Chapter 28

Term-structure models

Throughout this discussion, $\{W(t);\ 0 \le t \le T^*\}$ is a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{F(t);\ 0 \le t \le T^*\}$ is the filtration generated by W.

Suppose we are given an adapted *interest rate process* $\{r(t);\ 0 \le t \le T^*\}$. We define the accumulation factor

$$\beta(t) = \exp\left\{\int_0^t r(u)\ du\right\}, \quad 0 \le t \le T^*.$$

In a term-structure model, we take the zero-coupon bonds ("zeroes") of various maturities to be the primitive assets. We assume these bonds are default-free and pay \$1 at maturity. For $0 \le t \le T \le T^*$, let

$$B(t,T) \ = \ \text{price at time } t \text{ of the zero-coupon bond paying \$1 at time } T.$$

Theorem 0.67 (Fundamental Theorem of Asset Pricing) *A term structure model is free of arbitrage if and only if there is a probability measure $\widetilde{I\!P}$ on Ω (a risk-neutral measure) with the same probability-zero sets as $I\!P$ (i.e.,* equivalent *to $I\!P$), such that for each $T \in (0, T^*]$, the process*

$$\frac{B(t,T)}{\beta(t)}, \quad 0 \le t \le T,$$

is a martingale under $\widetilde{I\!P}$.

Remark 28.1 We shall always have

$$dB(t,T) = \mu(t,T)B(t,T)\ dt + \rho(t,T)B(t,T)\ dW(t), \quad 0 \le t \le T,$$

for some functions $\mu(t,T)$ and $\rho(t,T)$. Therefore

$$\begin{aligned} d\left(\frac{B(t,T)}{\beta(t)}\right) &= B(t,T)\ d\left(\frac{1}{\beta(t)}\right) + \frac{1}{\beta(t)}\ dB(t,T) \\ &= [\mu(t,T) - r(t)]\frac{B(t,T)}{\beta(t)}\ dt + \rho(t,T)\frac{B(t,T)}{\beta(t)}\ dW(t), \end{aligned}$$

so $I\!P$ is a risk-neutral measure if and only if $\mu(t,T)$, the mean rate of return of $B(t,T)$ under $I\!P$, is the interest rate $r(t)$. If the mean rate of return of $B(t,T)$ under $I\!P$ is not $r(t)$ at each time t and for each maturity T, we should change to a measure $\widetilde{I\!P}$ under which the mean rate of return is $r(t)$. If such a measure does not exist, then the model admits an arbitrage by trading in zero-coupon bonds.

28.1 Computing arbitrage-free bond prices: first method

Begin with a stochastic differential equation (SDE)

$$dX(t) = a(t, X(t))\,dt + b(t, X(t))\,dW(t).$$

The solution $X(t)$ is the *factor.* If we want to have n-factors, we let W be an n-dimensional Brownian motion and let X be an n-dimensional process. We let the interest rate $r(t)$ be a function of $X(t)$. In the usual one-factor models, we take $r(t)$ to be $X(t)$ (e.g., Cox-Ingersoll-Ross, Hull-White).

Now that we have an interest rate process $\{r(t);\ 0 \le t \le T^*\}$, we define the zero-coupon bond prices to be

$$\begin{aligned} B(t,T) &= \beta(t)\ I\!E\left[\frac{1}{\beta(T)}\middle|\mathcal{F}(t)\right] \\ &= I\!E\left[\exp\left\{-\int_t^T r(u)\,du\right\}\middle|\mathcal{F}(t)\right], \quad 0 \le t \le T \le T^*. \end{aligned}$$

We showed in Chapter 27 that

$$dB(t,T) = r(t)B(t,T)\,dt + \beta(t)\gamma(t)\,dW(t)$$

for some process γ. Since $B(t,T)$ has mean rate of return $r(t)$ under $I\!P$, $I\!P$ is a risk-neutral measure and there is no arbitrage.

