Chapter 24

An outside barrier option

Barrier process:

$$\frac{dY(t)}{Y(t)} = \lambda\ dt + \sigma_1\ dB_1(t).$$

Stock process:

$$\frac{dS(t)}{S(t)} = \mu\ dt + \rho\sigma_2\ dB_1(t) + \sqrt{1-\rho^2}\ \sigma_2\ dB_2(t),$$

where $\sigma_1 > 0,\ \sigma_2 > 0,\ -1 < \rho < 1$, and B_1 and B_2 are independent Brownian motions on some $(\Omega, \mathcal{F}, \mathbb{P})$. The option pays off:

$$(S(T) - K)^+ \mathbf{1}_{\{Y^*(T) < L\}}$$

at time T, where

$$0 < S(0) < K, \quad 0 < Y(0) < L,$$

$$Y^*(T) = \max_{0 \le t \le T} Y(t).$$

Remark 24.1 The option payoff depends on both the Y and S processes. In order to hedge it, we will need the money market and two other assets, which we take to be Y and S. The risk-neutral measure must make the discounted value of every traded asset be a martingale, which in this case means the discounted Y and S processes.

We want to find θ_1 and θ_2 and define

$$d\widetilde{B}_1 = \theta_1\ dt + dB_1, \quad d\widetilde{B}_2 = \theta_2\ dt + dB_2,$$

so that

$$\begin{aligned}\frac{dY}{Y} &= r\ dt + \sigma_1 d\widetilde{B}_1 \\ &= r\ dt + \sigma_1\theta_1\ dt + \sigma_1\ dB_1, \\ \frac{dS}{S} &= r\ dt + \rho\sigma_2\ d\widetilde{B}_1 + \sqrt{1-\rho^2}\ \sigma_2 d\widetilde{B}_2 \\ &= r\ dt + \rho\sigma_2\ \theta_1\ dt + \sqrt{1-\rho^2}\ \sigma_2\theta_2\ dt \\ &\quad + \rho\sigma_2\ dB_1 + \sqrt{1-\rho^2}\ \sigma_2\ dB_2.\end{aligned}$$

We must have

$$\lambda = r + \sigma_1\theta_1, \tag{0.1}$$

$$\mu = r + \rho\sigma_2\theta_1 + \sqrt{1-\rho^2}\ \sigma_2\theta_2. \tag{0.2}$$

We solve to get

$$\begin{aligned}\theta_1 &= \frac{\lambda - r}{\sigma_1}, \\ \theta_2 &= \frac{\mu - r - \rho\sigma_2\theta_1}{\sqrt{1-\rho^2}\ \sigma_2}.\end{aligned}$$

We shall see that the formulas for θ_1 and θ_2 do not matter. What matters is that (0.1) and (0.2) uniquely determine θ_1 and θ_2. This implies the existence and uniqueness of the risk-neutral measure. We define

$$\begin{aligned}Z(T) &= \exp\left\{-\theta_1 B_1(T) - \theta_2 B_2(T) - \tfrac{1}{2}(\theta_1^2 + \theta_2^2)T\right\}, \\ \widetilde{I\!P}(A) &= \int_A Z(T)\ dI\!P, \quad \forall A \in \mathcal{F}.\end{aligned}$$

Under $\widetilde{I\!P}$, $\widetilde{B}_1$ and $\widetilde{B}_2$ are independent Brownian motions (Girsanov's Theorem). $\widetilde{I\!P}$ is the unique risk-neutral measure.

Remark 24.2 Under both $I\!P$ and $\widetilde{I\!P}$, Y has volatility σ_1, S has volatility σ_2 and

$$\frac{dY\ dS}{Y S} = \rho\sigma_1\sigma_2\ dt,$$

i.e., the correlation between $\frac{dY}{Y}$ and $\frac{dS}{S}$ is ρ.

The value of the option at time zero is

$$v(0, S(0), Y(0)) = \widetilde{I\!E}\left[e^{-rT}(S(T) - K)^+ \mathbf{1}_{\{Y^*(T) < L\}}\right].$$

We need to work out a density which permits us to compute the right-hand side.

