Chapter 22

Summary of Arbitrage Pricing Theory

A *simple European derivative security* makes a random payment at a time fixed in advance. The *value at time* t of such a security is the amount of wealth needed at time t in order to replicate the security by trading in the market. The *hedging portfolio* is a specification of how to do this trading.

22.1 Binomial model, Hedging Portfolio

Let Ω be the set of all possible sequences of n coin-tosses. We have *no probabilities* at this point. Let $r \geq 0,\ u > r+1,\ d = 1/u$ be given. (See Fig. 2.1)

Evolution of the value of a portfolio:

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k).$$

Given a simple European derivative security $V(\omega_1, \omega_2)$, we want to start with a nonrandom X_0 and use a portfolio processes

$$\Delta_0,\ \Delta_1(H),\ \Delta_1(T)$$

so that

$$X_2(\omega_1, \omega_2) = V(\omega_1, \omega_2) \quad \forall \omega_1, \omega_2. \qquad \text{(four equations)}$$

There are four unknowns: X_0, Δ_0, $\Delta_1(H)$, $\Delta_1(T)$. Solving the equations, we obtain:

$$
\begin{aligned}
X_1(\omega_1) &= \frac{1}{1+r}\left[\frac{1+r-d}{u-d}\underbrace{X_2(\omega_1,H)}_{V(\omega_1,H)}+\frac{u-(1+r)}{u-d}\underbrace{X_2(\omega_1,T)}_{V(\omega_1,T)}\right],\\
X_0 &= \frac{1}{1+r}\left[\frac{1+r-d}{u-d}X_1(H)+\frac{u-(1+r)}{u-d}X_1(T)\right],\\
\Delta_1(\omega_1) &= \frac{X_2(\omega_1,H)-X_2(\omega_1,T)}{S_2(\omega_1,H)-S_2(\omega_1,T)},\\
\Delta_0 &= \frac{X_1(H)-X_1(T)}{S_1(H)-S_1(T)}.
\end{aligned}
$$

The probabilities of the stock price paths are irrelevant, because we have a hedge which works on *every path.* From a practical point of view, what matters is that the paths in the model include all the possibilities. We want to find a description of the paths in the model. They all have the property

$$
\begin{aligned}
(\log S_{k+1}-\log S_k)^2 &= \left(\log\frac{S_{k+1}}{S_k}\right)^2\\
&= (\pm\log u)^2\\
&= (\log u)^2.
\end{aligned}
$$

Let $\sigma = \log u > 0$. Then

$$\sum_{k=0}^{n-1}(\log S_{k+1}-\log S_k)^2 = \sigma^2 n.$$

The paths of $\log S_k$ accumulate quadratic variation at rate σ^2 per unit time.

If we change u, then we change σ, and the pricing and hedging formulas on the previous page will give different results.

We reiterate that the probabilities are only introduced as an aid to understanding and computation. Recall:

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k).$$

Define

$$\beta_k = (1+r)^k.$$

Then

$$\frac{X_{k+1}}{\beta_{k+1}} = \Delta_k\frac{S_{k+1}}{\beta_{k+1}} + \frac{X_k}{\beta_k} - \Delta_k\frac{S_k}{\beta_k},$$

i.e.,

$$\frac{X_{k+1}}{\beta_{k+1}} - \frac{X_k}{\beta_k} = \Delta_k\left(\frac{S_{k+1}}{\beta_{k+1}} - \frac{S_k}{\beta_k}\right).$$

In continuous time, we will have the analogous equation

$$d\left(\frac{X(t)}{\beta(t)}\right) = \Delta(t)\ d\left(\frac{S(t)}{\beta(t)}\right).$$

If we introduce a probability measure $\widetilde{I\!P}$ under which $\frac{S_k}{\beta_k}$ is a martingale, then $\frac{X_k}{\beta_k}$ will also be a martingale, regardless of the portfolio used. Indeed,

$$\begin{aligned}\widetilde{I\!E}\left[\frac{X_{k+1}}{\beta_{k+1}}\bigg|\mathcal{F}_k\right] &= \widetilde{I\!E}\left[\frac{X_k}{\beta_k}+\Delta_k\left(\frac{S_{k+1}}{\beta_{k+1}}-\frac{S_k}{\beta_k}\right)\bigg|\mathcal{F}_k\right]\\ &= \frac{X_k}{\beta_k}+\Delta_k\underbrace{\left(\widetilde{I\!E}\left[\frac{S_{k+1}}{\beta_{k+1}}\bigg|\mathcal{F}_k\right]-\frac{S_k}{\beta_k}\right)}_{=0}.\end{aligned}$$