28.2 Some interest-rate dependent assets

Coupon-paying bond: Payments $P_1, P_2, \ldots, P_n$ at times $T_1, T_2, \ldots, T_n$. Price at time t is

$$\sum_{\{k: t < T_k\}} P_k B(t, T_k).$$

Call option on a zero-coupon bond: Bond matures at time T. Option expires at time $T_1 < T$. Price at time t is

$$\beta(t)\ I\!E\left[\frac{1}{\beta(T_1)}(B(T_1,T) - K)^+\middle|\mathcal{F}(t)\right], \quad 0 \le t \le T_1.$$

28.3 Terminology

Definition 28.1 (Term-structure model) Any mathematical model which determines, at least theoretically, the stochastic processes

$$B(t,T), \quad 0 \le t \le T,$$

for all $T \in (0, T^*]$.

Definition 28.2 (Yield to maturity) For $0 \le t \le T \le T^*$, the *yield to maturity* $Y(t,T)$ is the $\mathcal{F}(t)$-measurable random-variable satisfying

$$B(t,T)\exp\{(T-t)Y(t,T)\} = 1,$$

or equivalently,

$$Y(t,T) = -\frac{1}{T-t}\log B(t,T).$$

Determining

$$B(t,T), \quad 0 \le t \le T \le T^*,$$

is equivalent to determining

$$Y(t,T), \quad 0 \le t \le T \le T^*.$$

28.4 Forward rate agreement

Let $0 \le t \le T < T+\epsilon \le T^*$ be given. Suppose you want to borrow \$1 at time T with repayment (plus interest) at time $T+\epsilon$, at an interest rate agreed upon at time t. To synthesize a *forward-rate agreement* to do this, at time t buy a T-maturity zero and short $\frac{B(t,T)}{B(t,T+\epsilon)}$ $(T+\epsilon)$-maturity zeroes. The value of this portfolio at time t is

$$B(t,T) - \frac{B(t,T)}{B(t,T+\epsilon)}B(t,T+\epsilon) = 0.$$

At time T, you receive \$1 from the T-maturity zero. At time $T+\epsilon$, you pay \$ $\frac{B(t,T)}{B(t,T+\epsilon)}$. The effective interest rate on the dollar you receive at time T is $R(t,T,T+\epsilon)$ given by

$$\frac{B(t,T)}{B(t,T+\epsilon)} = \exp\{\epsilon\, R(t,T,T+\epsilon)\},$$

or equivalently,

$$R(t,T,T+\epsilon) = -\frac{\log B(t,T+\epsilon) - \log B(t,T)}{\epsilon}.$$

The *forward rate* is

$$f(t,T) = \lim_{\epsilon \downarrow 0} R(t,T,T+\epsilon) = -\frac{\partial}{\partial T}\log B(t,T). \tag{4.1}$$

This is the instantaneous interest rate, agreed upon at time t, for money borrowed at time T.

Integrating the above equation, we obtain

$$\begin{aligned}\int_t^T f(t,u)\,du &= -\int_t^T \frac{\partial}{\partial u}\log B(t,u)\,du\\ &= -\log B(t,u)\Big|_{u=t}^{u=T}\\ &= -\log B(t,T),\end{aligned}$$

so

$$B(t,T) = \exp\left\{-\int_t^T f(t,u)\,du\right\}.$$

You can agree at time t to receive interest rate $f(t,u)$ at each time $u \in [t,T]$. If you invest \$ $B(t,T)$ at time t and receive interest rate $f(t,u)$ at each time u between t and T, this will grow to

$$B(t,T)\exp\left\{\int_t^T f(t,u)\,du\right\} = 1$$

at time T.

