Chapter 24

An outside barrier option

Barrier process:

\[ \frac{dY(t)}{Y(t)} = \lambda \, dt + \sigma_1 \, dB_1(t), \]

Stock process:

\[ \frac{dS(t)}{S(t)} = \mu \, dt + \rho \sigma_2 \, dB_1(t) + \sqrt{1 - \rho^2} \, \sigma_2 \, dB_2(t), \]

where \( \sigma_1 > 0, \, \sigma_2 > 0, \, -1 < \rho < 1, \) and \( B_1 \) and \( B_2 \) are independent Brownian motions on some \((\Omega, \mathcal{F}, \mathbb{P})\). The option pays off:

\[ (S(T) - K)^+ \, \mathbf{1}_{\{Y^*(T) < L\}} \]

at time \( T \), where

\[ 0 < S(0) < K, \quad 0 < Y(0) < L, \]

\[ Y^*(T) = \max_{0 \leq t \leq T} Y(t). \]

**Remark 24.1** The option payoff depends on both the \( Y \) and \( S \) processes. In order to hedge it, we will need the money market and two other assets, which we take to be \( Y \) and \( S \). The risk-neutral measure must make the discounted value of every traded asset be a martingale, which in this case means the discounted \( Y \) and \( S \) processes.

We want to find \( \theta_1 \) and \( \theta_2 \) and define

\[ d\bar{B}_1 = \theta_1 \, dt + dB_1, \quad d\bar{B}_2 = \theta_2 \, dt + dB_2, \]
so that
\[
\begin{align*}
\frac{dY}{Y} &= r \, dt + \sigma_1 \, dB_1 \\
&= r \, dt + \sigma_1 \theta_1 \, dt + \sigma_1 \, dB_1, \\
\frac{dS}{S} &= r \, dt + \rho \sigma_2 \, dB_1 + \sqrt{1 - \rho^2} \, \sigma_2 \, dB_2 \\
&= r \, dt + \rho \sigma_2 \, \theta_1 \, dt + \sqrt{1 - \rho^2} \, \sigma_2 \theta_2 \, dt \\
&\quad + \rho \sigma_2 \, dB_1 + \sqrt{1 - \rho^2} \, \sigma_2 \, dB_2.
\end{align*}
\]

We must have
\[
\begin{align*}
\lambda &= r + \sigma_1 \theta_1, \\
\mu &= r + \rho \sigma_2 \theta_1 + \sqrt{1 - \rho^2} \, \sigma_2 \theta_2. 
\end{align*}
\]

(0.1) (0.2)

We solve to get
\[
\begin{align*}
\theta_1 &= \frac{\lambda - r}{\sigma_1}, \\
\theta_2 &= \frac{\mu - r - \rho \sigma_2 \theta_1}{\sqrt{1 - \rho^2} \, \sigma_2}.
\end{align*}
\]

We shall see that the formulas for $\theta_1$ and $\theta_2$ do not matter. What matters is that (0.1) and (0.2) uniquely determine $\theta_1$ and $\theta_2$. This implies the existence and uniqueness of the risk-neutral measure. We define
\[
\begin{align*}
Z(T) &= \exp \left\{ -\theta_1 B_1(T) - \theta_2 B_2(T) - \frac{1}{2} (\theta_1^2 + \theta_2^2) T \right\}, \\
\tilde{\mathbb{P}}(A) &= \int_A Z(T) \, d\mathbb{P}, \quad \forall A \in \mathcal{F}.
\end{align*}
\]

Under $\tilde{\mathbb{P}}$, $B_1$ and $B_2$ are independent Brownian motions (Girsanov’s Theorem). $\tilde{\mathbb{P}}$ is the unique risk-neutral measure.

Remark 24.2 Under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$, $Y$ has volatility $\sigma_1$, $S$ has volatility $\sigma_2$ and
\[
\frac{dY}{dS} = \rho \sigma_1 \sigma_2 \, dt,
\]
i.e., the correlation between $\frac{dY}{dS}$ and $\frac{dS}{dS}$ is $\rho$.

