Chapter 17

Girsanov's theorem and the risk-neutral measure

(Please see Oksendal, 4th ed., pp 145–151.)

Theorem 0.52 (Girsanov, One-dimensional) *Let $B(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}(t), 0 \le t \le T$, be the accompanying filtration, and let $\theta(t), 0 \le t \le T$, be a process adapted to this filtration. For $0 \le t \le T$, define*

$$\widetilde{B}(t) = \int_0^t \theta(u)\, du + B(t),$$
$$Z(t) = \exp\left\{ -\int_0^t \theta(u)\, dB(u) - \tfrac{1}{2}\int_0^t \theta^2(u)\, du \right\},$$

and define a new probability measure by

$$\widetilde{I\!P}(A) = \int_A Z(T)\, dI\!P, \qquad \forall A \in \mathcal{F}.$$

Under $\widetilde{I\!P}$, the process $\widetilde{B}(t), 0 \le t \le T$, is a Brownian motion.

Caveat: This theorem requires a technical condition on the size of θ. If

$$I\!E \exp\left\{ \tfrac{1}{2}\int_0^T \theta^2(u)\, du \right\} < \infty,$$

everything is OK.

We make the following remarks:

$Z(t)$ **is a matingale.** In fact,

$$\begin{aligned} dZ(t) &= -\theta(t)Z(t)\, dB(t) + \tfrac{1}{2}\theta^2(t)Z(t)\, dB(t)\, dB(t) - \tfrac{1}{2}\theta^2(t)Z(t)\, dt \\ &= -\theta(t)Z(t)\, dB(t). \end{aligned}$$

$\widetilde{I\!P}$ is a probability measure. Since $Z(0) = 1$, we have $I\!E Z(t) = 1$ for every $t \geq 0$. In particular

$$\widetilde{I\!P}(\Omega) = \int_\Omega Z(T)\, dI\!P = I\!E Z(T) = 1,$$

so $\widetilde{I\!P}$ is a probability measure.

$\widetilde{I\!E}$ in terms of $I\!E$. Let $\widetilde{I\!E}$ denote expectation under $\widetilde{I\!P}$. If X is a random variable, then

$$\widetilde{I\!E} Z = I\!E\left[Z(T)X\right].$$

To see this, consider first the case $X = \mathbf{1}_A$, where $A \in \mathcal{F}$. We have

$$\widetilde{I\!E} X = \widetilde{I\!P}(A) = \int_A Z(T)\, dI\!P = \int_\Omega Z(T)\mathbf{1}_A\, dI\!P = I\!E\left[Z(T)X\right].$$

Now use Williams' "standard machine".

$\widetilde{I\!P}$ and $I\!P$. The intuition behind the formula

$$\widetilde{I\!P}(A) = \int_A Z(T)\, dI\!P \qquad \forall A \in \mathcal{F}$$

is that we want to have

$$\widetilde{I\!P}(\omega) = Z(T,\omega) I\!P(\omega),$$

but since $I\!P(\omega) = 0$ and $\widetilde{I\!P}(\omega) = 0$, this doesn't really tell us anything useful about $\widetilde{I\!P}$. Thus, we consider subsets of Ω, rather than individual elements of Ω.

Distribution of $\widetilde{B}(T)$. If θ is constant, then

$$\begin{aligned} Z(T) &= \exp\left\{-\theta B(T) - \tfrac{1}{2}\theta^2 T\right\} \\ \widetilde{B}(T) &= \theta T + B(T). \end{aligned}$$

Under $I\!P$, $B(T)$ is normal with mean 0 and variance T, so $\widetilde{B}(T)$ is normal with mean θT and variance T:

$$I\!P(\widetilde{B}(T) \in d\tilde{b}) = \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{(\tilde{b} - \theta T)^2}{2T}\right\}\, d\tilde{b}.$$

Removal of Drift from $\widetilde{B}(T)$. The change of measure from $I\!P$ to $\widetilde{I\!P}$ removes the drift from $\widetilde{B}(T)$. To see this, we compute

$$\begin{aligned} \widetilde{I\!E}\widetilde{B}(T) &= I\!E\left[Z(T)(\theta T + B(T))\right] \\ &= I\!E\left[\exp\left\{-\theta B(T) - \tfrac{1}{2}\theta^2 T\right\}\ (\theta T + B(T))\right] \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (\theta T + b) \exp\{-\theta b - \tfrac{1}{2}\theta^2 T\}\ \exp\left\{-\frac{b^2}{2T}\right\}\, db \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (\theta T + b) \exp\left\{-\frac{(b + \theta T)^2}{2T}\right\}\, db \\ (y = \theta T + b) &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} y \exp\left\{-\frac{y^2}{2}\right\}\, dy \quad \text{(Substitute } y = \theta T + b\text{)} \\ &= 0. \end{aligned}$$

