before,

$$de^{uX(t)} = ue^{uX(t)}\delta(t)\ dW(t) + \tfrac{1}{2}u^2 e^{uX(t)}\delta^2(t)\ dt.$$

Integrate from s to t to get

$$e^{uX(t)} = e^{uX(s)} + u\int_s^t \delta(v)e^{uX(v)}\ dW(v) + \tfrac{1}{2}u^2\int_s^t \delta^2(v)e^{uX(v)}\ dv.$$

Take $\mathbb{E}[\ldots|\mathcal{F}(s)]$ conditional expectations and use the martingale property

$$\begin{aligned}\mathbb{E}\left[\int_s^t \delta(v)e^{uX(v)}\ dW(v)\Big|\mathcal{F}(s)\right] &= \mathbb{E}\left[\int_0^t \delta(v)e^{uX(v)}\ dW(v)\Big|\mathcal{F}(s)\right] - \int_0^s \delta(v)e^{uX(v)}\ dW(v)\\ &= 0\end{aligned}$$

to get

$$\begin{aligned}\mathbb{E}\left[e^{uX(t)}\Big|\mathcal{F}(s)\right] &= e^{uX(s)} + \tfrac{1}{2}u^2\int_s^t \delta^2(v)\mathbb{E}\left[e^{uX(v)}\Big|\mathcal{F}(s)\right]\ dv\\ \frac{d}{dt}\mathbb{E}\left[e^{uX(t)}\Big|\mathcal{F}(s)\right] &= \tfrac{1}{2}u^2\delta^2(t)\mathbb{E}\left[e^{uX(t)}\Big|\mathcal{F}(s)\right],\quad t \ge s.\end{aligned}$$

The solution to this ordinary differential equation with initial time s is

$$\mathbb{E}\left[e^{uX(t)}\Big|\mathcal{F}(s)\right] = e^{uX(s)}\exp\left\{\tfrac{1}{2}u^2\int_s^t \delta^2(v)\ dv\right\},\quad t \ge s. \tag{1.2}$$

We now compute the m.g.f. for $(X(s), X(t))$, where $0 \le s \le t$:

$$\begin{aligned}\mathbb{E}\left[e^{u_1X(s)+u_2X(t)}\Big|\mathcal{F}(s)\right] &= e^{u_1X(s)}\mathbb{E}\left[e^{u_2X(t)}\Big|\mathcal{F}(s)\right]\\ &\overset{(1.2)}{=} e^{(u_1+u_2)X(s)}\exp\left\{\tfrac{1}{2}u_2^2\int_s^t \delta^2(v)\ dv\right\},\\ \mathbb{E}\left[e^{u_1X(s)+u_2X(t)}\right] &= \mathbb{E}\left\{\mathbb{E}\left[e^{u_1X(s)+u_2X(t)}\Big|\mathcal{F}(s)\right]\right\}\\ &= \mathbb{E}\left\{e^{(u_1+u_2)X(s)}\right\}.\exp\left\{\tfrac{1}{2}u_2^2\int_s^t \delta^2(v)\ dv\right\}\\ &\overset{(1.1)}{=} \exp\left\{\tfrac{1}{2}(u_1+u_2)^2\int_0^s \delta^2(v)\ dv\ +\ \tfrac{1}{2}u_2^2\int_s^t \delta^2(v)\ dv\right\}\\ &= \exp\left\{\tfrac{1}{2}(u_1^2+2u_1u_2)\int_0^s \delta^2(v)\ dv\ +\ \tfrac{1}{2}u_2^2\int_0^t \delta^2(v)\ dv\right\}\\ &= \exp\left\{\tfrac{1}{2}[u_1\ u_2]\begin{bmatrix}\int_0^s \delta^2 & \int_0^s \delta^2\\ \int_0^s \delta^2 & \int_0^t \delta^2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}\right\}.\end{aligned}$$

This shows that $(X(s), X(t))$ is jointly normal with $\mathbb{E}X(s) = \mathbb{E}X(t) = 0$,

$$\mathbb{E}X^2(s) = \int_0^s \delta^2(v)\ dv, \qquad \mathbb{E}X^2(t) = \int_0^t \delta^2(v)\ dv,$$
$$\mathbb{E}[X(s)X(t)] = \int_0^s \delta^2(v)\ dv.$$

