Chapter 25

American Options

This and the following chapters form part of the course *Stochastic Differential Equations for Finance II.*

25.1 Preview of perpetual American put

$$dS = rS\,dt + \sigma S\,dB$$

Intrinsic value at time t : $(K - S(t))^+$.

Let $L \in [0, K]$ be given. Suppose we exercise the first time the stock price is L or lower. We define

$$\begin{aligned}
\tau_L &= \min\{t \geq 0; S(t) \leq L\}, \\
v_L(x) &= I\!\!E e^{-r\tau_L}(K - S(\tau_L))^+ \\
&= \begin{cases} K - x & \text{if } x \leq L, \\ (K - L) I\!\!E e^{-r\tau_L} & \text{if } x > L. \end{cases}
\end{aligned}$$

The plan is to comute $v_L(x)$ and then maximize over L to find the optimal exercise price. We need to know the distribution of τ_L.

25.2 First passage times for Brownian motion: first method

(Based on the reflection principle)

Let B be a Brownian motion under $I\!\!P$, let $x > 0$ be given, and define

$$\tau = \min\{t \geq 0; B(t) = x\}.$$

τ is called the *first passage time to* x. We compute the distribution of τ.

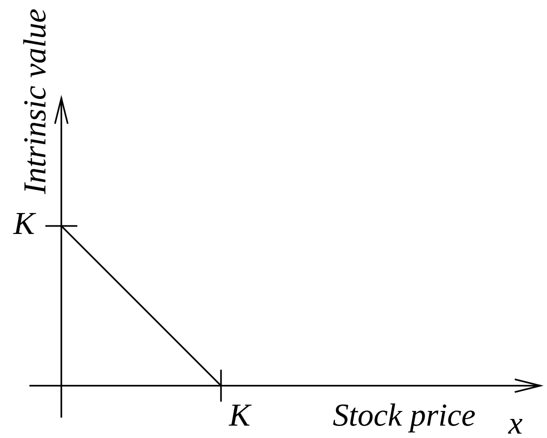

Figure 25.1: *Intrinsic value of perpetual American put*

Define

$$M(t) = \max_{0 \le u \le t} B(u).$$

From the first section of Chapter 20 we have

$$I\!P\{M(t) \in dm, B(t) \in db\} = \frac{2(2m-b)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2m-b)^2}{2t}\right\} dm\, db, \quad m > 0, b < m.$$

Therefore,

$$\begin{aligned} I\!P\{M(t) \ge x\} &= \int_x^\infty \int_{-\infty}^m \frac{2(2m-b)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2m-b)^2}{2t}\right\} db\, dm \\ &= \int_x^\infty \frac{2}{\sqrt{2\pi t}} \exp\left\{-\frac{(2m-b)^2}{2t}\right\}\Bigg|_{b=-\infty}^{b=m} dm \\ &= \int_x^\infty \frac{2}{\sqrt{2\pi t}} \exp\left\{-\frac{m^2}{2t}\right\} dm. \end{aligned}$$

We make the change of variable $z = \frac{m}{\sqrt{t}}$ in the integral to get

$$= \int_{x/\sqrt{t}}^\infty \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz.$$

Now

$$\tau \le t \Longleftrightarrow M(t) \ge x,$$

so

$$\begin{aligned}
\mathbb{P}\{\tau \in dt\} &= \frac{\partial}{\partial t}\mathbb{P}\{\tau \le t\}\, dt \\
&= \frac{\partial}{\partial t}\mathbb{P}\left\{M(t) \ge x\right\}\, dt \\
&= \left[\frac{\partial}{\partial t}\int_{x/\sqrt{t}}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz\right]\, dt \\
&= -\frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2t}\right\} . \frac{\partial}{\partial t}\left(\frac{x}{\sqrt{t}}\right)\, dt \\
&= \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}\, dt.
\end{aligned}$$

We also have the Laplace transform formula

$$\begin{aligned}
\mathbb{E}e^{-\alpha\tau} &= \int_0^{\infty} e^{-\alpha t}\,\mathbb{P}\{\tau \in dt\} \\
&= e^{-x\sqrt{2\alpha}}, \quad \alpha > 0. \quad \text{(See Homework)}
\end{aligned}$$

Reference: Karatzas and Shreve, Brownian Motion and Stochastic Calculus, pp 95-96.

25.3 Drift adjustment

Reference: Karatzas/Shreve, *Brownian motion and Stochastic Calculus*, pp 196–197.