28.5 Recovering the interest $r(t)$ from the forward rate

$$\begin{aligned}B(t,T) &= I\!E\left[\exp\left\{-\int_t^T r(u)\,du\right\}\Big|\mathcal{F}(t)\right],\\ \frac{\partial}{\partial T}B(t,T) &= I\!E\left[-r(T)\exp\left\{-\int_t^T r(u)\,du\right\}\Big|\mathcal{F}(t)\right],\\ \frac{\partial}{\partial T}B(t,T)\Big|_{T=t} &= I\!E\left[-r(t)\Big|\mathcal{F}(t)\right] = -r(t).\end{aligned}$$

On the other hand,

$$\begin{aligned}B(t,T) &= \exp\left\{-\int_t^T f(t,u)\,du\right\},\\ \frac{\partial}{\partial T}B(t,T) &= -f(t,T)\exp\left\{-\int_t^T f(t,u)\,du\right\},\\ \frac{\partial}{\partial T}B(t,T)\Big|_{T=t} &= -f(t,t).\end{aligned}$$

Conclusion: $r(t) = f(t,t)$.

28.6 Computing arbitrage-free bond prices: Heath-Jarrow-Morton method

For each $T \in (0, T^*]$, let the forward rate be given by

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW(u), \quad 0 \le t \le T.$$

Here $\{\alpha(u,T);\ 0 \le u \le T\}$ and $\{\sigma(u,T);\ 0 \le u \le T\}$ are adapted processes.
In other words,

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t).$$

Recall that

$$B(t,T) = \exp\left\{-\int_t^T f(t,u)\,du\right\}.$$

Now

$$\begin{aligned} d\left\{-\int_t^T f(t,u)\,du\right\} &= f(t,t)\,dt - \int_t^T df(t,u)\,du \\ &= r(t)\,dt - \int_t^T [\alpha(t,u)\,dt + \sigma(t,u)\,dW(t)]\,du \\ &= r(t)\,dt - \underbrace{\left[\int_t^T \alpha(t,u)\,du\right]}_{\alpha^*(t,T)} dt - \underbrace{\left[\int_t^T \sigma(t,u)\,du\right]}_{\sigma^*(t,T)} dW(t) \\ &= r(t)\,dt - \alpha^*(t,T)\,dt - \sigma^*(t,T)\,dW(t). \end{aligned}$$

Let

$$g(x) = e^x,\ g'(x) = e^x,\ g''(x) = e^x.$$

Then

$$B(t,T) = g\left(-\int_t^T f(t,u)\,du\right),$$

and

$$\begin{aligned} dB(t,T) &= dg\left(-\int_t^T f(t,u)\,du\right) \\ &= g'\left(-\int_t^T f(t,u)\,du\right)(r\,dt - \alpha^*\,dt - \sigma^*\,dW) \\ &\quad + \tfrac{1}{2} g''\left(-\int_t^T f(t,u)\,du\right)(\sigma^*)^2\,dt \\ &= B(t,T)\left[r(t) - \alpha^*(t,T) + \tfrac{1}{2}(\sigma^*(t,T))^2\right] dt \\ &\quad - \sigma^*(t,T)B(t,T)\,dW(t). \end{aligned}$$

28.7 Checking for absence of arbitrage

$I\!\!P$ is a risk-neutral measure if and only if

$$\alpha^*(t,T) = \tfrac{1}{2}\left(\sigma^*(t,T)\right)^2, \quad 0 \le t \le T \le T^*,$$

i.e.,

$$\int_t^T \alpha(t,u)\,du = \tfrac{1}{2}\left(\int_t^T \sigma(t,u)\,du\right)^2, \quad 0 \le t \le T \le T^*. \tag{7.1}$$

Differentiating this w.r.t. T, we obtain

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)\,du, \quad 0 \le t \le T \le T^*. \tag{7.2}$$

Not only does (7.1) imply (7.2), (7.2) also implies (7.1). This will be a homework problem.

Suppose (7.1) does not hold. Then $I\!\!P$ is not a risk-neutral measure, but there might still be a risk-neutral measure. Let $\{\theta(t);\ 0 \le t \le T^*\}$ be an adapted process, and define

$$\begin{aligned}
\widetilde{W}(t) &= \int_0^t \theta(u)\,du \;+\; W(t),\\
Z(t) &= \exp\left\{-\int_0^t \theta(u)\,dW(u) \;-\; \tfrac{1}{2}\int_0^t \theta^2(u)\,du\right\},\\
\widetilde{I\!\!P}(A) &= \int_A Z(T^*)\,dI\!\!P \quad \forall A \in \mathcal{F}(T^*).
\end{aligned}$$