The value of the option at time zero is
\[
v(0, S(0), Y(0)) = \tilde{\mathbb{E}} \left[ e^{-rT} (S(T) - K)^+ 1_{\{Y(T) < L\}} \right].
\]

We need to work out a density which permits us to compute the right-hand side.
Recall that the barrier process is

\[
\frac{dY}{Y} = rt + \sigma_1 d\tilde{B}_1,
\]

so

\[
Y(t) = Y(0) \exp \left\{ rt + \sigma_1 \tilde{B}_1(t) - \frac{1}{2} \sigma_1^2 t \right\}.
\]

Set

\[
\hat{\theta} = r/\sigma_1 - \sigma_1/2,
\]

\[
\hat{B}(t) = \hat{\theta} t + \tilde{B}_1(t),
\]

\[
\hat{M}(T) = \max_{0 \leq t \leq T} \hat{B}(t).
\]

Then

\[
Y(t) = Y(0) \exp \{ \sigma_1 \hat{B}(t) \},
\]

\[
Y^*(T) = Y(0) \exp \{ \sigma_1 \hat{M}(T) \}.
\]

The joint density of \( \hat{B}(T) \) and \( \hat{M}(T) \), appearing in Chapter 20, is

\[
\mathbb{P} \left\{ \hat{B}(T) \in d\hat{b}, \hat{M}(T) \in d\hat{m} \right\} = \frac{2(2\hat{m} - \hat{b})}{T \sqrt{2\pi T}} \exp \left\{ -\frac{(2\hat{m} - \hat{b})^2}{2T} + \hat{b} \hat{\theta} - \frac{1}{2} \hat{\theta}^2 T \right\} \ d\hat{b} \ d\hat{m},
\]

\( \hat{m} > 0, \hat{b} < \hat{m} \).

The stock process.

\[
\frac{dS}{S} = rt + \rho \sigma_2 dB_1 + \sqrt{1 - \rho^2} \sigma_2 dB_2,
\]

so

\[
S(T) = S(0) \exp \{ rt + \rho \sigma_2 B_1(T) - \frac{1}{2} \rho^2 \sigma_2^2 T + \sqrt{1 - \rho^2} \sigma_2 B_2(T) - \frac{1}{2} (1 - \rho^2) \sigma_2^2 T \}
\]

\[
= S(0) \exp \{ rt - \frac{1}{2} \sigma_2^2 T + \rho \sigma_2 B_1(T) + \sqrt{1 - \rho^2} \sigma_2 B_2(T) \}
\]

From the above paragraph we have

\[
\tilde{B}_1(T) = -\hat{\theta} T + \hat{B}(T),
\]

so

\[
S(T) = S(0) \exp \{ rt + \rho \sigma_2 \hat{B}(T) - \frac{1}{2} \sigma_2^2 T - \rho \sigma_2 \hat{\theta} T + \sqrt{1 - \rho^2} \sigma_2 B_2(T) \}.
\]
24.1 Computing the option value

\[ v(0, S(0), Y(0)) = \mathbb{E} \left[ e^{-rT} (S(T) - K)^{+} 1_{[Y(T) < L]} \right] \]

\[ = e^{-rT} \mathbb{E} \left[ \left( S(0) \exp \left\{ \left( r - \frac{1}{2} \sigma_2^2 - \rho \sigma_2 \theta \right) T + \rho \sigma_2 \tilde{B}(T) + \sqrt{1 - \rho^2} \sigma_2 \tilde{B}_2(T) \right) - K \right)^{+} \right] \]

\[ \cdot 1_{[Y(0) \exp(\sigma_1 \tilde{M}(T)) < L]} \]

We know the joint density of \( (\tilde{B}(T), \tilde{M}(T)) \). The density of \( \tilde{B}_2(T) \) is

\[ \tilde{P}(\tilde{B}_2(T) \in \tilde{b}) = \frac{1}{\sqrt{2\pi T}} \exp \left\{ -\frac{\tilde{b}^2}{2T} \right\} \tilde{b}, \quad \tilde{b} \in \mathbb{R}. \]

Furthermore, the pair of random variables \( (\tilde{B}(T), \tilde{M}(T)) \) is independent of \( \tilde{B}_2(T) \) because \( \tilde{B}_1 \) and \( \tilde{B}_2 \) are independent under \( \tilde{P} \). Therefore, the joint density of the random vector \( (\tilde{B}_2(T), \tilde{B}(T), \tilde{M}(T)) \) is

\[ \tilde{P}(\tilde{B}_2(T) \in \tilde{b}, \tilde{B}(T) \in \tilde{b}, \tilde{M}(T) \in d\tilde{m}, \) \]

\[ = \tilde{P}(\tilde{B}_2(T) \in \tilde{b}) \tilde{P}(\tilde{B}(T) \in \tilde{b}) \tilde{P}(\tilde{M}(T) \in d\tilde{m}) \]