For $0 \le t < \infty$, define

$$\begin{aligned}
\widetilde{B}(t) &= \theta t + B(t), \\
Z(t) &= \exp\{-\theta B(t) - \tfrac{1}{2}\theta^2 t\}, \\
&= \exp\{-\theta \widetilde{B}(t) + \tfrac{1}{2}\theta^2 t\},
\end{aligned}$$

Define

$$\tilde{\tau} = \min\{t \ge 0; \widetilde{B}(t) = x\}.$$

We fix a finite time T and change the probability measure "only up to T". More specifically, with T fixed, define

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T)\, dP, \quad A \in \mathcal{F}(T).$$

Under $\widetilde{\mathbb{P}}$, the process $\widetilde{B}(t), 0 \le t \le T$, is a (nondrifted) Brownian motion, so

$$\begin{aligned}
\widetilde{\mathbb{P}}\{\tilde{\tau} \in dt\} &= \mathbb{P}\{\tau \in dt\} \\
&= \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}\, dt, \quad 0 < t \le T.
\end{aligned}$$

For $0 < t \le T$ we have

$$\begin{aligned}
\mathbb{P}\{\tilde{\tau} \le t\} &= \mathbb{E}\left[\mathbf{1}_{\{\tilde{\tau}\le t\}}\right] \\
&= \widetilde{\mathbb{E}}\left[\mathbf{1}_{\{\tilde{\tau}\le t\}}\frac{1}{Z(T)}\right] \\
&= \widetilde{\mathbb{E}}\left[\mathbf{1}_{\{\tilde{\tau}\le t\}}\exp\{\theta\widetilde{B}(T) - \tfrac{1}{2}\theta^2 T\}\right] \\
&= \widetilde{\mathbb{E}}\left[\mathbf{1}_{\{\tilde{\tau}\le t\}}\widetilde{\mathbb{E}}\left[\exp\{\theta\widetilde{B}(T) - \tfrac{1}{2}\theta^2 T\}\middle|\mathcal{F}(\tilde{\tau}\wedge t)\right]\right] \\
&= \widetilde{\mathbb{E}}\left[\mathbf{1}_{\{\tilde{\tau}\le t\}}\exp\{\theta\widetilde{B}(\tilde{\tau}\wedge t) - \tfrac{1}{2}\theta^2(\tilde{\tau}\wedge t)\}\right] \\
&= \widetilde{\mathbb{E}}\left[\mathbf{1}_{\{\tilde{\tau}\le t\}}\exp\{\theta x - \tfrac{1}{2}\theta^2\tilde{\tau}\}\right] \\
&= \int_0^t \exp\{\theta x - \tfrac{1}{2}\theta^2 s\}\widetilde{\mathbb{P}}\{\tilde{\tau}\in ds\} \\
&= \int_0^t \frac{x}{s\sqrt{2\pi s}}\exp\left\{\theta x - \tfrac{1}{2}\theta^2 s - \frac{x^2}{2s}\right\}\, ds \\
&= \int_0^t \frac{x}{s\sqrt{2\pi s}}\exp\left\{-\frac{(x-\theta s)^2}{2s}\right\}\, ds.
\end{aligned}$$

Therefore,

$$\mathbb{P}\{\tilde{\tau}\in dt\} = \frac{x}{t\sqrt{2\pi t}}\exp\left\{-\frac{(x-\theta t)^2}{2t}\right\}\, dt, \quad 0 < t \le T.$$

Since T is arbitrary, this must in fact be the correct formula for all $t > 0$.

25.4 Drift-adjusted Laplace transform

Recall the Laplace transform formula for

$$\tau = \min\{t \ge 0; B(t) = x\}$$

for nondrifted Brownian motion:

$$\mathbb{E}e^{-\alpha\tau} = \int_0^\infty \frac{x}{t\sqrt{2\pi t}}\exp\left\{-\alpha t - \frac{x^2}{2t}\right\} dt = e^{-x\sqrt{2\alpha}}, \quad \alpha > 0, x > 0.$$

For

$$\tilde{\tau} = \min\{t \ge 0; \theta t + B(t) = x\},$$

the Laplace transform is

$$\begin{aligned}
\mathbb{E}e^{-\alpha\tilde{\tau}} &= \int_0^\infty \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\alpha t - \frac{(x-\theta t)^2}{2t}\right\}\, dt \\
&= \int_0^\infty \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\alpha t - \frac{x^2}{2t} + x\theta - \tfrac{1}{2}\theta^2 t\right\}\, dt \\
&= e^{x\theta} \int_0^\infty \frac{x}{t\sqrt{2\pi t}} \exp\left\{-(\alpha + \tfrac{1}{2}\theta^2)t - \frac{x^2}{2t}\right\}\, dt \\
&= e^{x\theta - x\sqrt{2\alpha+\theta^2}}, \quad \alpha > 0, x > 0,
\end{aligned}$$

where in the last step we have used the formula for $\mathbb{E}e^{-\alpha\tau}$ with α replaced by $\alpha + \frac{1}{2}\theta^2$.