Suppose we want to have $X_2 = V$, where V is some $\mathcal{F}_2$-measurable random variable. Then we must have

$$\begin{aligned}\frac{1}{1+r}X_1 = \frac{X_1}{\beta_1} &= \widetilde{I\!E}\left[\frac{X_2}{\beta_2}\bigg|\mathcal{F}_1\right] = \widetilde{I\!E}\left[\frac{V}{\beta_2}\bigg|\mathcal{F}_1\right],\\ X_0 = \frac{X_0}{\beta_0} &= \widetilde{I\!E}\left[\frac{X_1}{\beta_1}\right] = \widetilde{I\!E}\left[\frac{V}{\beta_2}\right].\end{aligned}$$

To find the risk-neutral probability measure $\widetilde{I\!P}$ under which $\frac{S_k}{\beta_k}$ is a martingale, we denote $\tilde{p} = \widetilde{I\!P}\{\omega_k = H\}$, $\tilde{q} = \widetilde{I\!P}\{\omega_k = T\}$, and compute

$$\begin{aligned}\widetilde{I\!E}\left[\frac{S_{k+1}}{\beta_{k+1}}\bigg|\mathcal{F}_k\right] &= \tilde{p}u\frac{S_k}{\beta_{k+1}}+\tilde{q}d\frac{S_k}{\beta_{k+1}}\\ &= \frac{1}{1+r}[\tilde{p}u+\tilde{q}d]\frac{S_k}{\beta_k}.\end{aligned}$$

We need to choose $\tilde{p}$ and $\tilde{q}$ so that

$$\begin{aligned}\tilde{p}u+\tilde{q}d &= 1+r,\\ \tilde{p}+\tilde{q} &= 1.\end{aligned}$$

The solution of these equations is

$$\tilde{p} = \frac{1+r-d}{u-d},\quad \tilde{q} = \frac{u-(1+r)}{u-d}.$$

22.2 Setting up the continuous model

Now the stock price $S(t), 0 \le t \le T$, is a continuous function of t. We would like to hedge along every possible path of $S(t)$, but that is impossible. Using the binomial model as a guide, we choose $\sigma > 0$ and try to hedge along every path $S(t)$ for which the quadratic variation of $\log S(t)$ accumulates at rate σ^2 per unit time. These are the paths with volatility σ^2.

To generate these paths, we use Brownian motion, rather than coin-tossing. To introduce Brownian motion, we need a probability measure. However, the only thing about this probability measure which ultimately matters is the set of paths to which it assigns probability zero.

Let $B(t), 0 \leq t \leq T$, be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any $\rho \in I\!R$, the paths of

$$\rho t + \sigma B(t)$$

accumulate quadratic variation at rate σ^2 per unit time. We want to define

$$S(t) = S(0) \exp\{\rho t + \sigma B(t)\},$$

so that the paths of

$$\log S(t) = \log S(0) + \rho t + \sigma B(t)$$

accumulate quadratic variation at rate σ^2 per unit time. Surprisingly, the choice of ρ in this definition is irrelevant. Roughly, the reason for this is the following: Choose $\omega_1 \in \Omega$. Then, for $\rho_1 \in I\!R$,

$$\rho_1 t + \sigma B(t, \omega_1), \quad 0 \leq t \leq T,$$

is a continuous function of t. If we replace ρ_1 by ρ_2, then $\rho_2 t + \sigma B(t, \omega_1)$ is a different function. However, there is an $\omega_2 \in \Omega$ such that

$$\rho_1 t + \sigma B(t, \omega_1) = \rho_2 t + \sigma B(t, \omega_2), \quad 0 \leq t \leq T.$$

In other words, regardless of whether we use ρ_1 or ρ_2 in the definition of $S(t)$, we will see the same paths. The mathematically precise statement is the following:

> If a set of stock price paths has a positive probability when $S(t)$ is defined by
>
> $$S(t) = S(0) \exp\{\rho_1 t + \sigma B(t)\},$$
>
> then this set of paths has positive probability when $S(t)$ is defined by
>
> $$S(t) = S(0) \exp\{\rho_2 t + \sigma B(t)\}.$$
>
> Since we are interested in hedging along every path, except possibly for a set of paths which has probability zero, the choice of ρ is irrelevant.

