Chapter 16

Markov processes and the Kolmogorov equations

16.1 Stochastic Differential Equations

Consider the *stochastic differential equation*:

$$dX(t) = a(t, X(t))\, dt + \sigma(t, X(t))\, dB(t). \tag{SDE}$$

Here $a(t, x)$ and $\sigma(t, x)$ are given functions, usually assumed to be continuous in (t, x) and Lipschitz continuous in x,i.e., there is a constant L such that

$$|a(t, x) - a(t, y)| \le L|x - y|, \qquad |\sigma(t, x) - \sigma(t, y)| \le L|x - y|$$

for all t, x, y.

Let (t_0, x) be given. A *solution* to (SDE) with the *initial condition* (t_0, x) is a process $\{X(t)\}_{t \ge t_0}$ satisfying

$$\begin{aligned} X(t_0) &= x, \\ X(t) &= X(t_0) + \int_{t_0}^{t} a(s, X(s))\, ds + \int_{t_0}^{t} \sigma(s, X(s))\, dB(s), \qquad t \ge t_0 \end{aligned}$$

The solution process $\{X(t)\}_{t \ge t_0}$ will be adapted to the filtration $\{\mathcal{F}(t)\}_{t \ge 0}$ generated by the Brownian motion. If you know the path of the Brownian motion up to time t, then you can evaluate $X(t)$.

Example 16.1 (Drifted Brownian motion) Let a be a constant and $\sigma = 1$, so

$$dX(t) = a\, dt + dB(t).$$

If (t_0, x) is given and we start with the initial condition

$$X(t_0) = x,$$

then

$$X(t) = x + a(t - t_0) + (B(t) - B(t_0)), \qquad t \geq t_0.$$

To compute the differential w.r.t. t, treat t_0 and $B(t_0)$ as constants:

$$dX(t) = a\ dt + dB(t).$$

■

Example 16.2 (Geometric Brownian motion) Let r and σ be constants. Consider

$$dX(t) = rX(t)\ dt + \sigma X(t)\ dB(t).$$

Given the initial condition

$$X(t_0) = x,$$

the solution is

$$X(t) = x \exp\left\{\sigma(B(t) - B(t_0)) + (r - \tfrac{1}{2}\sigma^2)(t - t_0)\right\}.$$

Again, to compute the differential w.r.t. t, treat t_0 and $B(t_0)$ as constants:

$$\begin{aligned} dX(t) &= (r - \tfrac{1}{2}\sigma^2)X(t)\ dt + \sigma X(t)\ dB(t) + \tfrac{1}{2}\sigma^2 X(t)\ dt \\ &= rX(t)\ dt + \sigma X(t)\ dB(t). \end{aligned}$$

■

16.2 Markov Property

Let $0 \leq t_0 < t_1$ be given and let $h(y)$ be a function. Denote by

$$I\!E^{t_0,x} h(X(t_1))$$

the expectation of $h(X(t_1))$, given that $X(t_0) = x$. Now let $\xi \in I\!R$ be given, and start with initial condition

$$X(0) = \xi.$$

We have the *Markov property*

$$I\!E^{0,\xi}\left[h(X(t_1))\Big|\mathcal{F}(t_0)\right] = I\!E^{t_0,X(t_0)} h(X(t_1)).$$

In other words, if you observe the path of the driving Brownian motion from time 0 to time t_0, and based on this information, you want to estimate $h(X(t_1))$, the only relevant information is the value of $X(t_0)$. You imagine starting the (SDE) at time t_0 at value $X(t_0)$, and compute the expected value of $h(X(t_1))$.

16.3 Transition density

Denote by

$$p(t_0, t_1;\ x, y)$$

the density (in the y variable) of $X(t_1)$, conditioned on $X(t_0) = x$. In other words,

$$I\!E^{t_0,x} h(X(t_1)) = \int_{I\!R} h(y) p(t_0, t_1;\ x, y)\ dy.$$

The Markov property says that for $0 \le t_0 \le t_1$ and for every ξ,

$$I\!E^{0,\xi}\left[h(X(t_1))\Big|\mathcal{F}(t_0)\right] = \int_{I\!R} h(y) p(t_0, t_1;\ X(t_0), y)\ dy.$$

Example 16.3 (Drifted Brownian motion) Consider the SDE

$$dX(t) = a\ dt + dB(t).$$

Conditioned on $X(t_0) = x$, the random variable $X(t_1)$ is normal with mean $x + a(t_1 - t_0)$ and variance $(t_1 - t_0)$, i.e.,

$$p(t_0, t_1;\ x, y) = \frac{1}{\sqrt{2\pi(t_1 - t_0)}} \exp\left\{-\frac{(y - (x + a(t_1 - t_0)))^2}{2(t_1 - t_0)}\right\}.$$