Recall that the *barrier process* is

$$\frac{dY}{Y} = r\ dt + \sigma_1\ d\widetilde{B}_1,$$

so

$$Y(t) = Y(0)\exp\left\{rt + \sigma_1\widetilde{B}_1(t) - \tfrac{1}{2}\sigma_1^2 t\right\}.$$

Set

$$\begin{aligned}
\widehat{\theta} &= r/\sigma_1 - \sigma_1/2,\\
\widehat{B}(t) &= \widehat{\theta}t + \widetilde{B}_1(t),\\
\widehat{M}(T) &= \max_{0\le t\le T}\widehat{B}(t).
\end{aligned}$$

Then

$$\begin{aligned}
Y(t) &= Y(0)\exp\{\sigma_1\widehat{B}(t)\},\\
Y^*(T) &= Y(0)\exp\{\sigma_1\widehat{M}(T)\}.
\end{aligned}$$

The joint density of $\widehat{B}(T)$ and $\widehat{M}(T)$, appearing in Chapter 20, is

$$\begin{aligned}
&\widetilde{I\!P}\{\widehat{B}(T)\in d\hat{b}, \widehat{M}(T)\in d\hat{m}\}\\
&= \frac{2(2\hat{m}-\hat{b})}{T\sqrt{2\pi T}}\exp\left\{-\frac{(2\hat{m}-\hat{b})^2}{2T} + \widehat{\theta}\hat{b} - \tfrac{1}{2}\widehat{\theta}^2 T\right\}\ d\hat{b}\ d\hat{m},\\
&\hat{m} > 0, \hat{b} < \hat{m}.
\end{aligned}$$

The stock process.

$$\frac{dS}{S} = r\ dt + \rho\sigma_2 d\widetilde{B}_1 + \sqrt{1-\rho^2}\ \sigma_2 d\widetilde{B}_2,$$

so

$$\begin{aligned}
S(T) &= S(0)\exp\{rT + \rho\sigma_2\widetilde{B}_1(T) - \tfrac{1}{2}\rho^2\sigma_2^2 T + \sqrt{1-\rho^2}\ \sigma_2\widetilde{B}_2(T) - \tfrac{1}{2}(1-\rho^2)\sigma_2^2 T\}\\
&= S(0)\exp\{rT - \tfrac{1}{2}\sigma_2^2 T + \rho\sigma_2\widetilde{B}_1(T) + \sqrt{1-\rho^2}\ \sigma_2\widetilde{B}_2(T)\}
\end{aligned}$$

From the above paragraph we have

$$\widetilde{B}_1(T) = -\widehat{\theta}T + \widehat{B}(T),$$

so

$$S(T) = S(0)\exp\{rT + \rho\sigma_2\widehat{B}(T) - \tfrac{1}{2}\sigma_2^2 T - \rho\sigma_2\widehat{\theta}T + \sqrt{1-\rho^2}\ \sigma_2\widetilde{B}_2(T)\}$$

24.1 Computing the option value

$$
\begin{aligned}
v(0,S(0),Y(0)) &= \widetilde{I\!E}\left[e^{-rT}(S(T)-K)^{+}\mathbf{1}_{\{Y^*(T)<L\}}\right]\\
&= e^{-rT}\widetilde{I\!E}\Bigg[\left(S(0)\exp\left\{(r-\tfrac{1}{2}\sigma_2^2-\rho\sigma_2\widehat{\theta})T+\rho\sigma_2\widehat{B}(T)+\sqrt{1-\rho^2}\,\sigma_2\widetilde{B}_2(T)\right\}-K\right)^{+}\\
&\quad .\mathbf{1}_{\{Y(0)\exp[\sigma_1\widehat{M}(T)]<L\}}\Bigg]
\end{aligned}
$$

We know the joint density of $(\widehat{B}(T),\widehat{M}(T))$. The density of $\widetilde{B}_2(T)$ is

$$
\widetilde{I\!P}\{\widetilde{B}_2(T)\in d\tilde{b}\}=\frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{\tilde{b}^2}{2T}\right\}d\tilde{b},\quad \tilde{b}\in I\!R.
$$

Furthermore, the pair of random variables $(\widehat{B}(T),\widehat{M}(T))$ is *independent* of $\widetilde{B}_2(T)$ because $\widetilde{B}_1$ and $\widetilde{B}_2$ are independent under $\widetilde{I\!P}$. Therefore, the joint density of the random vector $(\widetilde{B}_2(T),\widehat{B}(T),\widehat{M}(T))$ is

$$
\widetilde{I\!P}\{\widetilde{B}_2(T)\in d\tilde{b},\widehat{B}(T)\in d\hat{b},\widehat{M}(T)\in d\hat{m},\}=\widetilde{I\!P}\{\widetilde{B}_2(T)\in d\tilde{b}\}.\widetilde{I\!P}\{\widehat{B}(T)\in d\hat{b},\widehat{M}(T)\in d\hat{m}\}
$$