The most *convenient* choice of ρ is

$$\rho = r - \tfrac{1}{2}\sigma^2,$$

so

$$S(t) = S(0) \exp\{rt + \sigma B(t) - \tfrac{1}{2}\sigma^2 t\},$$

and

$$e^{-rt} S(t) = S(0) \exp\{\sigma B(t) - \tfrac{1}{2}\sigma^2 t\}$$

is a martingale under $I\!P$. With this choice of ρ,

$$dS(t) = rS(t)\, dt + \sigma S(t)\, dB(t)$$

and $I\!P$ is the risk-neutral measure. If a different choice of ρ is made, we have

$$
\begin{aligned}
S(t) &= S(0)\exp\{\rho t + \sigma B(t)\}, \\
dS(t) &= \underbrace{(\rho + \tfrac{1}{2}\sigma^2)}_{\mu} S(t)\ dt + \sigma S(t)\ dB(t). \\
&= rS(t)\ dt + \sigma \underbrace{\left[\tfrac{\mu - r}{\sigma} dt + dB(t)\right]}_{d\widetilde{B}(t)}.
\end{aligned}
$$

$\widetilde{B}$ has the same paths as B. We can change to the risk-neutral measure $\widetilde{I\!P}$, under which $\widetilde{B}$ is a Brownian motion, and then proceed as if ρ had been chosen to be equal to $r - \frac{1}{2}\sigma^2$.

22.3 Risk-neutral pricing and hedging

Let $\widetilde{I\!P}$ denote the risk-neutral measure. Then

$$dS(t) = rS(t)\ dt + \sigma S(t)\ d\widetilde{B}(t),$$

where $\widetilde{B}$ is a Brownian motion under $\widetilde{I\!P}$. Set

$$\beta(t) = e^{rt}.$$

Then

$$d\left(\frac{S(t)}{\beta(t)}\right) = \sigma \frac{S(t)}{\beta(t)} d\widetilde{B}(t),$$

so $\frac{S(t)}{\beta(t)}$ is a martingale under $\widetilde{I\!P}$.

Evolution of the value of a portfolio:

$$dX(t) = \Delta(t) dS(t) + r(X(t) - \Delta(t)S(t))\ dt, \tag{3.1}$$

which is equivalent to

$$
\begin{aligned}
d\left(\frac{X(t)}{\beta(t)}\right) &= \Delta(t) d\left(\frac{S(t)}{\beta(t)}\right) && (3.2) \\
&= \Delta(t)\sigma \frac{S(t)}{\beta(t)} d\widetilde{B}(t).
\end{aligned}
$$

Regardless of the portfolio used, $\frac{X(t)}{\beta(t)}$ is a martingale under $\widetilde{I\!P}$.

Now suppose V is a given $\mathcal{F}(T)$-measurable random variable, the payoff of a simple European derivative security. We want to find the portfolio process $\Delta(T), 0 \le t \le T$, and initial portfolio value $X(0)$ so that $X(T) = V$. Because $\frac{X(t)}{\beta(t)}$ must be a martingale, we must have

$$\frac{X(t)}{\beta(t)} = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\middle|\mathcal{F}(t)\right], \quad 0 \le t \le T. \tag{3.3}$$

This is the *risk-neutral pricing formula.* We have the following sequence:

1. V is given,
2. Define $X(t), 0 \leq t \leq T$, by (3.3) (not by (3.1) or (3.2), because we do not yet have $\Delta(t)$).
3. Construct $\Delta(t)$ so that (3.2) (or equivalently, (3.1)) is satisfied by the $X(t), 0 \leq t \leq T$, defined in step 2.

