Chapter 26

Options on dividend-paying stocks

26.1 American option with convex payoff function

Theorem 1.64 *Consider the stock price process*

$$dS(t) = r(t)S(t)\ dt + \sigma(t)S(t)\ dB(t),$$

where r and σ are processes and $r(t) \geq 0, \quad 0 \leq t \leq T$, a.s. This stock pays no dividends. Let $h(x)$ be a convex function of $x \geq 0$, and assume $h(0) = 0$. (E.g., $h(x) = (x - K)^+$). An American contingent claim paying $h(S(t))$ if exercised at time t does not need to be exercised before expiration, i.e., waiting until expiration to decide whether to exercise entails no loss of value.

Proof: For $0 \leq \alpha \leq 1$ and $x \geq 0$, we have

$$\begin{aligned} h(\alpha x) &= h((1-\alpha)0 + \alpha x) \\ &\leq (1-\alpha)h(0) + \alpha h(x) \\ &= \alpha h(x). \end{aligned}$$

Let T be the time of expiration of the contingent claim. For $0 \leq t \leq T$,

$$0 \leq \frac{\beta(t)}{\beta(T)} = \exp\left\{-\int_t^T r(u)\ du\right\} \leq 1$$

and $S(T) \geq 0$, so

$$h\left(\frac{\beta(t)}{\beta(T)} S(T)\right) \leq \frac{\beta(t)}{\beta(T)} h(S(T)). \tag{*}$$

Consider a European contingent claim paying $h(S(T))$ at time T. The value of this claim at time $t \in [0, T]$ is

$$X(t) = \beta(t)\ I\!E\left[\frac{1}{\beta(T)} h(S(T)) \middle| \mathcal{F}(t)\right].$$

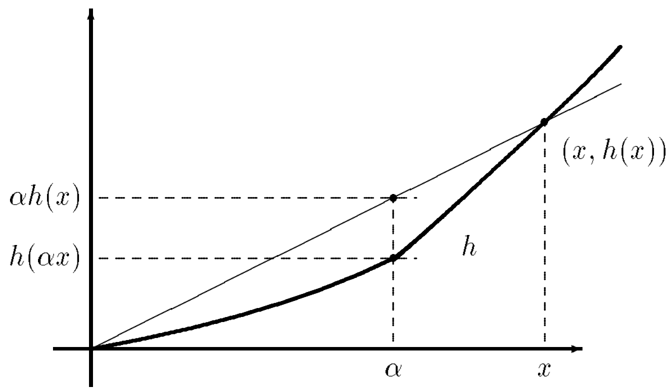

Figure 26.1: *Convex payoff function*

Therefore,

$$\begin{aligned}
\frac{X(t)}{\beta(t)} &= \frac{1}{\beta(t)} I\!E\left[\frac{\beta(t)}{\beta(T)} h(S(T)) \middle| \mathcal{F}(t)\right] \\
&\geq \frac{1}{\beta(t)} I\!E\left[h\left(\frac{\beta(t)}{\beta(T)} S(T)\right) \middle| \mathcal{F}(t)\right] \quad \text{(by (*))} \\
&\geq \frac{1}{\beta(t)} h\left(\beta(t)\ I\!E\left[\frac{S(T)}{\beta(T)} \middle| \mathcal{F}(t)\right]\right) \quad \text{(Jensen's inequality)} \\
&= \frac{1}{\beta(t)} h\left(\beta(t) \frac{S(t)}{\beta(t)}\right) \quad (\frac{S}{\beta} \text{ is a martingale}) \\
&= \frac{1}{\beta(t)} h(S(t)).
\end{aligned}$$

This shows that the value $X(t)$ of the European contingent claim dominates the intrinsic value $h(S(t))$ of the American claim. In fact, except in degenerate cases, the inequality

$$X(t) \geq h(S(t)), \quad 0 \leq t \leq T,$$

is strict, i.e., the American claim should not be exercised prior to expiration. ■

26.2 Dividend paying stock

Let r and σ be constant, let δ be a "dividend coefficient" satisfying

$$0 < \delta < 1.$$

Let $T > 0$ be an expiration time, and let $t_1 \in (0,T)$ be the time of dividend payment. The stock price is given by

$$S(t) = \begin{cases} S(0)\exp\{(r - \frac{1}{2}\sigma^2)t + \sigma B(t)\}, & 0 \le t \le t_1, \\ (1-\delta)S(t_1)\exp\{(r - \frac{1}{2}\sigma^2)(t - t_1) + \sigma(B(t) - B(t_1))\}, & t_1 < t \le T. \end{cases}$$

