Chapter 20

Pricing Exotic Options

20.1 Reflection principle for Brownian motion

Without drift.

Define

$$M(T) = \max_{0 \le t \le T} B(t).$$

Then we have:

$$\begin{aligned}
&\mathbb{P}\{M(T) > m, B(T) < b\} \\
&\qquad = \mathbb{P}\{B(T) > 2m - b\} \\
&\qquad = \frac{1}{\sqrt{2\pi T}} \int_{2m-b}^{\infty} \exp\left\{-\frac{x^2}{2T}\right\} dx, \quad m > 0,\ b < m
\end{aligned}$$

So the joint density is

$$\begin{aligned}
\mathbb{P}\{M(T) \in dm, B(T) \in db\} &= -\frac{\partial^2}{\partial m\, \partial b}\left(\frac{1}{\sqrt{2\pi T}} \int_{2m-b}^{\infty} \exp\left\{-\frac{x^2}{2T}\right\}\ dx\right) dm\ db \\
&= -\frac{\partial}{\partial m}\left(\frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{(2m-b)^2}{2T}\right\}\right) dm\ db, \\
&= \frac{2(2m-b)}{T\sqrt{2\pi T}} \exp\left\{-\frac{(2m-b)^2}{2T}\right\} dm\ db, \quad m > 0, b < m.
\end{aligned}$$

With drift. Let

$$\widetilde{B}(t) = \theta t + B(t),$$

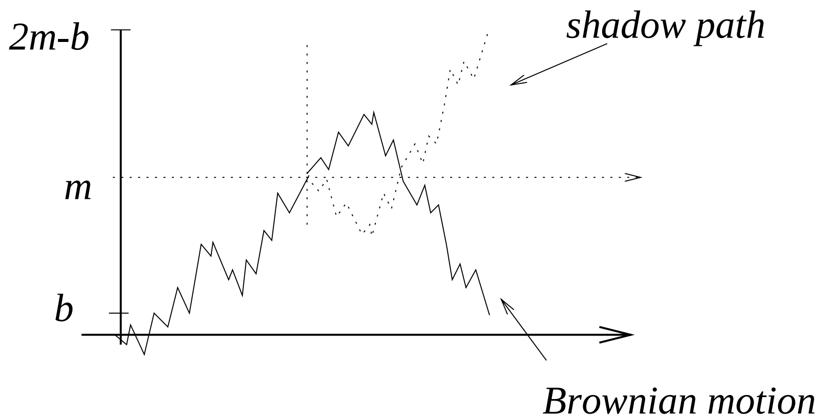

Figure 20.1: *Reflection Principle for Brownian motion without drift*

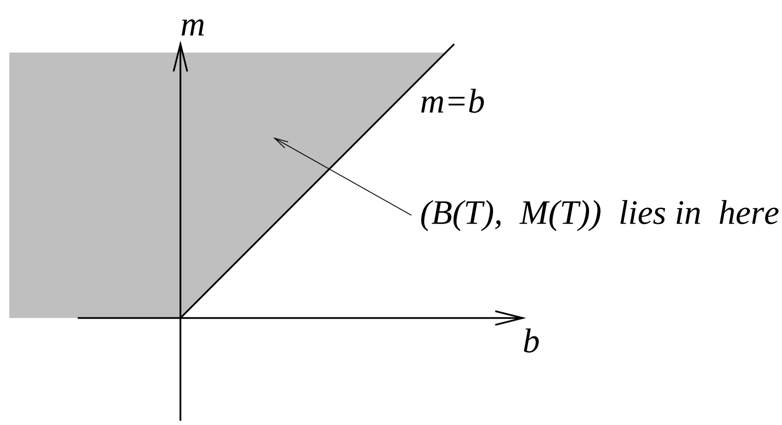

Figure 20.2: *Possible values of* $B(T), M(T)$.