To carry out step 3, we first use the tower property to show that $\frac{X(t)}{\beta(t)}$ defined by (3.3) is a martingale under $\widetilde{I\!P}$. We next use the corollary to the Martingale Representation Theorem (Homework Problem 4.5) to show that

$$d\left(\frac{X(t)}{\beta(t)}\right) = \gamma(t)\, d\widetilde{B}(t) \tag{3.4}$$

for some proecss γ. Comparing (3.4), which we know, and (3.2), which we want, we decide to define

$$\Delta(t) = \frac{\beta(t)\gamma(t)}{\sigma S(t)}. \tag{3.5}$$

Then (3.4) implies (3.2), which implies (3.1), which implies that $X(t), 0 \leq t \leq T$, is the value of the portfolio process $\Delta(t), 0 \leq t \leq T$.

From (3.3), the definition of X, we see that the hedging portfolio must begin with value

$$X(0) = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\right],$$

and it will end with value

$$X(T) = \beta(T)\widetilde{I\!E}\left[\frac{V}{\beta(T)}\middle|\mathcal{F}(T)\right] = \beta(T)\frac{V}{\beta(T)} = V.$$

Remark 22.1 Although we have taken r and σ to be constant, the risk-neutral pricing formula is still "valid" when r and σ are processes adapted to the filtration generated by B. If they depend on either $\widetilde{B}$ or on S, they are adapted to the filtration generated by B. The "validity" of the risk-neutral pricing formula means:

1. If you start with
$$X(0) = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\right],$$
then there is a hedging portfolio $\Delta(t), 0 \leq t \leq T$, such that $X(T) = V$;
2. At each time t, the value $X(t)$ of the hedging portfolio in 1 satisfies
$$\frac{X(t)}{\beta(t)} = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\middle|\mathcal{F}(t)\right].$$

Remark 22.2 In general, when there are multiple assets and/or multiple Brownian motions, the risk-neutral pricing formula is valid provided there is a *unique risk-neutral measure.* A probability measure is said to be risk-neutral provided

- it has the same probability-zero sets as the original measure;
- it makes all the discounted asset prices be martingales.

To see if the risk-neutral measure is unique, compute the differential of all discounted asset prices and check if there is more than one way to define $\widetilde{B}$ so that all these differentials have only $d\widetilde{B}$ terms.

22.4 Implementation of risk-neutral pricing and hedging

To get a computable result from the general risk-neutral pricing formula

$$\frac{X(t)}{\beta(t)} = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\middle|\mathcal{F}(t)\right],$$

one uses the Markov property. We need to identify some *state variables*, the stock price and possibly other variables, so that

$$X(t) = \beta(t)\widetilde{I\!E}\left[\frac{V}{\beta(T)}\middle|\mathcal{F}(t)\right]$$

is a function of these variables.

Example 22.1 Assume r and σ are constant, and $V = h(S(T))$. We can take the stock price to be the state variable. Define

$$v(t,x) = \widetilde{I\!E}^{t,x}\left[e^{-r(T-t)}h(S(T))\right].$$

Then

$$\begin{aligned} X(t) &= e^{rt}\,\widetilde{I\!E}\left[e^{-rT}h(S(T))\middle|\mathcal{F}(t)\right] \\ &= v(t,S(t)), \end{aligned}$$

and $\frac{X(t)}{\beta(t)} = e^{-rt}v(t,S(t))$ is a martingale under $\widetilde{I\!P}$. ■

Example 22.2 Assume r and σ are constant.

$$V = h\left(\int_0^T S(u)\ du\right).$$

Take $S(t)$ and $Y(t) = \int_0^t S(u)\ du$ to be the state variables. Define

$$v(t,x,y) = \widetilde{I\!E}^{t,x,y}\left[e^{-r(T-t)}h(Y(T))\right],$$

where

$$Y(T) = y + \int_t^T S(u)\ du.$$

Then

$$\begin{aligned} X(t) &= e^{rt}\ \widetilde{\mathbb{E}}\left[e^{-rT} h(S(T)) \middle| \mathcal{F}(t) \right] \\ &= v(t, S(t), Y(t)) \end{aligned}$$

and

$$\frac{X(t)}{\beta(t)} = e^{-rt} v(t, S(t), Y(t))$$

is a martingale under $\widetilde{\mathbb{P}}$. ■

Example 22.3 (Homework problem 4.2)

$$\begin{aligned} dS(t) &= r(t, Y(t))\ S(t)dt + \sigma(t, Y(t))S(t)\ d\widetilde{B}(t), \\ dY(t) &= \alpha(t, Y(t))\ dt + \gamma(t, Y(t))\ d\widetilde{B}(t), \\ V &= h(S(T)). \end{aligned}$$