We can also see that $\widetilde{I\!E}\widetilde{B}(T) = 0$ by arguing directly from the density formula

$$I\!P\left\{\widetilde{B}(t) \in d\tilde{b}\right\} = \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{(\tilde{b} - \theta T)^2}{2T}\right\} d\tilde{b}.$$

Because

$$\begin{aligned} Z(T) &= \exp\{-\theta B(T) - \tfrac{1}{2}\theta^2 T\} \\ &= \exp\{-\theta(\widetilde{B}(T) - \theta T) - \tfrac{1}{2}\theta^2 T\} \\ &= \exp\{-\theta \widetilde{B}(T) + \tfrac{1}{2}\theta^2 T\}, \end{aligned}$$

we have

$$\begin{aligned} \widetilde{I\!P}\left\{\widetilde{B}(T) \in d\tilde{b}\right\} &= I\!P\left\{\widetilde{B}(T) \in d\tilde{b}\right\} \exp\left\{-\theta\tilde{b} + \tfrac{1}{2}\theta^2 T\right\} \\ &= \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{(\tilde{b} - \theta T)^2}{2T} - \theta\tilde{b} + \tfrac{1}{2}\theta^2 T\right\} d\tilde{b}. \\ &= \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{\tilde{b}^2}{2T}\right\} d\tilde{b}. \end{aligned}$$

Under $\widetilde{I\!P}$, $\widetilde{B}(T)$ is normal with *mean zero* and variance T. Under $I\!P$, $\widetilde{B}(T)$ is normal with *mean* θT and variance T.

Means change, variances don't. When we use the Girsanov Theorem to change the probability measure, means change but variances do not. Martingales may be destroyed or created. Volatilities, quadratic variations and cross variations are unaffected. Check:

$$d\widetilde{B}\ d\widetilde{B} = (\theta(t)\ dt + dB(t))^2 = dB.dB = dt.$$

17.1 Conditional expectations under $\widetilde{I\!P}$

Lemma 1.53 *Let $0 \le t \le T$. If X is $\mathcal{F}(t)$-measurable, then*

$$\widetilde{I\!E}X = I\!E[X.Z(t)].$$

Proof:

$$\begin{aligned} \widetilde{I\!E}X &= I\!E[X.Z(T)] = I\!E\left[\ I\!E[X.Z(T)|\mathcal{F}(t)]\ \right] \\ &= I\!E\left[X\ I\!E[Z(T)|\mathcal{F}(t)]\ \right] \\ &= I\!E[X.Z(t)] \end{aligned}$$

because $Z(t), 0 \le t \le T$, is a martingale under $I\!P$. ∎

Lemma 1.54 (Baye's Rule) *If X is $\mathcal{F}(t)$-measurable and $0 \le s \le t \le T$, then*

$$\widetilde{I\!E}[X|\mathcal{F}(s)] = \frac{1}{Z(s)} I\!E[XZ(t)|\mathcal{F}(s)]. \tag{1.1}$$

Proof: It is clear that $\frac{1}{Z(s)} I\!E[XZ(t)|\mathcal{F}(s)]$ is $\mathcal{F}(s)$-measurable. We check the partial averaging property. For $A \in \mathcal{F}(s)$, we have

$$\begin{aligned}
\int_A \frac{1}{Z(s)} I\!E[XZ(t)|\mathcal{F}(s)]\, d\widetilde{I\!P} &= \widetilde{I\!E}\left[\mathbf{1}_A \frac{1}{Z(s)} I\!E[XZ(t)|\mathcal{F}(s)]\right] \\
&= I\!E\left[\mathbf{1}_A I\!E[XZ(t)|\mathcal{F}(s)]\right] \qquad \text{(Lemma 1.53)} \\
&= I\!E\left[I\!E[\mathbf{1}_A XZ(t)|\mathcal{F}(s)]\right] \qquad \text{(Taking in what is known)} \\
&= I\!E[\mathbf{1}_A XZ(t)] \\
&= \widetilde{I\!E}[\mathbf{1}_A X] \qquad \text{(Lemma 1.53 again)} \\
&= \int_A X\, d\widetilde{I\!P}.
\end{aligned}$$

■

Although we have proved Lemmas 1.53 and 1.54, we have not proved Girsanov's Theorem. We will not prove it completely, but here is the beginning of the proof.

