Chapter 27

Bonds, forward contracts and futures

Let $\{W(t), \mathcal{F}(t);\ 0 \le t \le T\}$ be a Brownian motion (Wiener process) on some $(\Omega, \mathcal{F}, \mathbb{P})$. Consider an asset, which we call a stock, whose price satisfies

$$dS(t) = r(t)S(t)\ dt + \sigma(t)S(t)\ dW(t).$$

Here, r and σ are adapted processes, and we have already switched to the risk-neutral measure, which we call $I\!\!P$. Assume that every martingale under $I\!\!P$ can be represented as an integral with respect to W.

Define the accumulation factor

$$\beta(t) = \exp\left\{\int_0^t r(u)\ du\right\}.$$

A zero-coupon bond, maturing at time T, pays 1 at time T and nothing before time T. According to the risk-neutral pricing formula, its value at time $t \in [0, T]$ is

$$\begin{aligned} B(t,T) &= \beta(t)\ I\!\!E\left[\frac{1}{\beta(T)}\middle|\mathcal{F}(t)\right] \\ &= I\!\!E\left[\frac{\beta(t)}{\beta(T)}\middle|\mathcal{F}(t)\right] \\ &= I\!\!E\left[\exp\left\{-\int_t^T r(u)\ du\right\}\middle|\mathcal{F}(t)\right]. \end{aligned}$$

Given $B(t,T)$ dollars at time t, one can construct a portfolio of investment in the stock and money market so that the portfolio value at time T is 1 almost surely. Indeed, for some process γ,

$$\begin{aligned}
B(t,T) &= \beta(t) \underbrace{\mathbb{E}\left[\frac{1}{\beta(T)}\middle|\mathcal{F}(t)\right]}_{\text{martingale}} \\
&= \beta(t)\left[\mathbb{E}\left(\frac{1}{\beta(T)}\right) + \int_0^t \gamma(u)\, dW(u)\right] \\
&= \beta(t)\left[B(0,T) + \int_0^t \gamma(u)\, dW(u)\right], \\
dB(t,T) &= r(t)\beta(t)\left[B(0,T) + \int_0^t \gamma(u)\, dW(u)\right] dt \;+\; \beta(t)\gamma(t)\, dW(t) \\
&= r(t)B(t,T)\, dt + \beta(t)\gamma(t)\, dW(t).
\end{aligned}$$

The value of a portfolio satisfies

$$\begin{aligned}
dX(t) &= \Delta(t)\, dS(t) + r(t)[X(t) - \Delta(t)S(t)]dt \\
&= r(t)X(t)\, dt + \Delta(t)\sigma(t)S(t)\, dW(t).
\end{aligned} \tag{*}$$

We set

$$\Delta(t) = \frac{\beta(t)\gamma(t)}{\sigma(t)S(t)}.$$

If, at any time t, $X(t) = B(t,T)$ and we use the portfolio $\Delta(u),\ t \le u \le T$, then we will have

$$X(T) = B(T,T) = 1.$$

If $r(t)$ is nonrandom for all t, then

$$\begin{aligned}
B(t,T) &= \exp\left\{-\int_t^T r(u)\, du\right\}, \\
dB(t,T) &= r(t)B(t,T)\, dt,
\end{aligned}$$

i.e., $\gamma = 0$. Then Δ given above is zero. If, at time t, you are given $B(t,T)$ dollars and you always invest only in the money market, then at time T you will have

$$B(t,T)\exp\left\{\int_t^T r(u)\, du\right\} = 1.$$

If $r(t)$ is random for all t, then γ is not zero. One generally has three different instruments: the stock, the money market, and the zero coupon bond. Any two of them are sufficient for hedging, and the two which are most convenient can depend on the instrument being hedged.