If $\tilde{\tau}(\omega) < \infty$, then

$$\lim_{\alpha\downarrow 0} e^{-\alpha\tilde{\tau}(\omega)} = 1;$$

if $\tilde{\tau}(\omega) = \infty$, then $e^{-\alpha\tilde{\tau}(\omega)} = 0$ for every $\alpha > 0$, so

$$\lim_{\alpha\downarrow 0} e^{-\alpha\tilde{\tau}(\omega)} = 0.$$

Therefore,

$$\lim_{\alpha\downarrow 0} e^{-\alpha\tilde{\tau}(\omega)} = \mathbf{1}_{\tilde{\tau}<\infty}.$$

Letting $\alpha\downarrow 0$ and using the Monotone Convergence Theorem in the Laplace transform formula

$$\mathbb{E}e^{-\alpha\tilde{\tau}} = e^{x\theta - x\sqrt{2\alpha+\theta^2}},$$

we obtain

$$\mathbb{P}\{\tilde{\tau} < \infty\} = e^{x\theta - x\sqrt{\theta^2}} = e^{x\theta - x|\theta|}.$$

If $\theta \geq 0$, then

$$\mathbb{P}\{\tilde{\tau} < \infty\} = 1.$$

If $\theta < 0$, then

$$\mathbb{P}\{\tilde{\tau} < \infty\} = e^{2x\theta} < 1.$$

(Recall that $x > 0$).

25.5 First passage times: Second method

(Based on martingales)

Let $\sigma > 0$ be given. Then

$$Y(t) = \exp\{\sigma B(t) - \tfrac{1}{2}\sigma^2 t\}$$

is a martingale, so $Y(t \wedge \tau)$ is also a martingale. We have

$$\begin{aligned}
1 &= Y(0 \wedge \tau) \\
&= I\!\!E Y(t \wedge \tau) \\
&= I\!\!E \exp\{\sigma B(t \wedge \tau) - \tfrac{1}{2}\sigma^2 (t \wedge \tau)\}. \\
&= \lim_{t \to \infty} I\!\!E \exp\{\sigma B(t \wedge \tau) - \tfrac{1}{2}\sigma^2 (t \wedge \tau)\}.
\end{aligned}$$

We want to take the limit inside the expectation. Since

$$0 \;\le\; \exp\{\sigma B(t \wedge \tau) - \tfrac{1}{2}\sigma^2 (t \wedge \tau)\} \;\le\; e^x,$$

this is justified by the Bounded Convergence Theorem. Therefore,

$$1 = I\!\!E \lim_{t \to \infty} \exp\{\sigma B(t \wedge \tau) - \tfrac{1}{2}\sigma^2 (t \wedge \tau)\}.$$

There are two possibilities. For those ω for which $\tau(\omega) < \infty$,

$$\lim_{t \to \infty} \exp\{\sigma B(t \wedge \tau) - \tfrac{1}{2}\sigma^2 (t \wedge \tau)\} = e^{\sigma x - \frac{1}{2}\sigma^2 \tau}.$$

For those ω for which $\tau(\omega) = \infty$,

$$\lim_{t \to \infty} \exp\{\sigma B(t \wedge \tau) - \tfrac{1}{2}\sigma^2 (t \wedge \tau)\} \le \lim_{t \to \infty} \exp\{\sigma x - \tfrac{1}{2}\sigma^2 t\} \;=\; 0.$$

Therefore,

$$\begin{aligned}
1 &= I\!\!E \lim_{t \to \infty} \exp\{\sigma B(t \wedge \tau) - \tfrac{1}{2}\sigma^2 (t \wedge \tau)\} \\
&= I\!\!E \left[e^{\sigma x - \frac{1}{2}\sigma^2 \tau} \mathbf{1}_{\tau < \infty} \right] \\
&= I\!\!E e^{\sigma x - \frac{1}{2}\sigma^2 \tau},
\end{aligned}$$

where we understand $e^{\sigma x - \frac{1}{2}\sigma^2 \tau}$ to be zero if $\tau = \infty$.

