Chapter 18

Martingale Representation Theorem

18.1 Martingale Representation Theorem

See Oksendal, 4th ed., Theorem 4.11, p.50.

Theorem 1.56 *Let $B(t), 0 \leq t \leq T$, be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}(t), 0 \leq t \leq T$, be the filtration* generated by this Brownian motion. *Let $X(t), 0 \leq t \leq T$, be a martingale (under $I\!\!P$) relative to this filtration. Then there is an adapted process $\delta(t), 0 \leq t \leq T$, such that*

$$X(t) = X(0) + \int_0^t \delta(u)\, dB(u), \qquad 0 \leq t \leq T.$$

In particular, the paths of X are continuous.

Remark 18.1 We already know that if $X(t)$ is a process satisfying

$$dX(t) = \delta(t)\, dB(t),$$

then $X(t)$ is a martingale. Now we see that if $X(t)$ is a martingale adapted to the filtration generated by the Brownian motion $B(t)$, i.e, the Brownian motion is the only source of randomness in X, then

$$dX(t) = \delta(t)\, dB(t)$$

for some $\delta(t)$.

18.2 A hedging application

Homework Problem 4.5. In the context of Girsanov's Theorem, suppse that $\mathcal{F}(t), 0 \leq t \leq T$, is the filtration generated by the Brownian motion B (under $I\!\!P$). Suppose that Y is a $\widetilde{I\!\!P}$-martingale. Then there is an adapted process $\gamma(t), 0 \leq t \leq T$, such that

$$Y(t) = Y(0) + \int_0^t \gamma(u)\, d\widetilde{B}(u), \qquad 0 \leq t \leq T.$$

$$
\begin{aligned}
dS(t) &= \mu(t)S(t)\,dt + \sigma(t)S(t)\,dB(t),\\
\beta(t) &= \exp\left\{\int_0^t r(u)\,du\right\},\\
\theta(t) &= \frac{\mu(t)-r(t)}{\sigma(t)},\\
\widetilde{B}(t) &= \int_0^t \theta(u)\,du + B(t),\\
Z(t) &= \exp\left\{-\int_0^t \theta(u)\,dB(u) - \tfrac{1}{2}\int_0^t \theta^2(u)\,du\right\},\\
\widetilde{I\!P}(A) &= \int_A Z(T)\,dI\!P, \qquad \forall A \in \mathcal{F}.
\end{aligned}
$$

Then

$$
d\left(\frac{S(t)}{\beta(t)}\right) = \frac{S(t)}{\beta(t)}\sigma(t)\,d\widetilde{B}(t).
$$

Let $\Delta(t), 0 \le t \le T,$ be a portfolio process. The corresponding wealth process $X(t)$ satisfies

$$
d\left(\frac{X(t)}{\beta(t)}\right) = \Delta(t)\sigma(t)\frac{S(t)}{\beta(t)}\,d\widetilde{B}(t),
$$

i.e.,

$$
\frac{X(t)}{\beta(t)} = X(0) + \int_0^t \Delta(u)\sigma(u)\frac{S(u)}{\beta(u)}\,d\widetilde{B}(u), \qquad 0 \le t \le T.
$$

Let V be an $\mathcal{F}(T)$-measurable random variable, representing the payoff of a contingent claim at time T. We want to choose $X(0)$ and $\Delta(t), 0 \le t \le T$, so that

$$
X(T) = V.
$$

Define the $\widetilde{I\!P}$-martingale

$$
Y(t) = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\middle|\mathcal{F}(t)\right], \qquad 0 \le t \le T.
$$

According to Homework Problem 4.5, there is an adapted process $\gamma(t), 0 \le t \le T$, such that

$$
Y(t) = Y(0) + \int_0^t \gamma(u)\,d\widetilde{B}(u), \qquad 0 \le t \le T.
$$

Set $X(0) = Y(0) = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\right]$ and choose $\Delta(u)$ so that

$$
\Delta(u)\sigma(u)\frac{S(u)}{\beta(u)} = \gamma(u).
$$

With this choice of $\Delta(u), 0 \le u \le T$, we have

$$\frac{X(t)}{\beta(t)} = Y(t) = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\bigg|\mathcal{F}(t)\right], \qquad 0 \le t \le T.$$

In particular,

$$\frac{X(T)}{\beta(T)} = \widetilde{I\!E}\left[\frac{V}{\beta(T)}\bigg|\mathcal{F}(T)\right] = \frac{V}{\beta(T)},$$

so

$$X(T) = V.$$

The Martingale Representation Theorem guarantees the existence of a hedging portfolio, although it does not tell us how to compute it. It also justifies the risk-neutral pricing formula

$$\begin{aligned} X(t) &= \beta(t)\widetilde{I\!E}\left[\frac{V}{\beta(T)}\bigg|\mathcal{F}(t)\right] \\ &= \frac{\beta(t)}{Z(t)} I\!E\left[\frac{Z(T)}{\beta(T)}V\bigg|\mathcal{F}(t)\right] \\ &= \frac{1}{\zeta(t)} I\!E\left[\zeta(T)V\bigg|\mathcal{F}(t)\right], \qquad 0 \le t \le T, \end{aligned}$$

where

$$\begin{aligned} \zeta(t) &= \frac{Z(t)}{\beta(t)} \\ &= \exp\left\{-\int_0^t \theta(u)\, dB(u) - \int_0^t (r(u) + \tfrac{1}{2}\theta^2(u))\, du\right\} \end{aligned}$$

18.3 d-dimensional Girsanov Theorem

Theorem 3.57 (d-dimensional Girsanov) • *$B(t) = (B_1(t), \ldots, B_d(t)), 0 \le t \le T$, a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$;*

