Chapter 18

Martingale Representation Theorem

18.1 Martingale Representation Theorem


**Theorem 1.56** Let $B(t), 0 \leq t \leq T$, be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}(t), 0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let $X(t), 0 \leq t \leq T$, be a martingale (under $\mathbb{P}$) relative to this filtration. Then there is an adapted process $\delta(t), 0 \leq t \leq T$, such that

$$X(t) = X(0) + \int_0^t \delta(u) \, dB(u), \quad 0 \leq t \leq T.$$  

In particular, the paths of $X$ are continuous.

**Remark 18.1** We already know that if $X(t)$ is a process satisfying

$$dX(t) = \delta(t) \, dB(t),$$

then $X(t)$ is a martingale. Now we see that if $X(t)$ is a martingale adapted to the filtration generated by the Brownian motion $B(t)$, i.e, the Brownian motion is the only source of randomness in $X$, then

$$dX(t) = \delta(t) \, dB(t)$$

for some $\delta(t)$.

18.2 A hedging application

**Homework Problem 4.5.** In the context of Girsanov’s Theorem, suppose that $\mathcal{F}(t), 0 \leq t \leq T$, is the filtration generated by the Brownian motion $B$ (under $\mathbb{P}$). Suppose that $Y$ is a $\mathbb{P}$-martingale. Then there is an adapted process $\gamma(t), 0 \leq t \leq T$, such that

$$Y(t) = Y(0) + \int_0^t \gamma(u) \, dB(u), \quad 0 \leq t \leq T.$$
\[ dS(t) = \mu(t)S(t) \, dt + \sigma(t)S(t) \, dB(t), \]
\[ \beta(t) = \exp\left\{ \int_0^t r(u) \, du \right\}, \]
\[ \theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}, \]
\[ \bar{B}(t) = \int_0^t \theta(u) \, du + B(t), \]
\[ Z(t) = \exp\left\{ -\int_0^t \theta(u) \, d\bar{B}(u) - \frac{1}{2} \int_0^t \theta^2(u) \, du \right\}, \]
\[ \mathbb{P}(A) = \int_A Z(T) \, d\mathbb{P}, \quad \forall A \in \mathcal{F}. \]

Then
\[ d\left( \frac{S(t)}{\beta(t)} \right) = \frac{S(t)}{\beta(t)} \sigma(t) \, d\bar{B}(t). \]

Let \( \Delta(t), 0 \leq t \leq T, \) be a portfolio process. The corresponding wealth process \( X(t) \) satisfies
\[ d\left( \frac{X(t)}{\beta(t)} \right) = \Delta(t)\sigma(t) \frac{S(t)}{\beta(t)} \, d\bar{B}(t), \]
i.e.,
\[ \frac{X(t)}{\beta(t)} = X(0) + \int_0^t \Delta(u) \sigma(u) \frac{S(u)}{\beta(u)} \, d\bar{B}(u), \quad \forall t \leq T. \]

Let \( V \) be an \( \mathcal{F}(T) \)-measurable random variable, representing the payoff of a contingent claim at time \( T \). We want to choose \( X(0) \) and \( \Delta(t), 0 \leq t \leq T, \) so that
\[ X(T) = V. \]

Define the \( \mathbb{P} \)-martingale
\[ Y(t) = \mathbb{E} \left[ \frac{V}{\beta(T)} \bigg| \mathcal{F}(t) \right], \quad \forall t \leq T. \]

According to Homework Problem 4.5, there is an adapted process \( \gamma(t), 0 \leq t \leq T, \) such that
\[ Y(t) = Y(0) + \int_0^t \gamma(u) \, dB(u), \quad \forall t \leq T. \]

Set \( X(0) = Y(0) = \mathbb{E} \left[ \frac{V}{\beta(T)} \right] \) and choose \( \Delta(u) \) so that
\[ \Delta(u) \sigma(u) \frac{S(u)}{\beta(u)} = \gamma(u), \]
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With this choice of $\Delta(u)$, $0 \leq u \leq T$, we have

$$\frac{X(t)}{\beta(t)} = Y(t) = \E \left[ \frac{V}{\beta(T)} \right]_{\mathcal{F}(t)}, \quad 0 \leq t \leq T.$$ 

In particular,

$$\frac{X(T)}{\beta(T)} = \E \left[ \frac{V}{\beta(T)} \right]_{\mathcal{F}(T)} = \frac{V}{\beta(T)},$$

so

$$X(T) = V.$$ 

The Martingale Representation Theorem guarantees the existence of a hedging portfolio, although it does not tell us how to compute it. It also justifies the risk-neutral pricing formula

$$X(t) = \beta(t) \E \left[ \frac{V}{\beta(T)} \right]_{\mathcal{F}(t)} = \frac{\beta(t)}{Z(t)} \E \left[ \frac{Z(T)}{\beta(T)} V \right]_{\mathcal{F}(t)} = \frac{1}{\zeta(t)} \E \left[ \zeta(T) V \right]_{\mathcal{F}(t)}, \quad 0 \leq t \leq T,$$

where

$$\zeta(t) = \frac{Z(t)}{\beta(t)} = \exp \left\{ - \int_0^t \theta(u) \, dB(u) - \int_0^t \left( r(u) + \frac{1}{2} \theta^2(u) \right) \, du \right\}.$$ 

18.3 $d$-dimensional Girsanov Theorem

**Theorem 3.57 ($d$-dimensional Girsanov)**

- $B(t) = (B_1(t), \ldots, B_d(t)), 0 \leq t \leq T$, a $d$-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$;
- $\mathcal{F}(t), 0 \leq t \leq T$, the accompanying filtration, perhaps larger than the one generated by $B$;
- $\theta(t) = (\theta_1(t), \ldots, \theta_d(t)), 0 \leq t \leq T$, $d$-dimensional adapted process.