where $B(t),\ 0 \le t \le T$, is a Brownian motion (without drift) on $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$\begin{aligned} Z(T) &= \exp\{-\theta B(T) - \tfrac{1}{2}\theta^2 T\} \\ &= \exp\{-\theta(B(T) + \theta T) + \tfrac{1}{2}\theta^2 T\} \\ &= \exp\{-\theta \widetilde{B}(t) + \tfrac{1}{2}\theta^2 T\}, \\ \widetilde{I\!P}(A) &= \int_A Z(T)\, dI\!P, \quad \forall A \in \mathcal{F}. \end{aligned}$$

Set$\widetilde{M}(T) = \max_{0 \le t \le T} \widetilde{B}(T)$.

Under $\widetilde{I\!P}$, $\widetilde{B}$ is a Brownian motion (without drift), so

$$\widetilde{I\!P}\{\widetilde{M}(T) \in d\tilde{m},\ \widetilde{B}(T) \in d\tilde{b}\} = \frac{2(2\tilde{m} - \tilde{b})}{T\sqrt{2\pi T}} \exp\left\{-\frac{(2\tilde{m} - \tilde{b})^2}{2T}\right\} d\tilde{m}\, d\tilde{b}, \quad \tilde{m} > 0,\ \tilde{b} < \tilde{m}.$$

Let $h(\tilde{m}, \tilde{b})$ be a function of two variables. Then

$$\begin{aligned} I\!E h(\widetilde{M}(T), \widetilde{B}(T)) &= \widetilde{I\!E} \frac{h(\widetilde{M}(T), \widetilde{B}(T))}{Z(T)} \\ &= \widetilde{I\!E}\left[h(\widetilde{M}(T), \widetilde{B}(T)) \exp\{\theta \widetilde{B}(T) - \tfrac{1}{2}\theta^2 T\}\right] \\ &= \int_{\tilde{m}=0}^{\tilde{m}=\infty} \int_{\tilde{b}=-\infty}^{\tilde{b}=\tilde{m}} h(\tilde{m}, \tilde{b})\ \exp\{\theta \tilde{b} - \tfrac{1}{2}\theta^2 T\}\ \widetilde{I\!P}\{\widetilde{M}(T) \in d\tilde{m},\ \widetilde{B}(T) \in d\tilde{b}\}. \end{aligned}$$

But also,

$$I\!E h(\widetilde{M}(T), \widetilde{B}(T)) = \int_{\tilde{m}=0}^{\tilde{m}=\infty} \int_{\tilde{b}=-\infty}^{\tilde{b}=\tilde{m}} h(\tilde{m}, \tilde{b})\ I\!P\{\widetilde{M}(T) \in d\tilde{m},\ \widetilde{B}(T) \in d\tilde{b}\}.$$

Since h is arbitrary, we conclude that

(MPR)

$$\begin{aligned} &I\!P\{\widetilde{M}(T) \in d\tilde{m},\ \widetilde{B}(T) \in d\tilde{b}\} \\ &= \exp\{\theta \tilde{b} - \tfrac{1}{2}\theta^2 T\}\ \widetilde{I\!P}\{\widetilde{M}(T) \in d\tilde{m},\ \widetilde{B}(T) \in d\tilde{b}\} \\ &= \frac{2(2\tilde{m} - \tilde{b})}{T\sqrt{2\pi T}} \exp\left\{-\frac{(2\tilde{m} - \tilde{b})^2}{2T}\right\} . \exp\{\theta \tilde{b} - \tfrac{1}{2}\theta^2 T\} d\tilde{m}\, d\tilde{b}, \quad \tilde{m} > 0,\ \tilde{b} < \tilde{m}. \end{aligned}$$