- *$\mathcal{F}(t), 0 \le t \le T$, the accompanying filtration, perhaps larger than the one generated by B;*
- *$\theta(t) = (\theta_1(t), \ldots, \theta_d(t)), 0 \le t \le T$, d-dimensional adapted process.*

For $0 \le t \le T$, define

$$\begin{aligned} \widetilde{B}_j(t) &= \int_0^t \theta_j(u)\, du + B_j(t), \qquad j = 1, \ldots, d, \\ Z(t) &= \exp\left\{-\int_0^t \theta(u).\, dB(u) - \tfrac{1}{2}\int_0^t ||\theta(u)||^2\, du\right\}, \\ \widetilde{I\!P}(A) &= \int_A Z(T)\, dI\!P. \end{aligned}$$

Then, under $\widetilde{I\!P}$, the process

$$\widetilde{B}(t) = (\widetilde{B}_1(t), \dots, \widetilde{B}_d(t)), \qquad 0 \le t \le T,$$

is a d-dimensional Brownian motion.

18.4 d-dimensional Martingale Representation Theorem

Theorem 4.58
- $B(t) = (B_1(t), \dots, B_d(t)), 0 \le t \le T$, *a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$;*
- $\mathcal{F}(t), 0 \le t \le T$, *the filtration* generated by the Brownian motion B.

If $X(t), 0 \le t \le T$, is a martingale (under $I\!P$) relative to $\mathcal{F}(t), 0 \le t \le T$, then there is a d-dimensional adpated process $\delta(t) = (\delta_1(t), \dots, \delta_d(t))$, such that

$$X(t) = X(0) + \int_0^t \delta(u).\, dB(u), \qquad 0 \le t \le T.$$

Corollary 4.59 *If we have a d-dimensional adapted process $\theta(t) = (\theta_1(t), \dots, \theta_d(t))$, then we can define $\widetilde{B}, Z$ and $\widetilde{I\!P}$ as in Girsanov's Theorem. If $Y(t), 0 \le t \le T$, is a martingale under $\widetilde{I\!P}$ relative to $\mathcal{F}(t), 0 \le t \le T$, then there is a d-dimensional adpated process $\gamma(t) = (\gamma_1(t), \dots, \gamma_d(t))$ such that*

$$Y(t) = Y(0) + \int_0^t \gamma(u).\, d\widetilde{B}(u), \qquad 0 \le t \le T.$$

18.5 Multi-dimensional market model

Let $B(t) = (B_1(t), \dots, B_d(t)),\ 0 \le t \le T$, be a d-dimensional Brownian motion on some $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t),\ 0 \le t \le T$, be the *filtration generated by* B. Then we can define the following:

Stocks

$$dS_i(t) = \mu_i(t) S_i(t)\ dt + S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)\ dB_j(t), \qquad i = 1, \dots, m$$

Accumulation factor

$$\beta(t) = \exp\left\{ \int_0^t r(u)\ du \right\}.$$

Here, $\mu_i(t), \sigma_{ij}(t)$ and $r(t)$ are adpated processes.

Discounted stock prices

$$d\left(\frac{S_i(t)}{\beta(t)}\right) = \underbrace{(\mu_i(t) - r(t))}_{\text{Risk Premium}} \frac{S_i(t)}{\beta(t)}\, dt + \frac{S_i(t)}{\beta(t)} \sum_{j=1}^{d} \sigma_{ij}(t)\, dB_j(t)$$

$$\stackrel{?}{=} \frac{S_i(t)}{\beta(t)} \sum_{j=1}^{d} \sigma_{ij}(t) \underbrace{[\theta_j(t) + dB_j(t)]}_{d\widetilde{B}_j(t)} \tag{5.1}$$

For 5.1 to be satisfied, we need to choose $\theta_1(t), \ldots, \theta_d(t)$, so that

$$\sum_{j=1}^{d} \sigma_{ij}(t)\theta_j(t) = \mu_i(t) - r(t), \qquad i = 1, \ldots, m. \tag{MPR}$$

Market price of risk. The market price of risk is an adapted process $\theta(t) = (\theta_1(t), \ldots, \theta_d(t))$ satisfying the system of equations (MPR) above. There are three cases to consider:

Case I: (Unique Solution). For Lebesgue-almost every t and $I\!\!P$-almost every ω, (MPR) has a *unique solution* $\theta(t)$. Using $\theta(t)$ in the d-dimensional Girsanov Theorem, we define a *unique risk-neutral probability measure* $\widetilde{I\!\!P}$. Under $\widetilde{I\!\!P}$, every discounted stock price is a martingale. Consequently, the discounted wealth process corresponding to any portfolio process is a $\widetilde{I\!\!P}$-martingale, and this implies that the market admits no arbitrage. Finally, the Martingale Representation Theorem can be used to show that every contingent claim can be hedged; the market is said to be *complete*.

Case II: (No solution.) If (MPR) has no solution, then there is *no risk-neutral probability measure* and the market admits *arbitrage*.

Case III: (Multiple solutions). If (MPR) has multiple solutions, then there are *multiple risk-neutral probability measures*. The market admits *no arbitrage*, but there are contingent claims which cannot be hedged; the market is said to be *incomplete.*

Theorem 5.60 (Fundamental Theorem of Asset Pricing) Part I. *(Harrison and Pliska, Martingales and Stochastic integrals in the theory of continuous trading,* Stochastic Proc. and Applications 11 *(1981), pp 215-260.):*
If a market has a risk-neutral probability measure, then it admits no arbitrage.

Part II. *(Harrison and Pliska, A stochastic calculus model of continuous trading: complete markets,* Stochastic Proc. and Applications *15 (1983), pp 313-316):*
The risk-neutral measure is unique if and only if every contingent claim can be hedged.