Let $\alpha = \frac{1}{2}\sigma^2$, so $\sigma = \sqrt{2\alpha}$. We have again derived the Laplace transform formula

$$e^{-x\sqrt{2\alpha}} = I\!\!E e^{-\alpha\tau}, \quad \alpha > 0, x > 0,$$

for the first passage time for nondrifted Brownian motion.

25.6 Perpetual American put

$$\begin{aligned}
dS &= rS\,dt + \sigma S\,dB \\
S(0) &= x \\
S(t) &= x \exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma B(t)\} \\
&= x \exp\left\{ \sigma \left[\underbrace{\left(\frac{r}{\sigma} - \frac{\sigma}{2} \right)}_{\theta} t + B(t) \right] \right\}.
\end{aligned}$$

Intrinsic value of the put at time t: $(K - S(t))^+$.

Let $L \in [0, K]$ be given. Define for $x \geq L$,

$$\begin{aligned}\tau_L &= \min\{t \geq 0;\ S(t) = L\}\\ &= \min\{t \geq 0;\ \theta t + B(t) = \frac{1}{\sigma}\log\frac{L}{x}\}\\ &= \min\{t \geq 0;\ -\theta t - B(t) = \frac{1}{\sigma}\log\frac{x}{L}\}\end{aligned}$$

Define

$$\begin{aligned}v_L &= (K - L)\mathbb{E}e^{-r\tau_L}\\ &= (K - L)\exp\left\{-\frac{\theta}{\sigma}\log\frac{x}{L} - \frac{1}{\sigma}\log\frac{x}{L}\sqrt{2r+\theta^2}\right\}\\ &= (K - L)\left(\frac{x}{L}\right)^{-\frac{\theta}{\sigma}-\frac{1}{\sigma}\sqrt{2r+\theta^2}}.\end{aligned}$$

We compute the exponent

$$\begin{aligned}-\frac{\theta}{\sigma} - \frac{1}{\sigma}\sqrt{2r+\theta^2} &= -\frac{r}{\sigma^2} + \tfrac{1}{2} - \frac{1}{\sigma}\sqrt{2r + \left(\frac{r}{\sigma} - \sigma/2\right)^2}\\ &= -\frac{r}{\sigma^2} + \tfrac{1}{2} - \frac{1}{\sigma}\sqrt{2r + \frac{r^2}{\sigma^2} - r + \sigma^2/4}\\ &= -\frac{r}{\sigma^2} + \tfrac{1}{2} - \frac{1}{\sigma}\sqrt{\frac{r^2}{\sigma^2} + r + \sigma^2/4}\\ &= -\frac{r}{\sigma^2} + \tfrac{1}{2} - \frac{1}{\sigma}\sqrt{\left(\frac{r}{\sigma} + \sigma/2\right)^2}\\ &= -\frac{r}{\sigma^2} + \tfrac{1}{2} - \frac{1}{\sigma}\left(\frac{r}{\sigma} + \sigma/2\right)\\ &= -\frac{2r}{\sigma^2}.\end{aligned}$$

Therefore,

$$v_L(x) = \begin{cases}(K - x), & 0 \leq x \leq L,\\ (K - L)\left(\frac{x}{L}\right)^{-2r/\sigma^2}, & x \geq L.\end{cases}$$

The curves $(K - L)\left(\frac{x}{L}\right)^{-2r/\sigma^2}$, are all of the form Cx^{-2r/σ^2}.