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Remark 29.2 The hard part of the above argument, and the reason we use moment generating functions, is to prove the normality. The computation of means and variances does not require the use of moment generating functions. Indeed,

$$X(t) = \int_0^t \delta(u)\, dW(u)$$

is a martingale and $X(0) = 0$, so

$$m(t) = \mathbb{E}X(t) = 0 \quad \forall t \geq 0.$$

For fixed $s \geq 0$,

$$\mathbb{E}X^2(s) = \int_0^s \delta^2(v)\, dv$$

by the Itô isometry. For $0 \leq s \leq t$,

$$\begin{aligned} \mathbb{E}[X(s)(X(t) - X(s))] &= \mathbb{E}\left[\mathbb{E}\left\{X(s)(X(t) - X(s))\middle|\mathcal{F}(s)\right\}\right] \\ &= \mathbb{E}\left[X(s)\underbrace{\left(\mathbb{E}\left[X(t)\middle|\mathcal{F}(s)\right] - X(s)\right)}_{0}\right] \\ &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[X(s)X(t)] &= \mathbb{E}[X(s)(X(t) - X(s)) + X^2(s)] \\ &= \mathbb{E}X^2(s) = \int_0^s \delta^2(v)\, dv. \end{aligned}$$

If δ were a stochastic proess, the Itô isometry says

$$\mathbb{E}X^2(s) = \int_0^s \mathbb{E}\delta^2(v)\, dv$$

and the same argument used above shows that for $0 \leq s \leq t$,

$$\mathbb{E}[X(s)X(t)] = \mathbb{E}X^2(s) = \int_0^s \mathbb{E}\delta^2(v)\, dv.$$

However, when δ is stochastic, X is not necessarily a Gaussian process, so its distribution is not determined from its mean and covariance functions.

Remark 29.3 When δ is nonrandom,

$$X(t) = \int_0^t \delta(u)\, dW(u)$$

is also Markov. We proved this before, but note again that the Markov property follows immediately from (1.2). The equation (1.2) says that conditioned on $\mathcal{F}(s)$, the distribution of $X(t)$ depends only on $X(s)$; in fact, $X(t)$ is normal with mean $X(s)$ and variance $\int_s^t \delta^2(v)\, dv$.

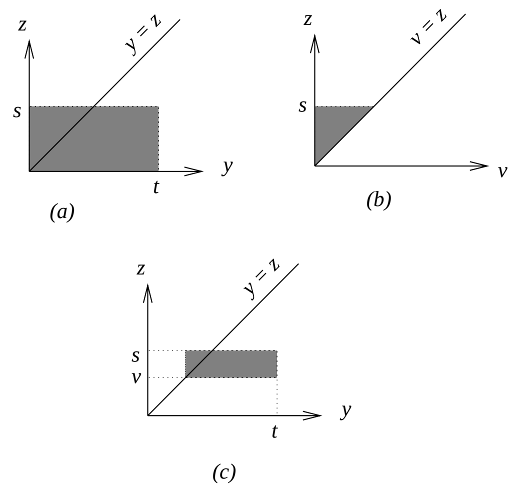

Figure 29.1: *Range of values of y, z, v for the integrals in the proof of Theorem 1.70.*

Theorem 1.70 *Let $W(t)$ be a Brownian motion, and let $\delta(t)$ and $h(t)$ be nonrandom functions. Define*

$$X(t) = \int_0^t \delta(u)\, dW(u), \quad Y(t) = \int_0^t h(u)X(u)\, du.$$

Then Y is a Gaussian process with mean function $m_Y(t) = 0$ and covariance function

$$\rho_Y(s,t) = \int_0^{s\wedge t} \delta^2(v) \left(\int_v^s h(y)\, dy\right) \left(\int_v^t h(y)\, dy\right)\, dv. \tag{1.3}$$

Proof: (Partial) Computation of $\rho_Y(s,t)$: Let $0 \le s \le t$ be given. It is shown in a homework problem that $(Y(s), Y(t))$ is a jointly normal pair of random variables. Here we observe that

$$m_Y(t) = \mathbb{E}Y(t) = \int_0^t h(u)\; \mathbb{E}X(u)\, du = 0,$$

and we verify that (1.3) holds.