27.1 Forward contracts

We continue with the set-up for zero-coupon bonds. The T-*forward price* of the stock at time $t \in [0,T]$ is the $\mathcal{F}(t)$-measurable price, agreed upon at time t, for purchase of a share of stock at time T, chosen so the forward contract has value zero at time t. In other words,

$$\mathbb{E}\left[\frac{1}{\beta(T)}\left(S(T)-F(t)\right)\Big|\mathcal{F}(t)\right]=0, \quad 0\le t\le T.$$

We solve for $F(t)$:

$$\begin{aligned} 0 &= \mathbb{E}\left[\frac{1}{\beta(T)}\left(S(T)-F(t)\right)\Big|\mathcal{F}(t)\right] \\ &= \mathbb{E}\left[\frac{S(T)}{\beta(T)}\Big|\mathcal{F}(t)\right]-\frac{F(t)}{\beta(t)}\mathbb{E}\left[\frac{\beta(t)}{\beta(T)}\Big|\mathcal{F}(t)\right] \\ &= \frac{S(t)}{\beta(t)}-\frac{F(t)}{\beta(t)}B(t,T). \end{aligned}$$

This implies that

$$F(t)=\frac{S(t)}{B(t,T)}.$$

Remark 27.1 (Value vs. Forward price) The T-forward price $F(t)$ is *not* the value at time t of the forward contract. The value of the contract at time t is zero. $F(t)$ is the price agreed upon at time t which will be paid for the stock at time T.

27.2 Hedging a forward contract

Enter a forward contract at time 0, i.e., agree to pay $F(0)=\frac{S(0)}{B(0,T)}$ for a share of stock at time T. At time zero, this contract has value 0. At later times, however, it does not. In fact, its value at time $t\in[0,T]$ is

$$\begin{aligned} V(t) &= \beta(t)\ \mathbb{E}\left[\frac{1}{\beta(T)}(S(T)-F(0))\Big|\mathcal{F}(t)\right] \\ &= \beta(t)\ \mathbb{E}\left[\frac{S(T)}{\beta(T)}\Big|\mathcal{F}(t)\right]-F(0)\ \mathbb{E}\left[\frac{\beta(t)}{\beta(T)}\Big|\mathcal{F}(t)\right] \\ &= \beta(t)\frac{S(t)}{\beta(t)}-F(0)B(t,T) \\ &= S(t)-F(0)B(t,T). \end{aligned}$$

This suggests the following hedge of a short position in the forward contract. At time 0, short $F(0)$ T-maturity zero-coupon bonds. This generates income

$$F(0)B(0,T)=\frac{S(0)}{B(0,T)}B(0,T)=S(0).$$

Buy one share of stock. This portfolio requires no initial investment. Maintain this position until time T, when the portfolio is worth

$$S(T) - F(0)B(T,T) = S(T) - F(0).$$

Deliver the share of stock and receive payment $F(0)$.

A short position in the forward could also be hedged using the stock and money market, but the implementation of this hedge would require a term-structure model.

27.3 Future contracts

Future contracts are designed to remove the risk of default inherent in forward contracts. Through the device of *marking to market*, the value of the future contract is maintained at zero at all times. Thus, either party can close out his/her position at any time.

Let us first consider the situation with discrete trading dates

$$0 = t_0 < t_1 < \ldots < t_n = T.$$

On each $[t_j, t_{j+1})$, r is constant, so

$$\begin{aligned}\beta(t_{k+1}) &= \exp\left\{\int_0^{t_{k+1}} r(u)\, du\right\} \\ &= \exp\left\{\sum_{j=0}^{k} r(t_j)(t_{j+1} - t_j)\right\}\end{aligned}$$

is $\mathcal{F}(t_k)$-measurable.

Enter a future contract at time t_k, taking the long position, when the future price is $\Phi(t_k)$. At time t_{k+1}, when the future price is $\Phi(t_{k+1})$, you receive a payment $\Phi(t_{k+1}) - \Phi(t_k)$. (If the price has fallen, you make the payment $-(\Phi(t_{k+1}) - \Phi(t_k))$.) The mechanism for receiving and making these payments is the *margin account* held by the broker.