The option value at time zero is

$$
\begin{aligned}
&v(0,S(0),Y(0))\\
&= e^{-rT}\int_0^{\frac{1}{\sigma_1}\log\frac{L}{Y(0)}}\int_{-\infty}^{\hat{m}}\int_{-\infty}^{\infty}\left(S(0)\exp\left\{(r-\tfrac{1}{2}\sigma_2^2-\rho\sigma_2\widehat{\theta})T+\rho\sigma_2\hat{b}+\sqrt{1-\rho^2}\sigma_2\tilde{b}\right\}-K\right)^{+}\\
&\quad .\frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{\tilde{b}^2}{2T}\right\}\\
&\quad .\frac{2(2\hat{m}-\hat{b})}{T\sqrt{2\pi T}}\exp\left\{-\frac{(2\hat{m}-\hat{b})^2}{2T}+\widehat{\theta}\hat{b}-\tfrac{1}{2}\widehat{\theta}^2T\right\}\\
&\quad .d\tilde{b}\;d\hat{b}\;d\hat{m}.
\end{aligned}
$$

The answer depends on $T,S(0)$ and $Y(0)$. It also depends on $\sigma_1,\sigma_2,\rho,r,K$ and L. It does not depend on λ,μ,θ_1, nor θ_2. The parameter $\widehat{\theta}$ appearing in the answer is $\widehat{\theta}=\frac{r}{\sigma_1}-\frac{\sigma_1}{2}$.

Remark 24.3 If we had not regarded Y as a traded asset, then we would not have tried to set its mean return equal to r. We would have had only one equation (see Eqs (0.1),(0.2))

$$
\mu=r+\rho\sigma_2\theta_1+\sqrt{1-\rho^2}\,\sigma_2\theta_2 \tag{1.1}
$$

to determine θ_1 and θ_2. The nonuniqueness of the solution alerts us that some options cannot be hedged. Indeed, any option whose payoff depends on Y cannot be hedged when we are allowed to trade only in the stock.

If we have an option whose payoff depends only on S, then Y is superfluous. Returning to the original equation for S,

$$\frac{dS}{S} = \mu\ dt + \rho\sigma_2\ dB_1 + \sqrt{1-\rho^2}\ \sigma_2\ dB_2,$$

we should set

$$dW = \rho\ dB_1 + \sqrt{1-\rho^2} dB_2,$$

so W is a Brownian motion under $I\!P$ (Levy's theorem), and

$$\frac{dS}{S} = \mu\ dt + \sigma_2 dW.$$

Now we have only Brownian motion, there will be only one θ, namely,

$$\theta = \frac{\mu - r}{\sigma_2},$$

so with $d\widetilde{W} = \theta\ dt + dW$, we have

$$\frac{dS}{S} = r\ dt + \sigma_2\ d\widetilde{W},$$

and we are on our way.

24.2 The PDE for the outside barrier option

Returning to the case of the option with payoff

$$(S(T) - K)^+ \mathbf{1}_{\{Y^*(T) < L\}},$$

we obtain a formula for

$$v(t,x,y) = e^{-r(T-t)} \widetilde{I\!E}^{t,x,y} \left[(S(T) - K)^+ \mathbf{1}_{\{\max_{t \le u \le T} Y(u)\ <\ L\}}, \right]$$

by replacing T, $S(0)$ and $Y(0)$ by $T-t$, x and y respectively in the formula for $v(0, S(0), Y(0))$. Now start at time 0 at $S(0)$ and $Y(0)$. Using the Markov property, we can show that the stochastic process

$$e^{-rt} v(t, S(t), Y(t))$$

is a martingale under $\widetilde{I\!P}$. We compute

$$\begin{aligned} & d\left[e^{-rt} v(t, S(t), Y(t))\right] \\ &= e^{-rt}\Big[\Big(-rv + v_t + rSv_x + rYv_y + \tfrac{1}{2}\sigma_2^2 S^2 v_{xx} + \rho\sigma_1\sigma_2 SY v_{xy} + \tfrac{1}{2}\sigma_1^2 Y^2 v_{yy} \Big)\ dt \\ &\quad + \rho\sigma_2 S v_x\ d\widetilde{B}_1 + \sqrt{1-\rho^2}\ \sigma_2 S v_x\ d\widetilde{B}_2 + \sigma_1 Y v_y d\widetilde{B}_1 \Big] \end{aligned}$$

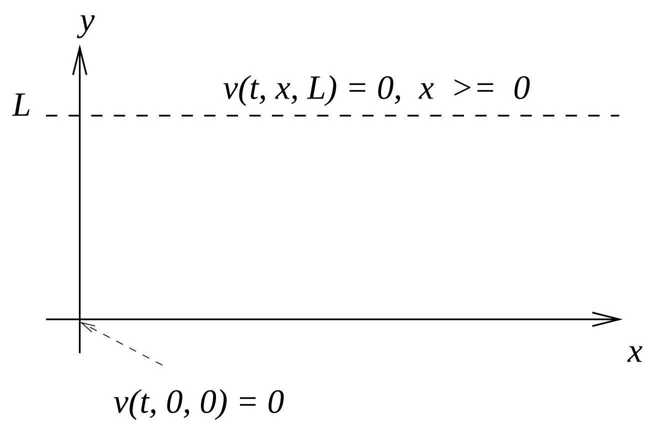

Figure 24.1: *Boundary conditions for barrier option. Note that* $t \in [0, T]$ *is fixed.*