Note that p depends on t_0 and t_1 only through their difference $t_1 - t_0$. This is always the case when $a(t, x)$ and $\sigma(t, x)$ don't depend on t. ■

Example 16.4 (Geometric Brownian motion) Recall that the solution to the SDE

$$dX(t) = rX(t)\ dt + \sigma X(t)\ dB(t),$$

with initial condition $X(t_0) = x$, is Geometric Brownian motion:

$$X(t_1) = x \exp\left\{\sigma(B(t_1) - B(t_0)) + (r - \tfrac{1}{2}\sigma^2)(t_1 - t_0)\right\}.$$

The random variable $B(t_1) - B(t_0)$ has density

$$I\!P\left\{B(t_1) - B(t_0) \in db\right\} = \frac{1}{\sqrt{2\pi(t_1 - t_0)}} \exp\left\{-\frac{b^2}{2(t_1 - t_0)}\right\}\ db,$$

and we are making the change of variable

$$y = x \exp\left\{\sigma b + (r - \tfrac{1}{2}\sigma^2)(t_1 - t_0)\right\}$$

or equivalently,

$$b = \frac{1}{\sigma}\left[\log\frac{y}{x} - (r - \tfrac{1}{2}\sigma^2)(t_1 - t_0)\right].$$

The derivative is

$$\frac{dy}{db} = \sigma y, \qquad \text{or equivalently,} \qquad db = \frac{dy}{\sigma y}.$$

Therefore,

$$\begin{aligned} p(t_0, t_1; x, y)\ dy &= \mathbb{P}\left\{X(t_1) \in dy\right\} \\ &= \frac{1}{\sigma y\sqrt{2\pi(t_1 - t_0)}} \exp\left\{-\frac{1}{2(t_1 - t_0)\sigma^2}\left[\log\frac{y}{x} - (r - \tfrac{1}{2}\sigma^2)(t_1 - t_0)\right]^2\right\}\ dy. \end{aligned}$$

Using the transition density and a fair amount of calculus, one can compute the expected payoff from a European call:

$$\begin{aligned} \mathbb{E}^{t,x}(X(T) - K)^+ &= \int_0^\infty (y - K)^+ p(t, T; x, y)\ dy \\ &= e^{r(T-t)} xN\left(\frac{1}{\sigma\sqrt{T - t}}\left[\log\frac{x}{K} + r(T - t) + \tfrac{1}{2}\sigma^2(T - t)\right]\right) \\ &\quad - KN\left(\frac{1}{\sigma\sqrt{T - t}}\left[\log\frac{x}{K} + r(T - t) - \tfrac{1}{2}\sigma^2(T - t)\right]\right) \end{aligned}$$

where

$$N(\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\eta} e^{-\frac{1}{2}x^2}\ dx = \frac{1}{\sqrt{2\pi}}\int_{-\eta}^{\infty} e^{-\frac{1}{2}x^2}\ dx.$$

Therefore,

$$\begin{aligned} \mathbb{E}^{0,\xi}\left[e^{-r(T-t)}(X(T) - K)^+ \middle| \mathcal{F}(t)\right] &= e^{-r(T-t)}\mathbb{E}^{t,X(t)}(X(T) - K)^+ \\ &= X(t)N\left(\frac{1}{\sigma\sqrt{T - t}}\left[\log\frac{X(t)}{K} + r(T - t) + \tfrac{1}{2}\sigma^2(T - t)\right]\right) \\ &\quad - e^{-r(T-t)}K\ N\left(\frac{1}{\sigma\sqrt{T - t}}\left[\log\frac{X(t)}{K} + r(T - t) - \tfrac{1}{2}\sigma^2(T - t)\right]\right) \end{aligned}$$

■

16.4 The Kolmogorov Backward Equation

Consider

$$dX(t) = a(t, X(t))\ dt + \sigma(t, X(t))\ dB(t),$$

and let $p(t_0, t_1; x, y)$ be the transition density. Then the Kolmogorov Backward Equation is:

$$-\frac{\partial}{\partial t_0}p(t_0, t_1;\ x, y) = a(t_0, x)\frac{\partial}{\partial x}p(t_0, t_1;\ x, y) + \tfrac{1}{2}\sigma^2(t_0, x)\frac{\partial^2}{\partial x^2}p(t_0, t_1;\ x, y). \tag{KBE}$$

The variables t_0 and x in (KBE) are called the *backward variables*.