The option value at time zero is

\[ v(0, S(0), Y(0)) \]

\[ = e^{-rT} \int_0^{\sigma_1 \log \frac{S(0)}{100}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S(0) \exp \left\{ \left( r - \frac{1}{2} \sigma_2^2 - \rho \sigma_2 \theta \right) T + \rho \sigma_2 \tilde{b} + \sqrt{1 - \rho^2} \sigma_2 \tilde{b}_2 \right) - K \right)^{+} \]

\[ \cdot \frac{1}{\sqrt{2\pi T}} \exp \left\{ -\frac{\tilde{b}^2}{2T} \right\} \]

\[ \cdot \frac{2(2\tilde{m} - \tilde{b})}{T \sqrt{2\pi T}} \exp \left\{ -\frac{(2\tilde{m} - \tilde{b})^2}{2T} + \tilde{b} \tilde{b} - \frac{1}{2} \tilde{b}^2 T \right\} \]

\[ . d\tilde{b} \, d\tilde{b} \, d\tilde{m}. \]

The answer depends on \( T, S(0) \) and \( Y(0) \). It also depends on \( \sigma_1, \sigma_2, \rho, r, K \) and \( L \). It does not depend on \( \lambda, \mu, \theta_1 \), nor \( \theta_2 \). The parameter \( \hat{\theta} \) appearing in the answer is

\[ \hat{\theta} = \frac{1}{\sigma_1^2} - \frac{\sigma_2^2}{2\sigma_1^2}. \]

Remark 24.3 If we had not regarded \( Y \) as a traded asset, then we would not have tried to set its mean return equal to \( r \). We would have had only one equation (see Eqs (0.1),(0.2))

\[ \mu = r + \rho \sigma_2 \theta_1 + \sqrt{1 - \rho^2} \sigma_2 \theta_2 \]

(1.1)

to determine \( \theta_1 \) and \( \theta_2 \). The nonuniqueness of the solution alerts us that some options cannot be hedged. Indeed, any option whose payoff depends on \( Y \) cannot be hedged when we are allowed to trade only in the stock.
If we have an option whose payoff depends only on \( S \), then \( Y \) is superfluous. Returning to the original equation for \( S \),

\[
\frac{dS}{S} = \mu \, dt + \rho \sigma_2 \, dB_1 + \sqrt{1 - \rho^2} \sigma_2 \, dB_2,
\]

we should set

\[
dW = \rho \, dB_1 + \sqrt{1 - \rho^2} \, dB_2,
\]

so \( W \) is a Brownian motion under \( \mathbb{P} \) (Levy’s theorem), and

\[
\frac{dS}{S} = \mu \, dt + \sigma_2 \, dW.
\]

Now we have only Brownian motion, there will be only one \( \theta \), namely,

\[
\theta = \frac{\mu - r}{\sigma_2},
\]

so with \( d\tilde{W} = \theta \, dt + dW \), we have

\[
\frac{dS}{S} = r \, dt + \sigma_2 \, d\tilde{W},
\]

and we are on our way.

### 24.2 The PDE for the outside barrier option

Returning to the case of the option with payoff

\[
(S(T) - K)^+ 1_{\{Y^*(T) < L\}},
\]

we obtain a formula for

\[
v(t, x, y) = e^{-r(T-t)} \mathbb{E}^\mathbb{P}_{x,y} \left[ (S(T) - K)^+ 1_{\{\max_{0 \leq s \leq T} Y(s) < L\}} \right],
\]

by replacing \( T, S(0) \) and \( Y(0) \) by \( T-t, x \) and \( y \) respectively in the formula for \( v(0, S(0), Y(0)) \). Now start at time 0 at \( S(0) \) and \( Y(0) \). Using the Markov property, we can show that the stochastic process

\[
e^{-r t} v(t, S(t), Y(t))
\]

is a martingale under \( \mathbb{P} \). We compute

\[
d \left[ e^{-r t} v(t, S(t), Y(t)) \right]
\]

\[
e^{-r t} \left[ (-r v + v_t + rS v_x + rY v_y + \frac{1}{2} \sigma_2^2 S^2 v_{xx} + \rho \sigma_1 \sigma_2 SY v_{xy} + \frac{1}{2} \sigma_1^2 Y^2 v_{yy}) \, dtight.
\]

\[+ \rho \sigma_2 S v_x \, dB_1 + \sqrt{1 - \rho^2} \sigma_2 S v_x \, dB_2 + \sigma_1 Y v_y \, dB_1]}

