Chapter 23

Recognizing a Brownian Motion

Theorem 0.62 (Levy) *Let $B(t), 0 \leq t \leq T$, be a process on $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a filtration $\mathcal{F}(t), 0 \leq t \leq T$, such that:*

1. *the paths of $B(t)$ are continuous,*
2. *B is a martingale,*
3. *$\langle B \rangle(t) = t, 0 \leq t \leq T$, (i.e., informally $dB(t)\ dB(t) = dt$).*

Then B is a Brownian motion.

Proof: (Idea) Let $0 \leq s < t \leq T$ be given. We need to show that $B(t) - B(s)$ is normal, with mean zero and variance $t - s$, and $B(t) - B(s)$ is independent of $\mathcal{F}(s)$. We shall show that the *conditional moment generating function* of $B(t) - B(s)$ is

$$\mathbb{E}\left[e^{u(B(t)-B(s))} \middle| \mathcal{F}(s) \right] = e^{\frac{1}{2}u^2(t-s)}.$$

Since the moment generating function characterizes the distribution, this shows that $B(t) - B(s)$ is normal with mean 0 and variance $t - s$, and conditioning on $\mathcal{F}(s)$ does not affect this, i.e., $B(t) - B(s)$ is independent of $\mathcal{F}(s)$.

We compute (this uses the continuity condition (1) of the theorem)

$$de^{uB(t)} = ue^{uB(t)}dB(t) + \tfrac{1}{2}u^2 e^{uB(t)}dB(t)\ dB(t),$$

so

$$e^{uB(t)} = e^{uB(s)} + \int_s^t ue^{uB(v)}\ dB(v) + \tfrac{1}{2}u^2 \int_s^t e^{uB(v)} \underbrace{dv.}_{\text{uses cond. 3}}$$

Now $\int_0^t ue^{uB(v)}dB(v)$ is a martingale (by condition 2), and so

$$\begin{aligned}
&\mathbb{E}\left[\int_s^t ue^{uB(v)}dB(v)\middle|\mathcal{F}(s)\right] \\
&= -\int_0^s ue^{uB(v)}dB(v) + \mathbb{E}\left[\int_0^t ue^{uB(v)}dB(v)\middle|\mathcal{F}(s)\right] \\
&= 0.
\end{aligned}$$

It follows that

$$\mathbb{E}\left[e^{uB(t)}\middle|\mathcal{F}(s)\right] = e^{uB(s)} + \tfrac{1}{2}u^2\int_s^t \mathbb{E}\left[e^{uB(v)}\middle|\mathcal{F}(s)\right]\,dv.$$

We define

$$\varphi(v) = \mathbb{E}\left[e^{uB(v)}\middle|\mathcal{F}(s)\right],$$

so that

$$\varphi(s) = e^{uB(s)}$$

and

$$\begin{aligned}
\varphi(t) &= e^{uB(s)} + \tfrac{1}{2}u^2\int_s^t \varphi(v)\,dv, \\
\varphi'(t) &= \tfrac{1}{2}u^2\varphi(t), \\
\varphi(t) &= ke^{\frac{1}{2}u^2 t}.
\end{aligned}$$

Plugging in s, we get

$$e^{uB(s)} = ke^{\frac{1}{2}u^2 s} \Longrightarrow k = e^{uB(s)-\frac{1}{2}u^2 s}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[e^{uB(t)}\middle|\mathcal{F}(s)\right] &= \varphi(t) = e^{uB(s)+\frac{1}{2}u^2(t-s)}, \\
\mathbb{E}\left[e^{u(B(t)-B(s))}\middle|\mathcal{F}(s)\right] &= e^{\frac{1}{2}u^2(t-s)}.
\end{aligned}$$

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23.1 Identifying volatility and correlation

Let B_1 and B_2 be independent Brownian motions and

$$\frac{dS_1}{S_1} = r\ dt + \sigma_{11}\ dB_1 + \sigma_{12}\ dB_2,$$
$$\frac{dS_2}{S_2} = r\ dt + \sigma_{21}\ dB_1 + \sigma_{22}\ dB_2,$$

Define

$$\sigma_1 = \sqrt{\sigma_{11}^2 + \sigma_{12}^2},$$
$$\sigma_2 = \sqrt{\sigma_{21}^2 + \sigma_{22}^2},$$
$$\rho = \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2}.$$