20.2 Up and out European call.

Let $0 < K < L$ be given. The payoff at time T is

$$(S(T) - K)^+ \mathbf{1}_{\{S^*(T) < L\}},$$

where

$$S^*(T) = \max_{0 \le t \le T} S(t).$$

To simplify notation, assume that $I\!\!P$ is already the risk-neutral measure, so the value at time zero of the option is

$$v(0, S(0)) = e^{-rT} I\!\!E \left[(S(T) - K)^+ \mathbf{1}_{\{S^*(T) < L\}} \right].$$

Because $I\!\!P$ is the risk-neutral measure,

$$\begin{aligned}
dS(t) &= rS(t)\, dt + \sigma S(t)\, dB(t) \\
S(t) &= S_0 \exp\{\sigma B(t) + (r - \tfrac{1}{2}\sigma^2)t\} \\
&= S_0 \exp\left\{ \sigma \left[B(t) + \underbrace{\left(\frac{r}{\sigma} - \frac{\sigma}{2} \right)}_{\theta} t \right] \right\} \\
&= S_0 \exp\{\sigma \widetilde{B}(t)\},
\end{aligned}$$

where

$$\begin{aligned}
\theta &= \left(\frac{r}{\sigma} - \frac{\sigma}{2} \right), \\
\widetilde{B}(t) &= \theta t + B(t).
\end{aligned}$$

Consequently,

$$S^*(t) = S_0 \exp\{\sigma \widetilde{M}(t)\},$$

where,

$$\widetilde{M}(t) = \max_{0 \le u \le t} \widetilde{B}(u).$$

We compute,

$$\begin{aligned}
v(0, S(0)) &= e^{-rT} I\!\!E \left[(S(T) - K)^+ \mathbf{1}_{\{S^*(T) < L\}} \right] \\
&= e^{-rT} I\!\!E \left[\left(S(0) \exp\{\sigma \widetilde{B}(T)\} - K \right)^+ \mathbf{1}_{\{S(0)\exp\{\sigma \widetilde{M}(T)\} < L\}} \right] \\
&= e^{-rT} I\!\!E \left[\left(S(0) \exp\{\sigma \widetilde{B}(T)\} - K \right) \mathbf{1}_{\left\{ \widetilde{B}(T) > \underbrace{\frac{1}{\sigma} \log \frac{K}{S(0)}}_{\tilde{b}},\ \widetilde{M}(T) < \underbrace{\frac{1}{\sigma} \log \frac{L}{S(0)}}_{\tilde{m}} \right\}} \right]
\end{aligned}$$

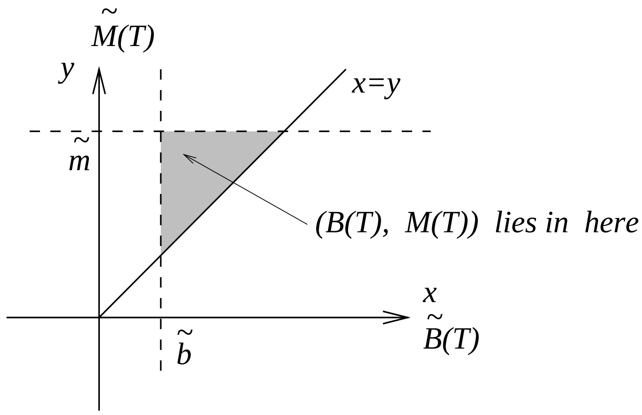

Figure 20.3: *Possible values of* $\widetilde{B}(T), \widetilde{M}(T)$.

