Chapter 19

A two-dimensional market model

Let $B(t) = (B_1(t), B_2(t)), 0 \le t \le T$, be a two-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}(t), 0 \le t \le T$, be the filtration generated by B.

In what follows, all processes can depend on t and ω, but are adapted to $\mathcal{F}(t), 0 \le t \le T$. To simplify notation, we omit the arguments whenever there is no ambiguity.

Stocks:

$$\begin{aligned} dS_1 &= S_1 \left[\mu_1 \; dt + \sigma_1 \; dB_1\right], \\ dS_2 &= S_2 \left[\mu_2 \; dt + \rho\sigma_2 \; dB_1 + \sqrt{1-\rho^2}\; \sigma_2 \; dB_2\right]. \end{aligned}$$

We assume $\sigma_1 > 0, \; \sigma_2 > 0, \; -1 \le \rho \le 1$. Note that

$$\begin{aligned} dS_1 \; dS_2 &= S_1^2 \sigma_1^2 \; dB_1 \; dB_1 = \sigma_1^2 S_1^2 \; dt, \\ dS_2 \; dS_2 &= S_2^2 \rho^2 \sigma_2^2 \; dB_1 \; dB_1 + S_2^2 (1-\rho^2) \sigma_2^2 \; dB_2 \; dB_2 \\ &= \sigma_2^2 S_2^2 \; dt, \\ dS_1 \; dS_2 &= S_1 \sigma_1 S_2 \rho \sigma_2 \; dB_1 \; dB_1 = \rho\sigma_1\sigma_2 S_1 S_2 \; dt. \end{aligned}$$

In other words,

- $\frac{dS_1}{S_1}$ has instantaneous variance σ_1^2,
- $\frac{dS_2}{S_2}$ has instantaneous variance σ_2^2,
- $\frac{dS_1}{S_1}$ and $\frac{dS_2}{S_2}$ have instantaneous covariance $\rho\sigma_1\sigma_2$.

Accumulation factor:

$$\beta(t) = \exp\left\{\int_0^t r \; du\right\}.$$

The market price of risk equations are

$$\begin{aligned} \sigma_1\theta_1 &= \mu_1 - r \\ \rho\sigma_2\theta_1 + \sqrt{1-\rho^2}\sigma_2\theta_2 &= \mu_2 - r \end{aligned} \qquad \text{(MPR)}$$

The solution to these equations is

$$\theta_1 = \frac{\mu_1 - r}{\sigma_1},$$
$$\theta_2 = \frac{\sigma_1(\mu_2 - r) - \rho\sigma_2(\mu_1 - r)}{\sigma_1\sigma_2\sqrt{1-\rho^2}},$$

provided $-1 < \rho < 1$.

Suppose $-1 < \rho < 1$. Then (MPR) has a unique solution (θ_1, θ_2); we define

$$Z(t) = \exp\left\{-\int_0^t \theta_1 \, dB_1 - \int_0^t \theta_2 \, dB_2 - \tfrac{1}{2}\int_0^t (\theta_1^2 + \theta_2^2) \, du\right\},$$
$$\widetilde{I\!P}(A) = \int_A Z(T) \, dI\!P, \qquad \forall A \in \mathcal{F}.$$

$\widetilde{I\!P}$ is the *unique* risk-neutral measure. Define

$$\widetilde{B}_1(t) = \int_0^t \theta_1 \, du + B_1(t),$$
$$\widetilde{B}_2(t) = \int_0^t \theta_2 \, du + B_2(t).$$

Then

$$dS_1 = S_1\left[r \, dt + \sigma_1 \, d\widetilde{B}_1\right],$$
$$dS_2 = S_2\left[r \, dt + \rho\sigma_2 \, d\widetilde{B}_1 + \sqrt{1-\rho^2}\sigma_2 d\widetilde{B}_2\right].$$

We have changed the mean rates of return of the stock prices, but not the variances and covariances.

19.1 Hedging when $-1 < \rho < 1$

$$\begin{aligned}
dX &= \Delta_1 \, dS_1 + \Delta_2 \, dS_2 + r(X - \Delta_1 S_1 - \Delta_2 S_2) \, dt \\
d\left(\frac{X}{\beta}\right) &= \frac{1}{\beta}(dX - rX \, dt) \\
&= \frac{1}{\beta}\Delta_1(dS_1 - rS_1 \, dt) + \frac{1}{\beta}\Delta_2(dS_2 - rS_2 \, dt) \\
&= \frac{1}{\beta}\Delta_1 S_1 \sigma_1 \, d\widetilde{B}_1 + \frac{1}{\beta}\Delta_2 S_2\left[\rho\sigma_2 \, d\widetilde{B}_1 + \sqrt{1-\rho^2}\sigma_2 \, d\widetilde{B}_2\right].
\end{aligned}$$

Let V be $\mathcal{F}(T)$-measurable. Define the $\widetilde{I\!P}$-martingale

$$Y(t) = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\middle|\mathcal{F}(t)\right], \qquad 0 \le t \le T.$$

The Martingale Representation Corollary implies

$$Y(t) = Y(0) + \int_0^t \gamma_1 \, d\widetilde{B}_1 + \int_0^t \gamma_2 \, d\widetilde{B}_2.$$