We want to choose the largest possible constant. The constant is

$$C = (K - L)L^{2r/\sigma^2},$$

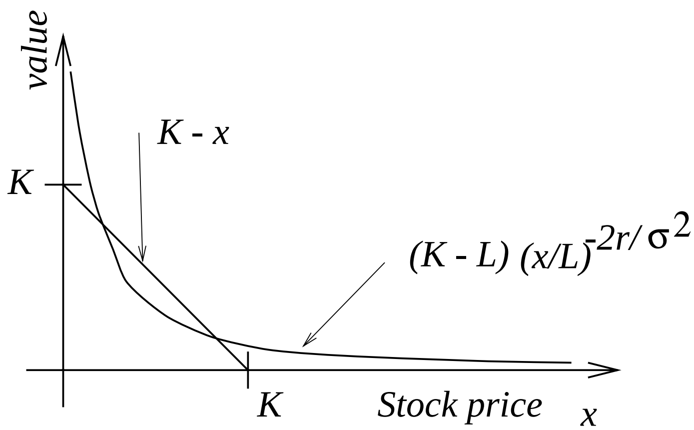

Figure 25.2: *Value of perpetual American put*

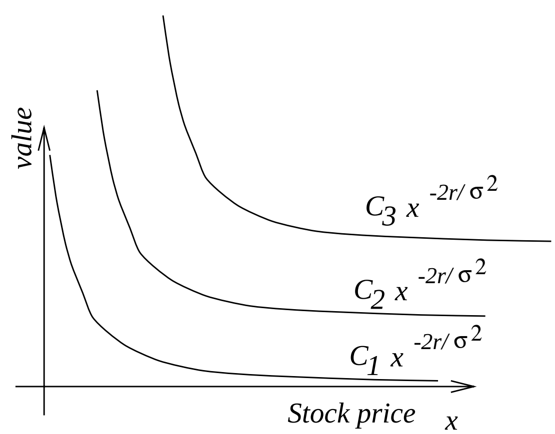

Figure 25.3: *Curves.*

and

$$\begin{aligned}\frac{\partial C}{\partial L} &= -L^{\frac{2r}{\sigma^2}} + \frac{2r}{\sigma^2}(K-L)L^{\frac{2r}{\sigma^2}-1}\\ &= L^{\frac{2r}{\sigma^2}}\left[-1+\frac{2r}{\sigma^2}(K-L)\frac{1}{L}\right]\\ &= L^{\frac{2r}{\sigma^2}}\left[-\left(1+\frac{2r}{\sigma^2}\right)+\frac{2r}{\sigma^2}\frac{K}{L}\right].\end{aligned}$$

We solve

$$-\left(1+\frac{2r}{\sigma^2}\right)+\frac{2r}{\sigma^2}\frac{K}{L}=0$$

to get

$$L=\frac{2rK}{\sigma^2+2r}.$$

Since $0 < 2r < \sigma^2 + 2r$, we have

$$0 < L < K.$$

Solution to the perpetual American put pricing problem (see Fig. 25.4):

$$v(x)=\begin{cases}(K-x), & 0 \le x \le L^*,\\ (K-L^*)\left(\frac{x}{L^*}\right)^{-2r/\sigma^2}, & x \ge L^*,\end{cases}$$

where

$$L^*=\frac{2rK}{\sigma^2+2r}.$$

Note that

$$v'(x)=\begin{cases}-1, & 0 \le x < L^*,\\ -\frac{2r}{\sigma^2}(K-L)^*(L^*)^{2r/\sigma^2}x^{-2r/\sigma^2-1}, & x > L^*.\end{cases}$$

We have

$$\begin{aligned}\lim_{x\downarrow L^*} v'(x) &= -2\frac{r}{\sigma^2}(K-L^*)\frac{1}{L^*}\\ &= -2\frac{r}{\sigma^2}\left(K-\frac{2rK}{\sigma^2+2r}\right)\frac{\sigma^2+2r}{2rK}\\ &= -2\frac{r}{\sigma^2}\left(\frac{\sigma^2+2r-2r}{\sigma^2+2r}\right)\frac{\sigma^2+2r}{2r}\\ &= -1\\ &= \lim_{x\uparrow L^*} v'(x).\end{aligned}$$

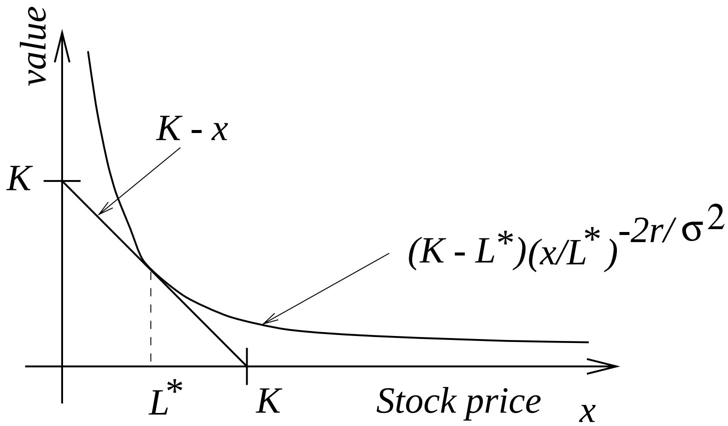

Figure 25.4: *Solution to perpetual American put.*

25.7 Value of the perpetual American put

Set

$$\gamma = \frac{2r}{\sigma^2}, \quad L^* = \frac{2rK}{\sigma^2 + 2r} = \frac{\gamma}{\gamma+1}K.$$