We consider only the case

$$S(0) \le K < L, \quad \text{so} \quad 0 \le \tilde{b} < \tilde{m}.$$

The other case, $K < S(0) \le L$ leads to $\tilde{b} < 0 \le \tilde{m}$ and the analysis is similar.
We compute $\int_{\tilde{b}}^{\tilde{m}} \int_x^{\tilde{m}} \dots dy\ dx$:

$$\begin{aligned}
v(0,S(0)) &= e^{-rT} \int_{\tilde{b}}^{\tilde{m}} \int_x^{\tilde{m}} (S(0)\exp\{\sigma x\} - K) \frac{2(2y-x)}{T\sqrt{2\pi T}} \exp\left\{ -\frac{(2y-x)^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \right\} dy\ dx \\
&= -e^{-rT} \int_{\tilde{b}}^{\tilde{m}} (S(0)\exp\{\sigma x\} - K) \frac{1}{\sqrt{2\pi T}} \exp\left\{ -\frac{(2y-x)^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \right\} \Bigg|_{y=x}^{y=\tilde{m}} dx \\
&= e^{-rT} \int_{\tilde{b}}^{\tilde{m}} (S(0)\exp\{\sigma x\} - K) \frac{1}{\sqrt{2\pi T}} \Bigg[\exp\left\{ -\frac{x^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \right\} \\
&\qquad - \exp\left\{ -\frac{(2\tilde{m}-x)^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \right\} \Bigg] dx \\
&= \frac{1}{\sqrt{2\pi T}} e^{-rT} S(0) \int_{\tilde{b}}^{\tilde{m}} \exp\left\{ \sigma x - \frac{x^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \right\}\ dx \\
&\quad - \frac{1}{\sqrt{2\pi T}} e^{-rT} K \int_{\tilde{b}}^{\tilde{m}} \exp\left\{ -\frac{x^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \right\}\ dx \\
&\quad - \frac{1}{\sqrt{2\pi T}} e^{-rT} S(0) \int_{\tilde{b}}^{\tilde{m}} \exp\left\{ \sigma x - \frac{(2\tilde{m}-x)^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \right\}\ dx \\
&\quad + \frac{1}{\sqrt{2\pi T}} e^{-rT} K \int_{\tilde{b}}^{\tilde{m}} \exp\left\{ -\frac{(2\tilde{m}-x)^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \right\}\ dx.
\end{aligned}$$

The standard method for all these integrals is to complete the square in the exponent and then recognize a cumulative normal distribution. We carry out the details for the first integral and just give the result for the other three. The exponent in the first integrand is

$$\begin{aligned}
&\sigma x - \frac{x^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T \\
&= -\frac{1}{2T}(x - \sigma T - \theta T)^2 + \tfrac{1}{2}\sigma^2 T + \sigma\theta T \\
&= -\frac{1}{2T}\left(x - \frac{rT}{\sigma} - \frac{\sigma T}{2}\right)^2 + rT.
\end{aligned}$$

In the first integral we make the change of variable

$$y = (x - rT/\sigma - \sigma T/2)/\sqrt{T}, \quad dy = dx/\sqrt{T},$$

to obtain

$$\begin{aligned}
&\frac{e^{-rT}S(0)}{\sqrt{2\pi T}} \int_{\tilde{b}}^{\tilde{m}} \exp\left\{\sigma x - \frac{x^2}{2T} + \theta x - \tfrac{1}{2}\theta^2 T\right\} dx \\
&= \frac{1}{\sqrt{2\pi T}} S(0) \int_{\tilde{b}}^{\tilde{m}} \exp\left\{-\frac{1}{2T}\left(x - \frac{rT}{\sigma} - \frac{\sigma T}{2}\right)^2\right\}\, dx \\
&= \frac{1}{\sqrt{2\pi T}} S(0). \int_{\frac{\tilde{b}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}}^{\frac{\tilde{m}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}} \exp\{-\frac{y^2}{2}\}\, dy \\
&= S(0)\left[N\left(\frac{\tilde{m}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right) - N\left(\frac{\tilde{b}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right)\right].
\end{aligned}$$