We have

$$\begin{aligned} d\left(\frac{X}{\beta}\right) &= \left(\frac{1}{\beta}\Delta_1 S_1 \sigma_1 + \frac{1}{\beta}\Delta_2 S_2 \rho \sigma_2\right) \, d\widetilde{B}_1 \\ &\qquad + \frac{1}{\beta}\Delta_2 S_2 \sqrt{1-\rho^2}\sigma_2 \, d\widetilde{B}_2, \\ dY &= \gamma_1 \, d\widetilde{B}_1 + \gamma_2 \, d\widetilde{B}_2. \end{aligned}$$

We solve the equations

$$\begin{aligned} \frac{1}{\beta}\Delta_1 S_1 \sigma_1 + \frac{1}{\beta}\Delta_2 S_2 \rho \sigma_2 &= \gamma_1 \\ \frac{1}{\beta}\Delta_2 S_2 \sqrt{1-\rho^2}\sigma_2 &= \gamma_2 \end{aligned}$$

for the hedging portfolio (Δ_1, Δ_2). With this choice of (Δ_1, Δ_2) and setting

$$X(0) = Y(0) = \widetilde{I\!E}\frac{V}{\beta(T)},$$

we have $X(t) = Y(t), \; 0 \le t \le T$, and in particular,

$$X(T) = V.$$

Every $\mathcal{F}(T)$-measurable random variable can be hedged; the market is *complete*.

19.2 Hedging when $\rho = 1$

The case $\rho = -1$ is analogous. Assume that $\rho = 1$. Then

$$\begin{aligned} dS_1 &= S_1[\mu_1 \, dt + \sigma_1 \, dB_1] \\ dS_2 &= S_2[\mu_2 \, dt + \sigma_2 \, dB_1] \end{aligned}$$

The stocks are perfectly correlated.

The market price of risk equations are

$$\begin{aligned} \sigma_1\theta_1 &= \mu_1 - r \\ \sigma_2\theta_1 &= \mu_2 - r \end{aligned} \qquad \text{(MPR)}$$

The process θ_2 is free. There are two cases:

Case I: $\frac{\mu_1 - r}{\sigma_1} \neq \frac{\mu_2 - r}{\sigma_2}$. There is no solution to (MPR), and consequently, there is no risk-neutral measure. This market admits arbitrage. Indeed

$$\begin{aligned} d\left(\frac{X}{\beta}\right) &= \frac{1}{\beta}\Delta_1(dS_1 - rS_1\ dt) + \frac{1}{\beta}\Delta_2(dS_2 - rS_2\ dt) \\ &= \frac{1}{\beta}\Delta_1 S_1[(\mu_1 - r)\ dt + \sigma_1\ dB_1] + \frac{1}{\beta}\Delta_2 S_2[(\mu_2 - r)\ dt + \sigma_2\ dB_1] \end{aligned}$$

Suppose $\frac{\mu_1 - r}{\sigma_1} > \frac{\mu_2 - r}{\sigma_2}$. Set

$$\Delta_1 = \frac{1}{\sigma_1 S_1}, \quad \Delta_2 = -\frac{1}{\sigma_2 S_2}.$$

Then

$$\begin{aligned} d\left(\frac{X}{\beta}\right) &= \frac{1}{\beta}\left[\frac{\mu_1 - r}{\sigma_1}dt + dB_1\right] - \frac{1}{\beta}\left[\frac{\mu_2 - r}{\sigma_2}dt + dB_1\right] \\ &= \frac{1}{\beta}\underbrace{\left[\frac{\mu_1 - r}{\sigma_1} - \frac{\mu_2 - r}{\sigma_2}\right]}_{\text{Positive}} dt \end{aligned}$$

Case II: $\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$. The market price of risk equations

$$\begin{aligned} \sigma_1\theta_1 &= \mu_1 - r \\ \sigma_2\theta_1 &= \mu_2 - r \end{aligned}$$

have the solution

$$\theta_1 = \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2},$$

θ_2 is free; there are infinitely many risk-neutral measures. Let $\widetilde{I\!P}$ be one of them.

Hedging:

$$\begin{aligned} d\left(\frac{X}{\beta}\right) &= \frac{1}{\beta}\Delta_1 S_1[(\mu_1 - r)\ dt + \sigma_1\ dB_1] + \frac{1}{\beta}\Delta_2 S_2[(\mu_2 - r)\ dt + \sigma_2\ dB_1] \\ &= \frac{1}{\beta}\Delta_1 S_1\sigma_1[\theta_1\ dt + dB_1] + \frac{1}{\beta}\Delta_2 S_2\sigma_2[\theta_1\ dt + dB_1] \\ &= \left(\frac{1}{\beta}\Delta_1 S_1\sigma_1 + \frac{1}{\beta}\Delta_2 S_2\sigma_2\right)\ d\widetilde{B}_1. \end{aligned}$$

Notice that $\widetilde{B}_2$ does not appear.

Let V be an $\mathcal{F}(T)$-measurable random variable. If V depends on B_2, then it can probably not be hedged. For example, if

$$V = h(S_1(T), S_2(T)),$$

and σ_1 or σ_2 depend on B_2, then there is trouble.

More precisely, we define the $\widetilde{I\!P}$-martingale

$$Y(t) = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\middle|\mathcal{F}(t)\right], \qquad 0 \le t \le T.$$

We can write

$$Y(t) = Y(0) + \int_0^t \gamma_1 \, d\widetilde{B}_1 + \int_0^t \gamma_2 \, d\widetilde{B}_2,$$

so

$$dY = \gamma_1 \, d\widetilde{B}_1 + \gamma_2 \, d\widetilde{B}_2.$$

To get $d\left(\frac{X}{\beta}\right)$ to match dY, we must have

$$\gamma_2 = 0.$$