Consider an American call on this stock. At times $t \in (t_1, T)$, it is not optimal to exercise, so the value of the call is given by the usual Black-Scholes formula

$$v(t,x) = xN(d_+(T-t,x)) \;-\; Ke^{-r(T-t)}N(d_-(T-t,x)), \quad t_1 < t \le T,$$

where

$$d_\pm(T-t,x) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + (T-t)(r \pm \sigma^2/2)\right].$$

At time t_1, immediately *after* payment of the dividend, the value of the call is

$$v(t_1, (1-\delta)S(t_1)).$$

At time t_1, immediately *before* payment of the dividend, the value of the call is

$$w(t_1, S(t_1)),$$

where

$$w(t_1, x) = \max\left\{(x-K)^+,\; v(t_1,(1-\delta)x\right\}.$$

Theorem 2.65 *For $0 \le t \le t_1$, the value of the American call is $w(t, S(t))$, where*

$$w(t,x) = I\!E^{t,x}\left[e^{-r(t_1-t)}w(t_1, S(t_1))\right].$$

This function satisfies the usual Black-Scholes equation

$$-rw + w_t + rxw_x + \tfrac{1}{2}\sigma^2x^2w_{xx} = 0, \quad 0 \le t \le t_1,\ x \ge 0,$$

(where $w = w(t,x)$) with terminal condition

$$w(t_1,x) = \max\left\{(x-K)^+,\; v(t_1,(1-\delta)x)\right\},\ x \ge 0,$$

and boundary condition

$$w(t,0) = 0, \quad 0 \le t \le T.$$

The hedging portfolio is

$$\Delta(t) = \begin{cases} w_x(t, S(t)), & 0 \le t \le t_1, \\ v_x(t, S(t)), & t_1 < t \le T. \end{cases}$$

Proof: We only need to show that an American contingent claim with payoff $w(t_1, S(t_1))$ at time t_1 need not be exercised before time t_1. According to Theorem 1.64, it suffices to prove

1. $w(t_1, 0) = 0$,

2. $w(t_1, x)$ is convex in x.

Since $v(t_1, 0) = 0$, we have immediately that

$$w(t_1, 0) = \max\left\{(0-K)^+,\ v(t_1, (1-\delta)0)\right\} = 0.$$

To prove that $w(t_1, x)$ is convex in x, we need to show that $v(t_1, (1-\delta)x)$ is convex is x. Obviously, $(x-K)^+$ is convex in x, and the maximum of two convex functions is convex. The proof of the convexity of $v(t_1, (1-\delta)x)$ in x is left as a homework problem. ■

26.3 Hedging at time t_1

Let $x = S(t_1)$.

Case I: $v(t_1, (1-\delta)x) \geq (x-K)^+$.
The option need not be exercised at time t_1 (should not be exercised if the inequality is strict). We have

$$\begin{aligned} w(t_1, x) &= v(t_1, (1-\delta)x), \\ \Delta(t_1) = w_x(t_1, x) &= (1-\delta)v_x(t_1, (1-\delta)x) = (1-\delta)\Delta(t_1+), \end{aligned}$$

where

$$\Delta(t_1+) = \lim_{t \downarrow t_1} \Delta(t)$$

is the number of shares of stock held by the hedge immediately after payment of the dividend. The post-dividend position can be achieved by reinvesting in stock the dividends received on the stock held in the hedge. Indeed,

$$\begin{aligned} \Delta(t_1+) &= \frac{1}{1-\delta}\Delta(t_1) = \Delta(t_1) + \frac{\delta}{1-\delta}\Delta(t_1) \\ &= \Delta(t_1) + \frac{\delta\Delta(t_1)S(t_1)}{(1-\delta)S(t_1)} \\ &= \text{\# of shares held when dividend is paid} + \frac{\text{dividends received}}{\text{price per share when dividend is reinvested}} \end{aligned}$$

Case II: $v(t_1, (1-\delta)x) < (x-K)^+$.
The owner of the option should exercise before the dividend payment at time t_1 and receive $(x-K)$. The hedge has been constructed so the seller of the option has $x-K$ before the dividend payment at time t_1. If the option is not exercised, its value drops from $x-K$ to $v(t_1, (1-\delta)x)$, and the seller of the option can pocket the difference and continue the hedge.