Putting all four integrals together, we have

$$\begin{aligned}
v(0, S(0)) = S(0)&\left[N\left(\frac{\tilde{m}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right) - N\left(\frac{\tilde{b}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right)\right] \\
&- e^{-rT}K\left[N\left(\frac{\tilde{m}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right) - N\left(\frac{\tilde{b}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right)\right] \\
&- S(0)\left[N\left(\frac{\tilde{m}}{\sqrt{T}} + \frac{r\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right) - N\left(\frac{(2\tilde{m} - \tilde{b})}{\sqrt{T}} + \frac{r\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right)\right] \\
&+ \exp\left\{-rT + 2\tilde{m}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\right\}\left[N\left(\frac{\tilde{m}}{\sqrt{T}} + \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right) - \right. \\
&\quad \left. N\left(\frac{(2\tilde{m} - \tilde{b})}{\sqrt{T}} + \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right)\right],
\end{aligned}$$

where

$$\tilde{b} = \frac{1}{\sigma}\log\frac{K}{S(0)}, \quad \tilde{m} = \frac{1}{\sigma}\log\frac{L}{S(0)}.$$

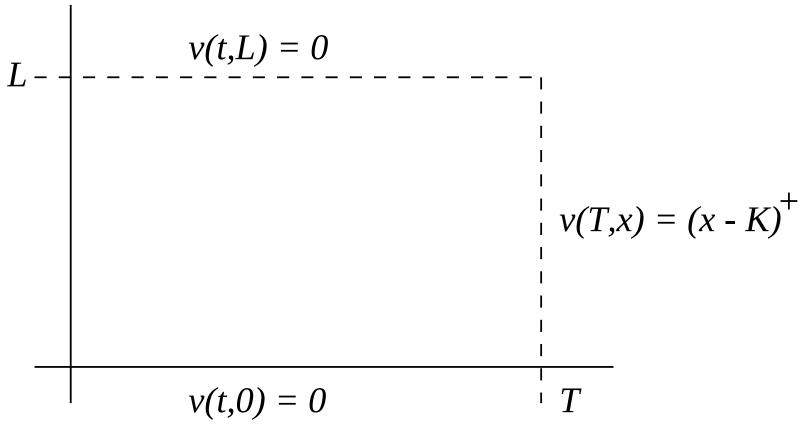

Figure 20.4: *Initial and boundary conditions.*

If we let $L \to \infty$ we obtain the classical Black-Scholes formula

$$\begin{aligned} v(0, S(0)) &= S(0)\left[1 - N\left(\frac{\tilde{b}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right)\right] \\ &\quad - e^{-rT}K\left[1 - N\left(\frac{\tilde{b}}{\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right)\right] \\ &= S(0)N\left(\frac{1}{\sigma\sqrt{T}}\log\frac{S(0)}{K} + \frac{r\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right) \\ &\quad - e^{-rT}KN\left(\frac{1}{\sigma\sqrt{T}}\log\frac{S(0)}{K} + \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right). \end{aligned}$$

If we replace T by $T - t$ and replace $S(0)$ by x in the formula for $v(0, S(0))$, we obtain a formula for $v(t, x)$, the value of the option at the time t if $S(t) = x$. We have actually derived the formula under the assumption $x \le K \le L$, but a similar albeit longer formula can also be derived for $K < x \le L$. We consider the function

$$v(t, x) = \mathbb{E}^{t,x}\left[e^{-r(T-t)}(S(T) - K)^{+}\mathbf{1}_{\{S^*(T)<L\}}\right], \quad 0 \le t \le T,\ 0 \le x \le L.$$

This function satisfies the *terminal condition*

$$v(T, x) = (x - K)^{+}, \quad 0 \le x < L$$

and the *boundary conditions*

$$\begin{aligned} v(t, 0) &= 0, \quad 0 \le t \le T, \\ v(t, L) &= 0, \quad 0 \le t \le T. \end{aligned}$$

We show that v satisfies the Black-Scholes equation

$$-rv + v_t + rxv_x + \tfrac{1}{2}\sigma^2 x^2 v_{xx}, \quad 0 \le t < T,\ 0 \le x \le L.$$

Let $S(0) > 0$ be given and define the *stopping time*

$$\tau = \min\{t \geq 0;\ S(t) = L\}.$$

Theorem 2.61 *The process*

$$e^{-r(t\wedge\tau)}v(t\wedge\tau, S(t\wedge\tau)), \quad 0 \leq t \leq T,$$

is a martingale.