Define processes W_1 and W_2 by

$$dW_1 = \frac{\sigma_{11}\ dB_1 + \sigma_{12}\ dB_2}{\sigma_1}$$
$$dW_2 = \frac{\sigma_{21}\ dB_1 + \sigma_{22}\ dB_2}{\sigma_2}.$$

Then W_1 and W_2 have continuous paths, are martingales, and

$$\begin{aligned} dW_1\ dW_1 &= \frac{1}{\sigma_1^2}(\sigma_{11}dB_1 + \sigma_{12}dB_2)^2 \\ &= \frac{1}{\sigma_1^2}(\sigma_{11}^2 dB_1\ dB_1 + \sigma_{12}^2 dB_2\ dB_2) \\ &= dt, \end{aligned}$$

and similarly

$$dW_2\ dW_2 = dt.$$

Therefore, W_1 and W_2 are Brownian motions. The stock prices have the representation

$$\frac{dS_1}{S_1} = r\ dt + \sigma_1\ dW_1,$$
$$\frac{dS_2}{S_2} = r\ dt + \sigma_2\ dW_2.$$

The Brownian motions W_1 and W_2 are correlated. Indeed,

$$\begin{aligned} dW_1\ dW_2 &= \frac{1}{\sigma_1\sigma_2}(\sigma_{11}dB_1 + \sigma_{12}dB_2)(\sigma_{21}dB_1 + \sigma_{22}dB_2) \\ &= \frac{1}{\sigma_1\sigma_2}(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})\ dt \\ &= \rho\ dt. \end{aligned}$$

23.2 Reversing the process

Suppose we are given that

$$\frac{dS_1}{S_1} = r\ dt + \sigma_1 dW_1,$$
$$\frac{dS_2}{S_2} = r\ dt + \sigma_2 dW_2,$$

where W_1 and W_2 are Brownian motions with correlation coefficient ρ. We want to find

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

so that

$$\begin{aligned} \Sigma\Sigma' &= \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11}^2 + \sigma_{12}^2 & \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} \\ \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} & \sigma_{21}^2 + \sigma_{22}^2 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \end{aligned}$$

A simple (but not unique) solution is (see Chapter 19)

$$\sigma_{11} = \sigma_1, \qquad \sigma_{12} = 0,$$
$$\sigma_{21} = \rho\sigma_2, \qquad \sigma_{22} = \sqrt{1-\rho^2}\ \sigma_2.$$

This corresponds to

$$\begin{aligned} \sigma_1\ dW_1 &= \sigma_1 dB_1 \Longrightarrow dB_1 = dW_1, \\ \sigma_2\ dW_2 &= \rho\sigma_2\ dB_1 + \sqrt{1-\rho^2}\sigma_2\ dB_2 \\ &\Longrightarrow dB_2 = \frac{dW_2 - \rho\ dW_1}{\sqrt{1-\rho^2}}, \quad (\rho \neq \pm 1) \end{aligned}$$

If $\rho = \pm 1$, then there is no B_2 and $dW_2 = \rho\ dB_1 = \rho\ dW_1$.

Continuing in the case $\rho \neq \pm 1$, we have

$$\begin{aligned} dB_1\ dB_1 &= dW_1\ dW_1 = dt, \\ dB_2\ dB_2 &= \frac{1}{1-\rho^2}\left(dW_2\ dW_2 - 2\rho\ dW_1\ dW_2 + \rho^2 dW_2\ dW_2\right) \\ &= \frac{1}{1-\rho^2}\left(dt - 2\rho^2\ dt + \rho^2\ dt\right) \\ &= dt, \end{aligned}$$

so both B_1 and B_2 are Brownian motions. Furthermore,

$$\begin{aligned} dB_1\ dB_2 &= \frac{1}{\sqrt{1-\rho^2}}\,(dW_1\ dW_2 - \rho dW_1\ dW_1) \\ &= \frac{1}{\sqrt{1-\rho^2}}\,(\rho\ dt - \rho\ dt) = 0. \end{aligned}$$

We can now apply an **Extension of Levy's Theorem** that says that Brownian motions with zero cross-variation are independent, to conclude that B_1, B_2 are independent Brownians.