Chapter 21

Asian Options

Stock:

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dB(t).$$

Payoff:

$$V = h\left(\int_0^T S(t)\,dt\right)$$

Value of the payoff at time zero:

$$X(0) = I\!E\left[e^{-rT} h\left(\int_0^T S(t)\,dt\right)\right].$$

Introduce an *auxiliary process* $Y(t)$ by specifying

$$dY(t) = S(t)\,dt.$$

With the initial conditions

$$S(t) = x, \quad Y(t) = y,$$

we have the solutions

$$S(T) = x\exp\left\{\sigma(B(T) - B(t)) + (r - \tfrac{1}{2}\sigma^2)(T - t)\right\},$$
$$Y(T) = y + \int_t^T S(u)\,du.$$

Define the undiscounted expected payoff

$$u(t, x, y) = I\!E^{t,x,y} h(Y(T)), \quad 0 \le t \le T,\ x \ge 0,\ y \in I\!R.$$

21.1 Feynman-Kac Theorem

The function u satisfies the PDE

$$u_t + rxu_x + \frac{1}{2}\sigma^2 x^2 u_{xx} + xu_y = 0, \quad 0 \le t \le T,\ x \ge 0,\ y \in I\!R,$$

the terminal condition

$$u(T, x, y) = h(y), \quad x \ge 0,\ y \in I\!R,$$

and the boundary condition

$$u(t, 0, y) = h(y), \quad 0 \le t \le T,\ y \in I\!R.$$

One can solve this equation. Then

$$v\left(t, S(t), \int_0^t S(u)\ du\right)$$

is the option value at time t, where

$$v(t, x, y) = e^{-r(T-t)} u(t, x, y).$$

The PDE for v is

$$\begin{aligned} -rv + v_t + rxv_x + \tfrac{1}{2}\sigma^2 x^2 v_{xx} + xv_y &= 0, \\ v(T, x, y) &= h(y), \\ v(t, 0, y) &= e^{-r(T-t)} h(y). \end{aligned} \tag{1.1}$$

One can solve this equation rather than the equation for u.

21.2 Constructing the hedge

Start with the stock price $S(0)$. The differential of the value $X(t)$ of a portfolio $\Delta(t)$ is

$$\begin{aligned} dX &= \Delta\ dS + r(X - \Delta S)\ dt \\ &= \Delta S(r\ dt + \sigma\ dB) + rX\ dt - r\Delta S\ dt \\ &= \Delta\sigma S\ dB + rX\ dt. \end{aligned}$$

We want to have

$$X(t) = v\left(t, S(t), \int_0^t S(u)\ du\right),$$

so that

$$\begin{aligned} X(T) &= v\left(T, S(0), \int_0^T S(u)\ du\right), \\ &= h\left(\int_0^T S(u)\ du\right). \end{aligned}$$

The differential of the value of the option is

$$\begin{aligned} dv\left(t, S(t), \int_0^t S(u)\,du\right) &= v_t dt + v_x dS + v_y S\,dt + \tfrac{1}{2} v_{xx}\,dS\,dS \\ &= (v_t + rSv_x + Sv_y + \tfrac{1}{2}\sigma^2 S^2 v_{xx})\,dt + \sigma S v_x\,dB \\ &= rv(t, S(t))\,dt + v_x(t, S(t))\,\sigma\,S(t)\,dB(t). \quad \text{(From Eq. 1.1)} \end{aligned}$$

Compare this with

$$dX(t) = rX(t)\,dt + \Delta(t)\,\sigma\,S(t)\,dB(t).$$

Take $\Delta(t) = v_x(t, S(t))$. If $X(0) = v(0, S(0), 0)$, then

$$X(t) = v\left(t, S(t), \int_0^t S(u)\,du\right), \quad 0 \le t \le T,$$

because both these processes satisfy the same stochastic differential equation, starting from the same initial condition.

21.3 Partial average payoff Asian option

Now suppose the payoff is

$$V = h\left(\int_\tau^T S(t)\,dt\right),$$

where $0 < \tau < T$. We compute

$$v(\tau, x, y) = I\!E^{\tau,x,y} e^{-r(T-\tau)} h(Y(T))$$

just as before. For $0 \le t \le \tau$, we compute next the value of a derivative security which pays off

$$v(\tau, S(\tau), 0)$$

at time τ. This value is

$$w(t, x) = I\!E^{t,x} e^{-r(\tau-t)} v(\tau, S(\tau), 0).$$

The function w satisfies the Black-Scholes PDE

$$-rw + w_t + rxw_x + \tfrac{1}{2}\sigma^2 x^2 w_{xx} = 0, \quad 0 \le t \le \tau,\ x \ge 0,$$

with terminal condition

$$w(\tau, x) = v(\tau, x, 0), \quad x \ge 0,$$

and boundary condition

$$w(t, 0) = e^{-r(T-t)} h(0), \quad 0 \le t \le T.$$

The hedge is given by

$$\Delta(t) = \begin{cases} w_x(t, S(t)), & 0 \le t \le \tau, \\ v_x\left(t, S(t), \int_\tau^t S(u)\,du\right), & \tau < t \le T. \end{cases}$$

Remark 21.1 While no closed-form for the Asian option price is known, the Laplace transform (in the variable $\frac{\sigma^2}{4}(T - t)$) has been computed. See H. Geman and M. Yor, *Bessel processes, Asian options, and perpetuities,* Math. Finance 3 (1993), 349–375.