We have

$$
\begin{aligned}
\rho_Y(s,t) &= \mathbb{E}\left[Y(s)Y(t)\right] \\
&= \mathbb{E}\left[\int_0^s h(y)X(y)\,dy.\int_0^t h(z)X(z)\,dz\right] \\
&= \mathbb{E}\int_0^s\int_0^t h(y)h(z)X(y)X(z)\,dy\,dz \\
&= \int_0^s\int_0^t h(y)h(z)\mathbb{E}\left[X(y)X(z)\right]\,dy\,dz \\
&= \int_0^s\int_0^t h(y)h(z)\int_0^{y\wedge z}\delta^2(v)\,dv\,dy\,dz \\
&= \int_0^s\int_z^t h(y)h(z)\left(\int_0^z \delta^2(v)\,dv\right)\,dy\,dz \\
&\quad + \int_0^s\int_y^s h(y)h(z)\left(\int_0^y \delta^2(v)\,dv\right)\,dz\,dy \quad \text{(See Fig. 29.1(a))} \\
&= \int_0^s h(z)\left(\int_z^t h(y)\,dy\right)\left(\int_0^z \delta^2(v)\,dv\right)\,dz \\
&\quad + \int_0^s h(y)\left(\int_y^s h(z)\,dz\right)\left(\int_0^y \delta^2(v)\,dv\right)\,dy \\
&= \int_0^s\int_0^z h(z)\delta^2(v)\left(\int_z^t h(y)\,dy\right)\,dv\,dz \\
&\quad + \int_0^s\int_0^y h(y)\delta^2(v)\left(\int_y^s h(z)\,dz\right)\,dv\,dy \\
&= \int_0^s\int_v^s h(z)\delta^2(v)\left(\int_z^t h(y)\,dy\right)\,dz\,dv \\
&\quad + \int_0^s\int_v^s h(y)\delta^2(v)\left(\int_y^s h(z)\,dz\right)\,dy\,dv \quad \text{(See Fig. 29.1(b))} \\
&= \int_0^s \delta^2(v)\left(\int_v^s\int_z^t h(y)h(z)\,dy\,dz\right)\,dv \\
&\quad + \int_0^s \delta^2(v)\left(\int_v^s\int_y^s h(y)h(z)\,dz\,dy\right)\,dv \\
&= \int_0^s \delta^2(v)\left(\int_v^s\int_v^t h(y)h(z)\,dy\,dz\right)\,dv \quad \text{(See Fig. 29.1(c))} \\
&= \int_0^s \delta^2(v)\left(\int_v^s h(y)\,dy\right)\left(\int_v^t h(z)\,dz\right)\,dv \\
&= \int_0^s \delta^2(v)\left(\int_v^s h(y)\,dy\right)\left(\int_v^t h(y)\,dy\right)\,dv
\end{aligned}
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Remark 29.4 Unlike the process $X(t) = \int_0^t \delta(u)\ dW(u)$, the process $Y(t) = \int_0^t X(u)\ du$ is neither Markov nor a martingale. For $0 \leq s < t$,

$$\begin{aligned} I\!E[Y(t)|\mathcal{F}(s)] &= \int_0^s h(u)X(u)\,du + I\!E\left[\int_s^t h(u)X(u)\,du \middle| \mathcal{F}(s)\right] \\ &= Y(s) + \int_s^t h(u) I\!E[X(u) \Big| \mathcal{F}(s)]\,du \\ &= Y(s) + \int_s^t h(u)X(s)\,du \\ &= Y(s) + X(s)\int_s^t h(u)\,du, \end{aligned}$$

where we have used the fact that X is a martingale. The conditional expectation $I\!E[Y(t)|\mathcal{F}(s)]$ is not equal to $Y(s)$, nor is it a function of $Y(s)$ alone.