Take $S(t)$ and $Y(t)$ to be the state variables. Define

$$v(t, x, y) = \widetilde{\mathbb{E}}^{t,x,y} \left[\underbrace{\exp\left\{ -\int_t^T r(u, Y(u))\ du \right\}}_{\frac{\beta(t)}{\beta(T)}} h(S(T)) \right].$$

Then

$$\begin{aligned} X(t) &= \beta(t)\widetilde{\mathbb{E}}\left[\frac{h(S(T))}{\beta(T)} \middle| \mathcal{F}(t) \right] \\ &= \widetilde{\mathbb{E}}\left[\exp\left\{ -\int_t^T r(u, Y(u))\ du \right\} h(S(T)) \middle| \mathcal{F}(t) \right] \\ &= v(t, S(t), Y(t)), \end{aligned}$$

and

$$\frac{X(t)}{\beta(t)} = \exp\left\{ -\int_0^t r(u, Y(u))\ du \right\} v(t, S(t), Y(t))$$

is a martingale under $\widetilde{\mathbb{P}}$. ■

In every case, we get an expression involving v to be a martingale. We take the differential and set the dt term to zero. This gives us a partial differential equation for v, and this equation must hold wherever the state processes can be. The $d\widetilde{B}$ term in the differential of the equation is the differential of a martingale, and since the martingale is

$$\frac{X(t)}{\beta(t)} = X(0) + \int_0^t \Delta(u)\sigma\frac{S(u)}{\beta(u)} d\widetilde{B}(u)$$

we can solve for $\Delta(t)$. This is the argument which uses (3.4) to obtain (3.5).

Example 22.4 (Continuation of Example 22.3)

$$\frac{X(t)}{\beta(t)} = \underbrace{\exp\left\{-\int_0^t r(u, Y(u))\ du\right\}}_{1/\beta(t)} v(t, S(t), Y(t))$$

is a martingale under $\widetilde{IP}$. We have

$$\begin{aligned} d\left(\frac{X(t)}{\beta(t)}\right) &= \frac{1}{\beta(t)}\Big[-r(t, Y(t))v(t, S(t), Y(t))\ dt \\ &\qquad + v_t\,dt + v_x\,dS + v_y\,dY \\ &\qquad + \tfrac{1}{2}v_{xx}\,dS\ dS + v_{xy}\,dS\ dY + \tfrac{1}{2}v_{yy}\,dY\ dY\Big] \\ &= \frac{1}{\beta(t)}\Big[\left(-rv + v_t + rSv_x + \alpha v_y + \tfrac{1}{2}\sigma^2 S^2 v_{xx} + \sigma\gamma S v_{xy} + \tfrac{1}{2}\gamma^2 v_{yy}\right) dt \\ &\qquad + \left(\sigma S v_x + \gamma v_y\right) d\widetilde{B}\Big] \end{aligned}$$

The partial differential equation satisfied by v is

$$-rv + v_t + rxv_x + \alpha v_y + \tfrac{1}{2}\sigma^2 x^2 v_{xx} + \sigma\gamma x v_{xy} + \tfrac{1}{2}\gamma^2 v_{yy} = 0$$

where it should be noted that $v = v(t, x, y)$, and all other variables are functions of (t, y). We have

$$d\left(\frac{X(t)}{\beta(t)}\right) = \frac{1}{\beta(t)}[\sigma S v_x + \gamma v_y]\ d\widetilde{B}(t),$$

where $\sigma = \sigma(t, Y(t))$, $\gamma = \gamma(t, Y(t))$, $v = v(t, S(t), Y(t))$, and $S = S(t)$. We want to choose $\Delta(t)$ so that (see (3.2))

$$d\left(\frac{X(t)}{\beta(t)}\right) = \Delta(t)\sigma(t, Y(t))\frac{S(t)}{\beta(t)}\ d\widetilde{B}(t).$$

Therefore, we should take $\Delta(t)$ to be

$$\Delta(t) = v_x(t, S(t), Y(t)) + \frac{\gamma(t, Y(t))}{\sigma(t, Y(t))\ S(t)} v_y(t, S(t), Y(t)).$$

■