For $0 \leq t \leq T$, define

$$\bar{B}_j(t) = \int_0^t \theta_j(u) \, du + B_j(t), \quad j = 1, \ldots, d,$$

$$Z(t) = \exp \left\{ - \int_0^t \theta(u) \, dB(u) - \frac{1}{2} \int_0^t \|\theta(u)\|^2 \, du \right\},$$

$$\bar{\mathbb{P}}(A) = \int_A Z(T) \, d\mathbb{P}.$$
Then, under \( \tilde{\mathbb{P}} \), the process 
\[
\bar{B}(t) = (\bar{B}_1(t),\ldots,\bar{B}_d(t)), \quad 0 \leq t \leq T,
\]
is a \( d \)-dimensional Brownian motion.

### 18.4 \( d \)-dimensional Martingale Representation Theorem

**Theorem 4.58**  
- \( B(t) = (B_1(t),\ldots,B_d(t)), 0 \leq t \leq T \), a \( d \)-dimensional Brownian motion on \( (\Omega, \mathcal{F}, \mathbb{P}) \);  
- \( \mathcal{F}(t), 0 \leq t \leq T \), the filtration generated by the Brownian motion \( B \).  

If \( X(t), 0 \leq t \leq T \), is a martingale (under \( \mathbb{P}' \)) relative to \( \mathcal{F}(t), 0 \leq t \leq T \), then there is a \( d \)-dimensional adapted process \( \delta(t) = (\delta_1(t),\ldots,\delta_d(t)) \), such that  
\[
X(t) = X(0) + \int_0^t \delta(u) \cdot dB(u), \quad 0 \leq t \leq T.
\]

**Corollary 4.59**  
If we have a \( d \)-dimensional adapted process \( \theta(t) = (\theta_1(t),\ldots,\theta_d(t)) \), then we can define \( \bar{B}, Z \) and \( \mathbb{P}' \) as in Girsanov’s Theorem. If \( Y(t), 0 \leq t \leq T \), is a martingale under \( \mathbb{P}' \) relative to \( \mathcal{F}(t), 0 \leq t \leq T \), then there is a \( d \)-dimensional adapted process \( \gamma(t) = (\gamma_1(t),\ldots,\gamma_d(t)) \) such that  
\[
Y(t) = Y(0) + \int_0^t \gamma(u) \cdot d \bar{B}(u), \quad 0 \leq t \leq T.
\]

### 18.5 Multi-dimensional market model

Let \( B(t) = (B_1(t),\ldots,B_d(t)), 0 \leq t \leq T \), be a \( d \)-dimensional Brownian motion on some \( (\Omega, \mathcal{F}, \mathbb{P}) \), and let \( \mathcal{F}(t), 0 \leq t \leq T \), be the filtration generated by \( B \). Then we can define the following:

**Stocks**
\[
dS_i(t) = \mu_i(t)S_i(t) \, dt + S_i(t) \sum_{j=1}^d \sigma_{ij}(t) \, dB_j(t), \quad i = 1,\ldots,m
\]

**Accumulation factor**
\[
\beta(t) = \exp \left\{ \int_0^t r(u) \, du \right\}.
\]

Here, \( \mu_i(t), \sigma_{ij}(t) \) and \( r(t) \) are adapted processes.
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Discounted stock prices

\[
d \frac{S_i(t)}{\beta(t)} = (\mu_i(t) - r(t)) \frac{S_i(t)}{\beta(t)} \, dt + \frac{S_i(t)}{\beta(t)} \sum_{j=1}^{d} \sigma_{ij}(t) \, dB_j(t)
\]

\[
=r \frac{S_i(t)}{\beta(t)} \sum_{j=1}^{d} \sigma_{ij}(t) \left[ \theta_j(t) + dB_j(t) \right] 
\]

For 5.1 to be satisfied, we need to choose \( \theta_1(t), \ldots, \theta_d(t) \), so that

\[
\sum_{j=1}^{d} \sigma_{ij}(t) \theta_j(t) = \mu_i(t) - r(t), \quad i = 1, \ldots, m.
\]

**Market price of risk.** The market price of risk is an adapted process \( \theta(t) = (\theta_1(t), \ldots, \theta_d(t)) \) satisfying the system of equations (MPR) above. There are three cases to consider:

**Case I:** (Unique Solution). For Lebesgue-almost every \( t \) and \( P \)-almost every \( \omega \), (MPR) has a unique solution \( \theta(t) \). Using \( \theta(t) \) in the \( d \)-dimensional Girsanov Theorem, we define a unique risk-neutral probability measure \( \tilde{P} \). Under \( \tilde{P} \), every discounted stock price is a martingale. Consequently, the discounted wealth process corresponding to any portfolio process is a \( \tilde{P} \)-martingale, and this implies that the market admits no arbitrage. Finally, the Martingale Representation Theorem can be used to show that every contingent claim can be hedged; the market is said to be complete.

**Case II:** (No solution.) If (MPR) has no solution, then there is no risk-neutral probability measure and the market admits arbitrage.

**Case III:** (Multiple solutions). If (MPR) has multiple solutions, then there are multiple risk-neutral probability measures. The market admits no arbitrage, but there are contingent claims which cannot be hedged; the market is said to be incomplete.


If a market has a risk-neutral probability measure, then it admits no arbitrage.


The risk-neutral measure is unique if and only if every contingent claim can be hedged.