Setting the dt term equal to 0, we obtain the PDE

$$\begin{aligned} -rv + v_t + rxv_x + ryv_y + \tfrac{1}{2}\sigma_2^2 x^2 v_{xx} \\ + \rho\sigma_1\sigma_2 xy v_{xy} + \tfrac{1}{2}\sigma_1^2 y^2 v_{yy} = 0, \\ 0 \le t < T, \quad x \ge 0, \quad 0 \le y \le L. \end{aligned}$$

The terminal condition is

$$v(T, x, y) = (x - K)^+, \quad x \ge 0,\ 0 \le y < L,$$

and the boundary conditions are

$$\begin{aligned} v(t, 0, 0) = 0, \quad 0 \le t \le T, \\ v(t, x, L) = 0, \quad 0 \le t \le T, \quad x \ge 0. \end{aligned}$$

$x = 0$ $-rv + v_t + ryv_y + \frac{1}{2}\sigma_1^2 y^2 v_{yy} = 0$	$y = 0$ $-rv + v_t + rxv_x + \frac{1}{2}\sigma_2^2 x^2 v_{xx} = 0$
This is the usual Black-Scholes formula in y.	This is the usual Black-Scholes formula in x.
The boundary conditions are $v(t, 0, L) = 0,\ v(t, 0, 0) = 0;$ the terminal condition is $v(T, 0, y) = (0 - K)^+ = 0, \quad y \geq 0.$	The boundary condition is $v(t, 0, 0) = e^{-r(T-t)}(0 - K)^+ = 0;$ the terminal condition is $v(T, x, 0) = (x - K)^+, \quad x \geq 0.$
On the $x = 0$ boundary, the option value is $v(t, 0, y) = 0, \quad 0 \leq y \leq L.$	On the $y = 0$ boundary, the barrier is irrelevant, and the option value is given by the usual Black-Scholes formula for a European call.

24.3 The hedge

After setting the dt term to 0, we have the equation

$$\begin{aligned} &d\left[e^{-rt}v(t, S(t), Y(t))\right] \\ &= e^{-rt}\left[\rho\sigma_2 S v_x\ d\widetilde{B}_1 + \sqrt{1-\rho^2}\ \sigma_2 S v_x\ d\widetilde{B}_2 + \sigma_1 Y v_y d\widetilde{B}_1\right], \end{aligned}$$

where $v_x = v_x(t, S(t), Y(t))$, $v_y = v_y(t, S(t), Y(t))$, and $\widetilde{B}_1, \widetilde{B}_2, S, Y$ are functions of t. Note that

$$\begin{aligned} d\left[e^{-rt}S(t)\right] &= e^{-rt}\left[-rS(t)\ dt + dS(t)\right] \\ &= e^{-rt}\left[\rho\sigma_2 S(t)\ d\widetilde{B}_1(t) + \sqrt{1-\rho^2}\ \sigma_2 S(t)\ d\widetilde{B}_2(t)\right]. \\ d\left[e^{-rt}Y(t)\right] &= e^{-rt}\left[-rY(t)\ dt + dY(t)\right] \\ &= e^{-rt}\sigma_1 Y(t)\ d\widetilde{B}_1(t). \end{aligned}$$

Therefore,

$$d\left[e^{-rt}v(t, S(t), Y(t))\right] = v_x d[e^{-rt}S] + v_y d[e^{-rt}Y].$$

Let $\Delta_2(t)$ denote the number of shares of stock held at time t, and let $\Delta_1(t)$ denote the number of "shares" of the barrier process Y. The value $X(t)$ of the portfolio has the differential

$$dX = \Delta_2 dS + \Delta_1 dY + r[X - \Delta_2 S - \Delta_1 Y]\ dt.$$

This is equivalent to

$$d[e^{-rt}X(t)] = \Delta_2(t)d[e^{-rt}S(t)] + \Delta_1(t)d[e^{-rt}Y(t)].$$

To get $X(t) = v(t, S(t), Y(t))$ for all t, we must have

$$X(0) = v(0, S(0), Y(0))$$

and

$$\Delta_2(t) = v_x(t, S(t), Y(t)),$$
$$\Delta_1(t) = v_y(t, S(t), Y(t)).$$