Then

$$\begin{aligned}
dB(t,T) &= B(t,T)\left[r(t) - \alpha^*(t,T) + \tfrac{1}{2}(\sigma^*(t,T))^2\right]\,dt\\
&\quad - \sigma^*(t,T)B(t,T)\,dW(t)\\
&= B(t,T)\left[r(t) - \alpha^*(t,T) + \tfrac{1}{2}(\sigma^*(t,T))^2 + \sigma^*(t,T)\theta(t)\right]\,dt\\
&\quad - \sigma^*(t,T)B(t,T)\,d\widetilde{W}(t), \quad 0 \le t \le T.
\end{aligned}$$

In order for $B(t,T)$ to have mean rate of return $r(t)$ under $\widetilde{I\!\!P}$, we must have

$$\alpha^*(t,T) = \tfrac{1}{2}(\sigma^*(t,T))^2 + \sigma^*(t,T)\theta(t), \quad 0 \le t \le T \le T^*. \tag{7.3}$$

Differentiation w.r.t. T yields the equivalent condition

$$\alpha(t,T) = \sigma(t,T)\sigma^*(t,T) + \sigma(t,T)\theta(t), \quad 0 \le t \le T \le T^*. \tag{7.4}$$

Theorem 7.68 (Heath-Jarrow-Morton) *For each $T \in (0,T^*]$, let $\alpha(u,T),\ 0 \le u \le T$, and $\sigma(u,T), 0 \le u \le T$, be adapted processes, and assume $\sigma(u,T) > 0$ for all u and T. Let $f(0,T),\ 0 \le t \le T^*$, be a deterministic function, and define*

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW(u).$$

Then $f(t,T),\ 0 \le t \le T \le T^*$ is a family of forward rate processes for a term-structure model without arbitrage if and only if there is an adapted process $\theta(t),\ 0 \le t \le T^*$, satisfying (7.3), or equivalently, satisfying (7.4).

Remark 28.2 Under $I\!P$, the zero-coupon bond with maturity T has mean rate of return

$$r(t) - \alpha^*(t,T) + \tfrac{1}{2}(\sigma^*(t,T))^2$$

and volatility $\sigma^*(t,T)$. The excess mean rate of return, above the interest rate, is

$$-\alpha^*(t,T) + \tfrac{1}{2}(\sigma^*(t,T))^2,$$

and when normalized by the volatility, this becomes the *market price of risk*

$$\frac{-\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2}{\sigma^*(t,T)}.$$

The no-arbitrage condition is that this market price of risk at time t does not depend on the maturity T of the bond. We can then set

$$\theta(t) = -\left[\frac{-\alpha^*(t,T) + \frac{1}{2}(\sigma^*(t,T))^2}{\sigma^*(t,T)}\right],$$

and (7.3) is satisfied.

(The remainder of this chapter was taught Mar 21)

Suppose the market price of risk does not depend on the maturity T, so we can solve (7.3) for θ. Plugging this into the stochastic differential equation for $B(t,T)$, we obtain for every maturity T:

$$dB(t,T) = r(t)B(t,T)\,dt - \sigma^*(t,T)B(t,T)\,d\widetilde{W}(t).$$

Because (7.4) is equivalent to (7.3), we may plug (7.4) into the stochastic differential equation for $f(t,T)$ to obtain, for every maturity T:

$$\begin{aligned} df(t,T) &= [\sigma(t,T)\sigma^*(t,T) + \sigma(t,T)\theta(t)]\,dt + \sigma(t,T)\,dW(t) \\ &= \sigma(t,T)\sigma^*(t,T)\,dt + \sigma(t,T)\,d\widetilde{W}(t). \end{aligned}$$

28.8 Implementation of the Heath-Jarrow-Morton model

Choose

$$\begin{aligned} \sigma^*(t,T), &\quad 0 \le t \le T \le T^*, \\ \theta(t), &\quad 0 \le t \le T^*. \end{aligned}$$