By time $T = t_n$, you have received the sequence of payments

$$\Phi(t_{k+1}) - \Phi(t_k),\ \Phi(t_{k+2}) - \Phi(t_{k+1}),\ \ldots,\ \Phi(t_n) - \Phi(t_{n-1})$$

at times $t_{k+1}, t_{k+2}, \ldots, t_n$. The value at time $t = t_0$ of this sequence is

$$\beta(t)\ \mathbb{E}\left[\sum_{j=k}^{n-1} \frac{1}{\beta(t_{j+1})} \left(\Phi(t_{j+1}) - \Phi(t_j)\right) \Big| \mathcal{F}(t)\right].$$

Because it costs nothing to enter the future contract at time t, this expression must be zero almost surely.

The continuous-time version of this condition is

$$\beta(t)\ I\!E\left[\int_t^T \frac{1}{\beta(u)}\ d\Phi(u)\middle|\mathcal{F}(t)\right] = 0, \quad 0 \le t \le T.$$

Note that $\beta(t_{j+1})$ appearing in the discrete-time version is $\mathcal{F}(t_j)$-measurable, as it should be when approximating a stochastic integral.

Definition 27.1 The T-*future price* of the stock is any $\mathcal{F}(t)$-adapted stochastic process

$$\{\Phi(t);\ 0 \le t \le T\},$$

satisfying

$$\Phi(T) = S(T) \text{ a.s., and} \tag{a}$$

$$I\!E\left[\int_t^T \frac{1}{\beta(u)}\ d\Phi(u)\middle|\mathcal{F}(t)\right] = 0, \quad 0 \le t \le T. \tag{b}$$

Theorem 3.66 *The unique process satisfying (a) and (b) is*

$$\Phi(t) = I\!E\left[S(T)\middle|\mathcal{F}(t)\right], 0 \le t \le T.$$

Proof: We first show that (b) holds if and only if Φ is a martingale. If Φ is a martingale, then $\int_0^t \frac{1}{\beta(u)}\ d\Phi(u)$ is also a martingale, so

$$\begin{aligned} I\!E\left[\int_t^T \frac{1}{\beta(u)}\ d\Phi(u)\middle|\mathcal{F}(t)\right] &= I\!E\left[\int_0^t \frac{1}{\beta(u)}\ d\Phi(u)\middle|\mathcal{F}(t)\right] - \int_0^t \frac{1}{\beta(u)}\ d\Phi(u) \\ &= 0. \end{aligned}$$

On the other hand, if (b) holds, then the martingale

$$M(t) = I\!E\left[\int_0^T \frac{1}{\beta(u)}\ d\Phi(u)\middle|\mathcal{F}(t)\right]$$

satisfies

$$\begin{aligned} M(t) &= \int_0^t \frac{1}{\beta(u)}\ d\Phi(u) + I\!E\left[\int_t^T \frac{1}{\beta(u)}\ d\Phi(u)\middle|\mathcal{F}(t)\right] \\ &= \int_0^t \frac{1}{\beta(u)}\ d\Phi(u), \quad 0 \le t \le T. \end{aligned}$$

this implies

$$\begin{aligned} dM(t) &= \frac{1}{\beta(t)}\ d\Phi(t), \\ d\Phi(t) &= \beta(t)\ dM(t), \end{aligned}$$

and so Φ is a martingale (its differential has no dt term).

Now define

$$\Phi(t) = I\!\!E\left[S(T)\middle|\mathcal{F}(t)\right], \quad 0 \le t \le T.$$

Clearly (a) is satisfied. By the tower property, Φ is a martingale, so (b) is also satisfied. Indeed, this Φ is the only martingale satisfying (a). ∎

27.4 Cash flow from a future contract

With a forward contract, entered at time 0, the buyer agrees to pay $F(0)$ for an asset valued at $S(T)$. The only payment is at time T.