If $0 \le x < L^*$, then $v(x) = K - x$. If $L^* \le x < \infty$, then

$$v(x) = \underbrace{(K - L^*)(L^*)^\gamma}_{C} x^{-\gamma} \tag{7.1}$$

$$= I\!E^x \left[e^{-r\tau}(K - L^*)^+ \mathbf{1}_{\{\tau < \infty\}} \right], \tag{7.2}$$

where

$$S(0) = x \tag{7.3}$$

$$\tau = \min\{t \ge 0;\ S(t) = L^*\}. \tag{7.4}$$

If $0 \le x < L^*$, then

$$-rv(x) + rxv'(x) + \tfrac{1}{2}\sigma^2 x^2 v''(x) \;=\; -r(K - x) + rx(-1) \;=\; -rK.$$

If $L^* \le x < \infty$, then

$$\begin{aligned}
&-rv(x) + rxv'(x) + \tfrac{1}{2}\sigma^2 x^2 v''(x) \\
&= C[-rx^{-\gamma} - rx\gamma x^{-\gamma-1} - \tfrac{1}{2}\sigma^2 x^2 \gamma(-\gamma-1)x^{-\gamma-2}] \\
&= Cx^{-\gamma}[-r - r\gamma - \tfrac{1}{2}\sigma^2\gamma(-\gamma-1)] \\
&= C(-\gamma-1)x^{-\gamma}\left[r - \tfrac{1}{2}\sigma^2\left(\frac{2r}{\sigma^2}\right)\right] \\
&= 0.
\end{aligned}$$

In other words, v solves the *linear complementarity problem:* (See Fig. 25.5).

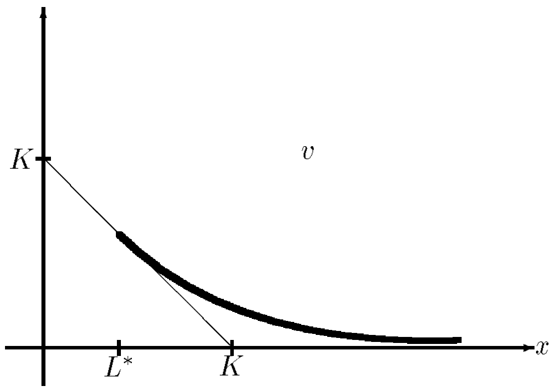

Figure 25.5: *Linear complementarity*

For all $x \in I\!\!R$, $x \neq L^*$,

$$rv - rxv' - \frac{1}{2}\sigma^2 x^2 v'' \geq 0, \quad \text{(a)}$$

$$v \geq (K - x)^+, \quad \text{(b)}$$

One of the inequalities (a) or (b) is an equality. (c)

The half-line $[0, \infty)$ is divided into two regions:

$$\mathcal{C} = \{x;\ v(x) > (K - x)^+\},$$
$$\mathcal{S} = \{x;\ rv - rxv' - \frac{1}{2}\sigma^2 x^2 v'' > 0\},$$

and L^* is the boundary between them. If the stock price is in $\mathcal{C}$, the owner of the put should not exercise (should "continue"). If the stock price is in $\mathcal{S}$ or at L^*, the owner of the put should exercise (should "stop").

25.8 Hedging the put

Let $S(0)$ be given. Sell the put at time zero for $v(S(0))$. Invest the money, holding $\Delta(t)$ shares of stock and consuming at rate $C(t)$ at time t. The value $X(t)$ of this portfolio is governed by

$$dX(t) = \Delta(t)\ dS(t)\ +\ r(X(t) - \Delta(t)S(t))\ dt\ -\ C(t)\ dt,$$

or equivalently,

$$d(e^{-rt}X(t)) = -e^{-rt}C(t)\ dt + e^{-rt}\Delta(t)\sigma S(t)\ dB(t).$$

The discounted value of the put satisfies

$$\begin{aligned} d\left(e^{-rt}v(S(t))\right) &= e^{-rt}\left[-rv(S(t)) + rS(t)v'(S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)v''(S(t))\right]\,dt \\ &\quad + e^{-rt}\sigma S(t)v'(S(t))\,dB(t) \\ &= -rKe^{-rt}\mathbf{1}_{\{S(t)<L^*\}}dt + e^{-rt}\sigma S(t)v'(S(t))\,dB(t). \end{aligned}$$