Proof: First note that

$$S^*(T) < L \iff \tau > T.$$

Let $\omega \in \Omega$ be given, and choose $t \in [0,T]$. If $\tau(\omega) \leq t$, then

$$I\!\!E\left[e^{-rT}(S(T)-K)^+\mathbf{1}_{\{S^*(T)<L\}}\middle|\mathcal{F}(t)\right](\omega) \;=\; 0.$$

But when $\tau(\omega) \leq t$, we have

$$v(t\wedge\tau(\omega), S(t\wedge\tau(\omega),\omega)) = v(t\wedge\tau(\omega), L) = 0,$$

so we may write

$$I\!\!E\left[e^{-rT}(S(T)-K)^+\mathbf{1}_{\{S^*(T)<L\}}\middle|\mathcal{F}(t)\right](\omega) \;=\; e^{-r(t\wedge\tau(\omega))}v\left(t\wedge\tau(\omega),\ S(t\wedge\tau(\omega),\omega)\right).$$

On the other hand, if $\tau(\omega) > t$, then the Markov property implies

$$\begin{aligned}
&I\!\!E\left[e^{-rT}(S(T)-K)^+\mathbf{1}_{\{S^*(T)<L\}}\middle|\mathcal{F}(t)\right](\omega)\\
&= I\!\!E^{t,S(t,\omega)}\left[e^{-rT}(S(T)-K)^+\mathbf{1}_{\{S^*(T)<L\}}\right]\\
&= e^{-rt}v(t,S(t,\omega))\\
&= e^{-r(t\wedge\tau(\omega))}v\left(t\wedge\tau,\ S(t\wedge\tau(\omega),\omega)\right).
\end{aligned}$$

In both cases, we have

$$e^{-r(t\wedge\tau)}v(t\wedge\tau, S(t\wedge\tau)) = I\!\!E\left[e^{-rT}(S(T)-K)^+\mathbf{1}_{\{S^*(T)<L\}}\middle|\mathcal{F}(t)\right].$$

Suppose $0 \leq u \leq t \leq T$. Then

$$\begin{aligned}
&I\!\!E\left[e^{-r(t\wedge\tau)}v(t\wedge\tau, S(t\wedge\tau))\middle|\mathcal{F}(u)\right]\\
&= I\!\!E\left[I\!\!E\left[e^{-rT}(S(T)-K)^+\mathbf{1}_{\{S^*(T)<L\}}\middle|\mathcal{F}(t)\right]\middle|\mathcal{F}(u)\right]\\
&= I\!\!E\left[e^{-rT}(S(T)-K)^+\mathbf{1}_{\{S^*(T)<L\}}\middle|\mathcal{F}(u)\right]\\
&= e^{-r(u\wedge\tau)}v\left(u\wedge\tau,\ S(u\wedge\tau)\right).
\end{aligned}$$

■

For $0 \leq t \leq T$, we compute the differential

$$d\left(e^{-rt}v(t,S(t))\right) = e^{-rt}(-rv + v_t + rSv_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx})\;dt + e^{-rt}\sigma S v_x\;dB.$$

Integrate from 0 to $t \wedge \tau$:

$$\begin{aligned} e^{-r(t\wedge\tau)}v\left(t\wedge\tau, S(t\wedge\tau)\right) &= v(0,S(0)) \\ &\quad + \int_0^{t\wedge\tau} e^{-ru}(-rv + v_t + rSv_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx})\;du \\ &\quad + \underbrace{\int_0^{t\wedge\tau} e^{-ru}\sigma S v_x\;dB.}_{\text{A stopped martingale is still a martingale}} \end{aligned}$$