These may be stochastic processes, but are usually taken to be deterministic functions. Define

$$\begin{aligned}\alpha(t,T) &= \sigma(t,T)\sigma^*(t,T) + \sigma(t,T)\theta(t),\\ \widetilde{W}(t) &= \int_0^t \theta(u)\,du + W(t),\\ Z(t) &= \exp\left\{-\int_0^t \theta(u)\,dW(u) - \tfrac{1}{2}\int_0^t \theta^2(u)\,du\right\},\\ \widetilde{I\!P}(A) &= \int_A Z(T^*)\,dI\!P \quad \forall A \in \mathcal{F}(T^*).\end{aligned}$$

Let $f(0,T),\ 0 \le T \le T^*$, be determined by the market; recall from equation (4.1):

$$f(0,T) = -\frac{\partial}{\partial T}\log B(0,T), \quad 0 \le T \le T^*.$$

Then $f(t,T)$ for $0 \le t \le T$ is determined by the equation

$$df(t,T) = \sigma(t,T)\sigma^*(t,T)\,dt + \sigma(t,T)\,d\widetilde{W}(t), \tag{8.1}$$

this determines the interest rate process

$$r(t) = f(t,t), \quad 0 \le t \le T^*, \tag{8.2}$$

and then the zero-coupon bond prices are determined by the initial conditions $B(0,T),\ 0 \le T \le T^*$, gotten from the market, combined with the stochastic differential equation

$$dB(t,T) = r(t)B(t,T)\,dt - \sigma^*(t,T)B(t,T)\,d\widetilde{W}(t). \tag{8.3}$$

Because all pricing of interest rate dependent assets will be done under the risk-neutral measure $\widetilde{I\!P}$, under which $\widetilde{W}$ is a Brownian motion, we have written (8.1) and (8.3) in terms of $\widetilde{W}$ rather than W. Written this way, it is apparent that neither $\theta(t)$ nor $\alpha(t,T)$ will enter subsequent computations. The only process which matters is $\sigma(t,T),\ 0 \le t \le T \le T^*$, and the process

$$\sigma^*(t,T) = \int_t^T \sigma(t,u)\,du, \quad 0 \le t \le T \le T^*, \tag{8.4}$$

obtained from $\sigma(t,T)$.

From (8.3) we see that $\sigma^*(t,T)$ is the volatility at time t of the zero coupon bond maturing at time T. Equation (8.4) implies

$$\sigma^*(T,T) = 0, \quad 0 \le T \le T^*. \tag{8.5}$$

This is because $B(T,T) = 1$ and so as t approaches T (from below), the volatility in $B(t,T)$ must vanish.

In conclusion, to implement the HJM model, it suffices to have the initial market data $B(0,T),\ 0 \le T \le T^*$, and the volatilities

$$\sigma^*(t,T), \quad 0 \le t \le T \le T^*.$$

We require that $\sigma^*(t,T)$ be differentiable in T and satisfy (8.5). We can then define

$$\sigma(t,T) = \frac{\partial}{\partial T}\sigma^*(t,T),$$

and (8.4) will be satisfied because

$$\sigma^*(t,T) = \sigma^*(t,T) - \sigma^*(t,t) = \int_t^T \frac{\partial}{\partial u}\sigma^*(t,u)\,du.$$

We then let $\widetilde{W}$ be a Brownian motion under a probability measure $\widetilde{I\!P}$, and we let $B(t,T),\ 0 \le t \le T \le T^*$, be given by (8.3), where $r(t)$ is given by (8.2) and $f(t,T)$ by (8.1). In (8.1) we use the initial conditions

$$f(0,T) = -\frac{\partial}{\partial T}\log B(0,T), \quad 0 \le T \le T^*.$$

Remark 28.3 It is customary in the literature to write W rather than $\widetilde{W}$ and $I\!P$ rather than $\widetilde{I\!P}$, so that $I\!P$ is the symbol used for the risk-neutral measure and no reference is ever made to the market measure. The only parameter which must be estimated from the market is the bond volatility $\sigma^*(t,T)$, and volatility is unaffected by the change of measure.