We should set

$$\begin{aligned} C(t) &= rK\mathbf{1}_{\{S(t)<L^*\}}, \\ \Delta(t) &= v'(S(t)). \end{aligned}$$

Remark 25.1 If $S(t) < L^*$, then

$$v(S(t)) = K - S(t), \quad \Delta(t) = v'(S(t)) = -1.$$

To hedge the put when $S(t) < L^*$, short one share of stock and hold K in the money market. As long as the owner does not exercise, you can consume the interest from the money market position, i.e.,

$$C(t) = rK\mathbf{1}_{\{S(t)<L^*\}}.$$

Properties of $e^{-rt}v(S(t))$:

1. $e^{-rt}v(S(t))$ is a supermartingale (see its differential above).
2. $e^{-rt}v(S(t)) \geq e^{-rt}(K - S(t))^+, \quad 0 \leq t < \infty$;
3. $e^{-rt}v(S(t))$ is the smallest process with properties 1 and 2.

Explanation of property 3. Let Y be a supermartingale satisfying

$$Y(t) \geq e^{-rt}(K - S(t))^+, \quad 0 \leq t < \infty. \tag{8.1}$$

Then property 3 says that

$$Y(t) \geq e^{-rt}v(S(t)), \quad 0 \leq t < \infty. \tag{8.2}$$

We use (8.1) to prove (8.2) for $t = 0$, i.e.,

$$Y(0) \geq v(S(0)). \tag{8.3}$$

If t is not zero, we can take t to be the initial time and $S(t)$ to be the initial stock price, and then adapt the argument below to prove property (8.2).

Proof of (8.3), assuming Y is a supermartingale satisfying (8.1):

Case I: $S(0) \leq L^*$. We have

$$Y(0) \underbrace{\geq}_{(8.1)} (K - S(0))^+ = v(S(0)).$$

Case II: $S(0) > L^*$: For $T > 0$, we have

$$\begin{aligned} Y(0) &\geq \mathbb{E} Y(\tau \wedge T) \quad \text{(Stopped supermartingale is a supermartingale)} \\ &\geq \mathbb{E}\left[Y(\tau \wedge T) \mathbf{1}_{\{\tau<\infty\}}\right]. \quad \text{(Since } Y \geq 0\text{)} \end{aligned}$$

Now let $T \to \infty$ to get

$$\begin{aligned} Y(0) &\geq \lim_{T \to \infty} \mathbb{E}\left[Y(\tau \wedge T) \mathbf{1}_{\{\tau<\infty\}}\right] \\ &\geq \mathbb{E}\left[Y(\tau) \mathbf{1}_{\{\tau<\infty\}}\right] \quad \text{(Fatou's Lemma)} \\ &\geq \mathbb{E}\left[e^{-r\tau}(K - \underbrace{S(\tau)}_{L^*})^+ \mathbf{1}_{\{\tau<\infty\}}\right] \quad \text{(by 8.1)} \\ &= v(S(0)). \quad \text{(See eq. 7.2)} \end{aligned}$$

25.9 Perpetual American contingent claim

Intinsic value: $h(S(t))$.

Value of the American contingent claim:

$$v(x) = \sup_{\tau} \mathbb{E}^x \left[e^{-r\tau} h(S(\tau))\right],$$

where the supremum is over all stopping times.

Optimal exercise rule: Any stopping time τ which attains the supremum.

Characterization of v:

1. $e^{-rt} v(S(t))$ is a supermartingale;
2. $e^{-rt} v(S(t)) \geq e^{-rt} h(S(t)), \quad 0 < t < \infty$;
3. $e^{-rt} v(S(t))$ is the smallest process with properties 1 and 2.

25.10 Perpetual American call

$$v(x) = \sup_{\tau} \mathbb{E}^x \left[e^{-r\tau} (S(\tau) - K)^+\right]$$

Theorem 10.63

$$v(x) = x \quad \forall x \geq 0.$$

Proof: For every t,

$$\begin{aligned} v(x) &\geq I\!\!E^x\left[e^{-rt}(S(t)-K)^+\right] \\ &\geq I\!\!E^x\left[e^{-rt}(S(t)-K)\right] \\ &= I\!\!E^x\left[e^{-rt}S(t)\right] - e^{-rt}K \\ &= x - e^{-rt}K. \end{aligned}$$

Let $t\to\infty$ to get $v(x) \geq x$.