Lemma 1.55 *Using the notation of Girsanov's Theorem, we have the martingale property*

$$\widetilde{I\!E}[\widetilde{B}(t)|\mathcal{F}(s)] = \widetilde{B}(s), \qquad 0 \le s \le t \le T.$$

Proof: We first check that $\widetilde{B}(t)Z(t)$ is a martingale under $I\!P$. Recall

$$\begin{aligned}
d\widetilde{B}(t) &= \theta(t)\, dt + dB(t), \\
dZ(t) &= -\theta(t)Z(t)\, dB(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(\widetilde{B}Z) &= \widetilde{B}\, dZ + Z\, d\widetilde{B} + d\widetilde{B}\, dZ \\
&= -\widetilde{B}\theta Z\, dB + Z\theta\, dt + Z\, dB - \theta Z\, dt \\
&= (-\widetilde{B}\theta Z + Z)\, dB.
\end{aligned}$$

Next we use Bayes' Rule. For $0 \le s \le t \le T$,

$$\begin{aligned}
\widetilde{I\!E}[\widetilde{B}(t)|\mathcal{F}(s)] &= \frac{1}{Z(s)} I\!E[\widetilde{B}(t)Z(t)|\mathcal{F}(s)] \\
&= \frac{1}{Z(s)} \widetilde{B}(s)Z(s) \\
&= \widetilde{B}(s).
\end{aligned}$$

■

Definition 17.1 (Equivalent measures) Two measures on the same probability space which have the same measure-zero sets are said to be *equivalent.*

The probability measures $I\!P$ and $\widetilde{I\!P}$ of the Girsanov Theorem are equivalent. Recall that $\widetilde{I\!P}$ is defined by

$$\widetilde{I\!P}(A) = \int Z(T)\ dI\!P, \qquad A \in \mathcal{F}.$$

If $I\!P(A) = 0$, then $\int_A Z(T)\ dI\!P = 0$. Because $Z(T) > 0$ for every ω, we can invert the definition of $\widetilde{I\!P}$ to obtain

$$I\!P(A) = \int_A \frac{1}{Z(T)}\ d\widetilde{I\!P}, \qquad A \in \mathcal{F}.$$

If $\widetilde{I\!P}(A) = 0$, then $\int_A \frac{1}{Z(T)}\ dI\!P = 0$.

17.2 Risk-neutral measure

As usual we are given the **Brownian motion:** $B(t), 0 \le t \le T$, with filtration $\mathcal{F}(t), 0 \le t \le T$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We can then define the following.

Stock price:

$$dS(t) = \mu(t)S(t)\ dt + \sigma(t)S(t)\ dB(t).$$

The processes $\mu(t)$ and $\sigma(t)$ are adapted to the filtration. The stock price model is completely general, subject only to the condition that the paths of the process are continuous.

Interest rate: $r(t), 0 \le t \le T$. The process $r(t)$ is adapted.

Wealth of an agent, starting with $X(0) = x$. We can write the wealth process differential in several ways:

$$\begin{aligned}
dX(t) &= \underbrace{\Delta(t)\ dS(t)}_{\text{Capital gains from Stock}} + \underbrace{r(t)[X(t) - \Delta(t)S(t)]\ dt}_{\text{Interest earnings}} \\
&= r(t)X(t)\ dt + \Delta(t)[dS(t) - rS(t)\ dt] \\
&= r(t)X(t)\ dt + \Delta(t)\underbrace{(\mu(t) - r(t))}_{\text{Risk premium}}S(t)\ dt + \Delta(t)\sigma(t)S(t)\ dB(t) \\
&= r(t)X(t)\ dt + \Delta(t)\sigma(t)S(t)\left[\underbrace{\frac{\mu(t) - r(t)}{\sigma(t)}}_{\text{Market price of risk}=\theta(t)}\ dt + dB(t)\right]
\end{aligned}$$