Because $e^{-r(t\wedge\tau)}v\left(t\wedge\tau, S(t\wedge\tau)\right)$ is also a martingale, the Riemann integral

$$\int_0^{t\wedge\tau} e^{-ru}(-rv + v_t + rSv_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx})\;du$$

is a martingale. Therefore,

$$-rv(u,S(u)) + v_t(u,S(u)) + rS(u)v_x(u,S(u)) + \tfrac{1}{2}\sigma^2 S^2(u)v_{xx}(u,S(u)) = 0, \quad 0 \leq u \leq t\wedge\tau.$$

The PDE

$$-rv + v_t + rxv_x + \tfrac{1}{2}\sigma^2 x^2 v_{xx} = 0, \quad 0 \leq t \leq T,\; 0 \leq x \leq L,$$

then follows.

The Hedge

$$d\left(e^{-rt}v(t,S(t))\right) = e^{-rt}\sigma S(t) v_x(t,S(t))\;dB(t), \quad 0 \leq t \leq \tau.$$

Let $X(t)$ be the wealth process corresponding to some portfolio $\Delta(t)$. Then

$$d(e^{-rt}X(t)) = e^{-rt}\Delta(t)\sigma S(t)\;dB(t).$$

We should take

$$X(0) = v(0,S(0))$$

and

$$\Delta(t) = v_x(t,S(t)), \quad 0 \leq t \leq T\wedge\tau.$$

Then

$$\begin{aligned} X(T\wedge\tau) &= v(T\wedge\tau, S(T\wedge\tau)) \\ &= \begin{cases} v(T,S(T)) = (S(T)-K)^+ & \text{if } \tau > T \\ v(\tau,L) = 0 & \text{if } \tau \leq T. \end{cases} \end{aligned}$$

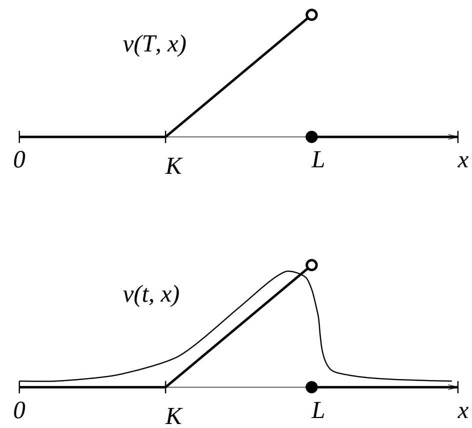

Figure 20.5: *Practial issue.*

20.3 A practical issue

For $t < T$ but t near T, $v(t,x)$ has the form shown in the bottom part of Fig. 20.5.

In particular, the hedging portfolio

$$\Delta(t) = v_x(t, S(t))$$

can become very negative near the knockout boundary. The hedger is in an unstable situation. He should take a large short position in the stock. If the stock does not cross the barrier L, he covers this short position with funds from the money market, pays off the option, and is left with zero. If the stock moves across the barrier, he is now in a region of $\Delta(t) = v_x(t, S(t))$ near zero. He should cover his short position with the money market. This is more expensive than before, because the stock price has risen, and consequently he is left with no money. However, the option has "knocked out", so no money is needed to pay it off.

Because a large short position is being taken, a small error in hedging can create a significant effect. Here is a possible resolution.

Rather than using the boundary condition

$$v(t, L) = 0, \quad 0 \le t \le T,$$

solve the PDE with the boundary condition

$$v(t, L) + \alpha L v_x(t, L) = 0, \quad 0 \le t \le T,$$

where α is a "tolerance parameter", say 1%. At the boundary, $Lv_x(t,L)$ is the dollar size of the short position. The new boundary condition guarantees:

1. $Lv_x(t,L)$ remains bounded;
2. The value of the portfolio is always sufficient to cover a hedging error of α times the dollar size of the short position.