Now start with $S(0) = x$ and define

$$Y(t) = e^{-rt}S(t).$$

Then:

1. Y is a supermartingale (in fact, Y is a martingale);
2. $Y(t) \geq e^{-rt}(S(t)-K)^+, \quad 0 \leq t < \infty$.

Therefore, $Y(0) \geq v(S(0))$, i.e.,

$$x \geq v(x).$$

■

Remark 25.2 No matter what τ we choose,

$$I\!\!E^x\left[e^{-r\tau}(S(\tau)-K)^+\right] \; < \; I\!\!E^x\left[e^{-r\tau}S(\tau)\right] \; \leq \; x \; = \; v(x).$$

There is no optimal exercise time.

25.11 Put with expiration

Expiration time: $T > 0$.

Intrinsic value: $(K - S(t))^+$.

Value of the put:

$$\begin{aligned} v(t,x) &= (\text{value of the put at time } t \text{ if } S(t)=x) \\ &= \underbrace{\sup_{t\leq\tau\leq T}}_{\tau:\text{stopping time}} I\!\!E^x e^{-r(\tau-t)}(K-S(\tau))^+. \end{aligned}$$

See Fig. 25.6. It can be shown that v, v_t, v_x are continuous across the boundary, while v_{xx} has a jump.

Let $S(0)$ be given. Then

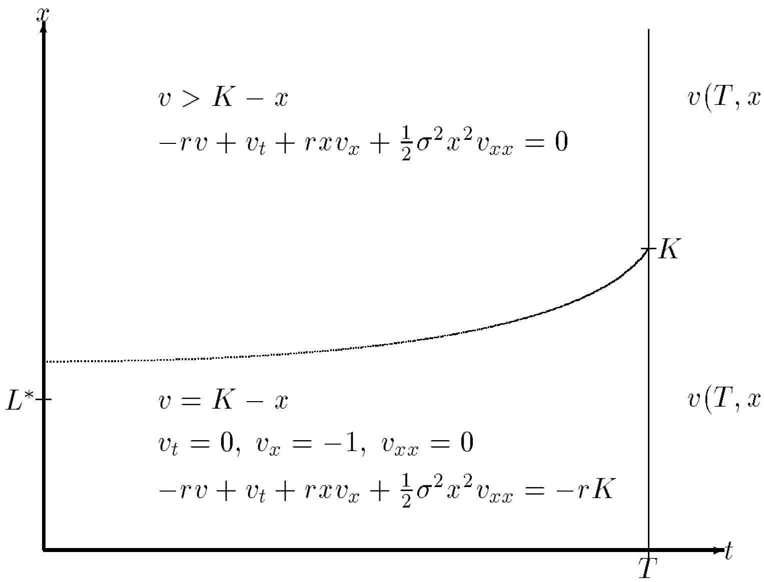

Figure 25.6: *Value of put with expiration*

1. $e^{-rt}v(t, S(t)), \quad 0 \leq t \leq T$, is a supermartingale;
2. $e^{-rt}v(t, S(t)) \geq e^{-rt}(K - S(t))^+, \quad 0 \leq t \leq T$;
3. $e^{-rt}v(t, S(t))$ is the smallest process with properties 1 and 2.

25.12 American contingent claim with expiration

Expiration time: $T > 0$.

Intrinsic value: $h(S(t))$.

Value of the contingent claim:

$$v(t,x) = \sup_{t \leq \tau \leq T} I\!E^x e^{-r(\tau - t)} h(S(\tau)).$$

Then

$$rv - v_t - rxv_x - \tfrac{1}{2}\sigma^2 x^2 v_{xx} \geq 0, \qquad \text{(a)}$$

$$v \geq h(x), \qquad \text{(b)}$$

At every point $(t,x) \in [0,T] \times [0,\infty)$, either (a) or (b) is an equality. (c)

Characterization of v: Let $S(0)$ be given. Then

1. $e^{-rt}v(t, S(t)), \ \ 0 \leq t \leq T$, is a supermartingale;
2. $e^{-rt}v(t, S(t)) \geq e^{-rt}h(S(t))$;
3. $e^{-rt}v(t, S(t))$ is the smallest process with properties 1 and 2.

The optimal exercise time is

$$\tau = \min\{t \geq 0; \ v(t, S(t)) = h(S(t))\}$$

If $\tau(\omega) = \infty$, then there is no optimal exercise time along the particular path ω.