Discounted processes:

$$\begin{aligned}
d\left(e^{-\int_0^t r(u)\,du}S(t)\right) &= e^{-\int_0^t r(u)\,du}[-r(t)S(t)\,dt + dS(t)] \\
d\left(e^{-\int_0^t r(u)\,du}X(t)\right) &= e^{-\int_0^t r(u)\,du}[-r(t)X(t)\,dt + dX(t)] \\
&= \Delta(t)d\left(e^{-\int_0^t r(u)\,du}S(t)\right).
\end{aligned}$$

Notation:

$$\begin{aligned}
\beta(t) &= e^{\int_0^t r(u)\,du}, \qquad & \frac{1}{\beta(t)} &= e^{-\int_0^t r(u)\,du}, \\
d\beta(t) &= r(t)\beta(t)\,dt, \qquad & d\left(\frac{1}{\beta(t)}\right) &= -\frac{r(t)}{\beta(t)}\,dt.
\end{aligned}$$

The discounted formulas are

$$\begin{aligned}
d\left(\frac{S(t)}{\beta(t)}\right) &= \frac{1}{\beta(t)}\left[-r(t)S(t)\,dt + dS(t)\right] \\
&= \frac{1}{\beta(t)}\left[(\mu(t) - r(t))S(t)\,dt + \sigma(t)S(t)\,dB(t)\right] \\
&= \frac{1}{\beta(t)}\sigma(t)S(t)\left[\theta(t)\,dt + dB(t)\right], \\
d\left(\frac{X(t)}{\beta(t)}\right) &= \Delta(t)\,d\left(\frac{S(t)}{\beta(t)}\right) \\
&= \frac{\Delta(t)}{\beta(t)}\,\sigma(t)S(t)\,[\theta(t)\,dt + dB(t)].
\end{aligned}$$

Changing the measure. Define

$$\widetilde{B}(t) = \int_0^t \theta(u)\,du + B(t).$$

Then

$$\begin{aligned}
d\left(\frac{S(t)}{\beta(t)}\right) &= \frac{1}{\beta(t)}\sigma(t)S(t)\,d\widetilde{B}(t), \\
d\left(\frac{X(t)}{\beta(t)}\right) &= \frac{\Delta(t)}{\beta(t)}\sigma(t)S(t)\,d\widetilde{B}(t).
\end{aligned}$$

Under $\widetilde{I\!P}$, $\frac{S(t)}{\beta(t)}$ and $\frac{X(t)}{\beta(t)}$ are martingales.

Definition 17.2 (Risk-neutral measure) A *risk-neutral measure* (sometimes called a *martingale measure*) is any probability measure, equivalent to the market measure $I\!P$, which makes all discounted asset prices martingales.

For the market model considered here,

$$\widetilde{I\!P}(A) = \int_A Z(T)\, dI\!P, \qquad A \in \mathcal{F},$$

where

$$Z(t) = \exp\left\{ -\int_0^t \theta(u)\, dB(u) - \tfrac{1}{2}\int_0^t \theta^2(u)\, du \right\},$$

is the unique risk-neutral measure. Note that because $\theta(t) = \frac{\mu(t)-r(t)}{\sigma(t)}$, we must assume that $\sigma(t) \neq 0$.

Risk-neutral valuation. Consider a contingent claim paying an $\mathcal{F}(T)$-measurable random variable V at time T.

Example 17.1

$$V = (S(T)-K)^+, \qquad \text{European call}$$

$$V = (K-S(T))^+, \qquad \text{European put}$$

$$V = \left(\frac{1}{T}\int_0^T S(u)\, du - K \right)^+, \qquad \text{Asian call}$$

$$V = \max_{0 \le t \le T} S(t), \qquad \text{Look back}$$

■

If there is a hedging portfolio, i.e., a process $\Delta(t)$, $0 \le t \le T$, whose corresponding wealth process satisfies $X(T) = V$, then

$$X(0) = \widetilde{I\!E}\left[\frac{V}{\beta(T)} \right].$$

This is because $\frac{X(t)}{\beta(t)}$ is a martingale under $\widetilde{I\!P}$, so

$$X(0) = \frac{X(0)}{\beta(0)} = \widetilde{I\!E}\left[\frac{X(T)}{\beta(T)} \right] = \widetilde{I\!E}\left[\frac{V}{\beta(T)} \right].$$