Setting the $dt$ term equal to 0, we obtain the PDE

$$-rv + vt + vxv_x + vyv_y + \frac{1}{2} \sigma^2 v_x v_{xx} + \rho \sigma \sigma_2 xy v_{xy} + \frac{1}{2} \sigma^2 y^2 v_{yy} = 0, \quad 0 \leq t < T, \quad x \geq 0, \quad 0 \leq y \leq L.$$  

The terminal condition is

$$v(T, x, y) = (x - K)^+, \quad x \geq 0, \quad 0 \leq y < L,$$

and the boundary conditions are

$$v(t, 0, 0) = 0, \quad 0 \leq t \leq T,$$

$$v(t, x, L) = 0, \quad 0 \leq t \leq T, \quad x \geq 0.$$
\[
x = 0 \\
-rv + vt + ryv_y + \frac{1}{2} \sigma_1^2 y^2 v_{yy} = 0
\]

This is the usual Black-Scholes formula in \( y \).

The boundary conditions are
\[
v(t, 0, L) = 0, \ v(t, 0, 0) = 0; \\
\text{the terminal condition is} \\
v(T, 0, y) = (0 - K)^+ = 0, \quad y \geq 0.
\]

On the \( x = 0 \) boundary, the option value is \( v(t, 0, y) = 0, \quad 0 \leq y \leq L \).

\[
y = 0 \\
-rv + vt + rxv_x + \frac{1}{2} \sigma_2^2 x^2 v_{xx} = 0
\]

This is the usual Black-Scholes formula in \( x \).

The boundary condition is
\[
v(t, 0, 0) = e^{-r(T-t)}(0 - K)^+ = 0; \\
\text{the terminal condition is} \\
v(T, x, 0) = (x - K)^+, \quad x \geq 0.
\]

On the \( y = 0 \) boundary, the barrier is irrelevant, and the option value is given by the usual Black-Scholes formula for a European call.

### 24.3 The hedge

After setting the \( dt \) term to 0, we have the equation
\[
d \left[ e^{-rt} v(t, S(t), Y(t)) \right]
= e^{-rt} \left[ \rho \sigma_2 S v_x \ d \bar{B}_1 + \sqrt{1 - \rho^2} \ \sigma_2 S v_x \ d \bar{B}_2 + \sigma_1 Y v_y \ d \bar{B}_1 \right],
\]
where \( v_x = v_x(t, S(t), Y(t)), \ v_y = v_y(t, S(t), Y(t)), \) and \( \bar{B}_1, \bar{B}_2, S, Y \) are functions of \( t \). Note that
\[
d \left[ e^{-rt} S(t) \right] = e^{-rt} \left[ -rS(t) \ dt + dS(t) \right]
= e^{-rt} \left[ \rho \sigma_2 S(t) \ d \bar{B}_1(t) + \sqrt{1 - \rho^2} \ \sigma_2 S(t) \ d \bar{B}_2(t) \right].
\]
\[
d \left[ e^{-rt} Y(t) \right] = e^{-rt} \left[ -rY(t) \ dt + dY(t) \right]
= e^{-rt} \sigma_1 Y(t) \ d \bar{B}_1(t).
\]

Therefore,
\[
d \left[ e^{-rt} v(t, S(t), Y(t)) \right] = v_x d \left[ e^{-rt} S \right] + v_y d \left[ e^{-rt} Y \right].
\]

Let \( \Delta_2(t) \) denote the number of shares of stock held at time \( t \), and let \( \Delta_1(t) \) denote the number of “shares” of the barrier process \( Y \). The value \( X(t) \) of the portfolio has the differential
\[
dX = \Delta_2 dS + \Delta_1 dY + r[X - \Delta_2 S - \Delta_1 Y] \ dt.
\]
This is equivalent to

\[ d[e^{-\tau t} X(t)] = \Delta_2(t) d[e^{-\tau t} S(t)] + \Delta_1(t) d[e^{-\tau t} Y(t)]. \]

To get \( X(t) = v(t, S(t), Y(t)) \) for all \( t \), we must have

\[ X(0) = v(0, S(0), Y(0)) \]

and

\[ \Delta_2(t) = v_x(t, S(t), Y(t)), \]
\[ \Delta_1(t) = v_y(t, S(t), Y(t)). \]