With a future contract, entered at time 0, the buyer receives a cash flow (which may at times be negative) between times 0 and T. If he still holds the contract at time T, then he pays $S(T)$ at time T for an asset valued at $S(T)$. The cash flow received between times 0 and T sums to

$$\int_0^T d\Phi(u) = \Phi(T) - \Phi(0) = S(T) - \Phi(0).$$

Thus, if the future contract holder takes delivery at time T, he has paid a total of

$$(\Phi(0) - S(T)) + S(T) = \Phi(0)$$

for an asset valued at $S(T)$.

27.5 Forward-future spread

Future price: $\Phi(t) = I\!\!E\left[S(T)\middle|\mathcal{F}(t)\right]$.

Forward price:

$$F(t) = \frac{S(t)}{B(t,T)} = \frac{S(t)}{\beta(t) I\!\!E\left[\frac{1}{\beta(T)}\middle|\mathcal{F}(t)\right]}.$$

Forward-future spread:

$$\begin{aligned}\Phi(0) - F(0) &= I\!\!E[S(T)] - \frac{S(0)}{I\!\!E\left[\frac{1}{\beta(T)}\right]} \\ &= \frac{1}{I\!\!E\left(\frac{1}{\beta(T)}\right)}\left[I\!\!E\left(\frac{1}{\beta(T)}\right) I\!\!E\left(S(T)\right) - I\!\!E\left(\frac{S(T)}{\beta(T)}\right)\right].\end{aligned}$$

If $\frac{1}{\beta(T)}$ and $S(T)$ are uncorrelated,

$$\Phi(0) = F(0).$$

If $\frac{1}{\beta(T)}$ and $S(T)$ are positively correlated, then

$$\Phi(0) \leq F(0).$$

This is the case that a rise in stock price tends to occur with a fall in the interest rate. The owner of the future tends to receive income when the stock price rises, but invests it at a declining interest rate. If the stock price falls, the owner usually must make payments on the future contract. He withdraws from the money market to do this just as the interest rate rises. In short, the long position in the future is hurt by positive correlation between $\frac{1}{\beta(T)}$ and $S(T)$. The buyer of the future is compensated by a reduction of the future price below the forward price.

27.6 Backwardation and contango

Suppose

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t).$$

Define $\theta = \frac{\mu - r}{\sigma}$, $\widetilde{W}(t) = \theta t + W(t)$,

$$\begin{aligned} Z(T) &= \exp\{-\theta W(T) - \tfrac{1}{2}\theta^2 T\} \\ \widetilde{I\!P}(A) &= \int_A Z(T)\, dI\!P, \quad \forall A \in \mathcal{F}(T). \end{aligned}$$

Then $\widetilde{W}$ is a Brownian motion under $\widetilde{I\!P}$, and

$$dS(t) = rS(t)\, dt + \sigma S(t)\, d\widetilde{W}(t).$$

We have

$$\begin{aligned} \beta(t) &= e^{rt} \\ S(t) &= S(0)\exp\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W(t)\} \\ &= S(0)\exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma \widetilde{W}(t)\} \end{aligned}$$

Because $\frac{1}{\beta(T)} = e^{-rT}$ is nonrandom, $S(T)$ and $\frac{1}{\beta(T)}$ are uncorrelated under $\widetilde{I\!P}$. Therefore,

$$\begin{aligned} \Phi(t) &= \widetilde{I\!E}[S(T) \big| \mathcal{F}(t)] \\ &= F(t) \\ &= \frac{S(t)}{B(t,T)} = e^{r(T-t)} S(t). \end{aligned}$$

The expected future spot price of the stock under $I\!P$ is

$$\begin{aligned} I\!ES(T) &= S(0)e^{\mu T} I\!E\left[\exp\left\{-\tfrac{1}{2}\sigma^2 T + \sigma W(T)\right\}\right] \\ &= e^{\mu T} S(0). \end{aligned}$$

The future price at time 0 is

$$\Phi(0) = e^{rT} S(0).$$

If $\mu > r$, then $\Phi(0) < \mathbb{E}S(T)$. This situation is called *normal backwardation* (see Hull). If $\mu < r$, then $\Phi(0) > \mathbb{E}S(T)$. This is called *contango.*