In the case that a and σ are functions of x alone, $p(t_0, t_1;\ x, y)$ depends on t_0 and t_1 only through their difference $\tau = t_1 - t_0$. We then write $p(\tau;\ x, y)$ rather than $p(t_0, t_1;\ x, y)$, and (KBE) becomes

$$\frac{\partial}{\partial \tau}p(\tau;\ x, y) = a(x)\frac{\partial}{\partial x}p(\tau;\ x, y) + \tfrac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}p(\tau;\ x, y). \tag{KBE'}$$

Example 16.5 (Drifted Brownian motion)

$$
\begin{aligned}
dX(t) &= a\;dt + dB(t) \\
p(\tau;\; x, y) &= \frac{1}{\sqrt{2\pi\tau}} \exp\left\{ -\frac{(y-(x+a\tau))^2}{2\tau} \right\}. \\
\frac{\partial}{\partial\tau} p = p_\tau &= \left(\frac{\partial}{\partial\tau} \frac{1}{\sqrt{2\pi\tau}} \right) \exp\left\{ -\frac{(y-x-a\tau)^2}{2\tau} \right\} \\
&\quad - \left(\frac{\partial}{\partial\tau} \frac{(y-x-a\tau)^2}{2\tau} \right) \frac{1}{\sqrt{2\pi\tau}} \exp\left\{ -\frac{(y-x-a\tau)^2}{2\tau} \right\} \\
&= \left[-\frac{1}{2\tau} + \frac{a(y-x-a\tau)}{\tau} + \frac{(y-x-a\tau)}{2\tau^2} \right] p. \\
\frac{\partial}{\partial x} p = p_x &= \frac{y-x-a\tau}{\tau} p. \\
\frac{\partial^2}{\partial x^2} p = p_{xx} &= \left(\frac{\partial}{\partial x} \frac{y-x-a\tau}{\tau} \right) p + \frac{y-x-a\tau}{\tau} p_x \\
&= -\frac{1}{\tau} p + \frac{(y-x-a\tau)^2}{\tau^2} p.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
ap_x + \tfrac{1}{2} p_{xx} &= \left[\frac{a(y-x-a\tau)}{\tau} - \frac{1}{2\tau} + \frac{(y-x-a\tau)^2}{2\tau^2} \right] p \\
&= p_\tau.
\end{aligned}
$$

This is the Kolmogorov backward equation. ■

Example 16.6 (Geometric Brownian motion)

$$dX(t) = rX(t)\;dt + \sigma X(t)\;dB(t).$$

$$p(\tau;\; x, y) = \frac{1}{\sigma y \sqrt{2\pi\tau}} \exp\left\{ -\frac{1}{2\tau\sigma^2} \left[\log\frac{y}{x} - (r - \tfrac{1}{2}\sigma^2)\tau \right]^2 \right\}.$$

It is true but very tedious to verify that p satisfies the KBE

$$p_\tau = rxp_x + \tfrac{1}{2}\sigma^2 x^2 p_{xx}.$$

■

16.5 Connection between stochastic calculus and KBE

Consider

$$dX(t) = a(X(t))\;dt + \sigma(X(t))\;dB(t). \tag{5.1}$$

Let $h(y)$ be a function, and define

$$v(t, x) = \mathbb{E}^{t,x} h(X(T)),$$

where $0 \le t \le T$. Then

$$\begin{aligned}
v(t,x) &= \int h(y)\ p(T-t;\ x,y)\ dy, \\
v_t(t,x) &= -\int h(y)\ p_\tau(T-t;\ x,y)\ dy, \\
v_x(t,x) &= \int h(y)\ p_x(T-t;\ x,y)\ dy, \\
v_{xx}(t,x) &= \int h(y)\ p_{xx}(T-t;\ x,y)\ dy.
\end{aligned}$$

Therefore, the Kolmogorov backward equation implies

$$\begin{aligned}
&v_t(t,x) + a(x)v_x(t,x) + \tfrac{1}{2}\sigma^2(x)v_{xx}(t,x) = \\
&\qquad \int h(y)\ \Big[-p_\tau(T-t;x,y) + a(x)p_x(T-t;x,y) + \tfrac{1}{2}\sigma^2(x)p_{xx}(T-t;x,y)\Big]\ dy = 0
\end{aligned}$$

Let $(0,\xi)$ be an initial condition for the SDE (5.1). We simplify notation by writing $I\!\!E$ rather than $I\!\!E^{0,\xi}$.

Theorem 5.50 *Starting at $X(0)=\xi$, the process $v(t,X(t))$ satisfies the martingale property:*

$$I\!\!E\left[v(t,X(t))\Big|\mathcal{F}(s)\right] = v(s,X(s)), \qquad 0 \le s \le t \le T.$$

Proof: According to the Markov property,

$$I\!\!E\left[h(X(T))\Big|\mathcal{F}(t)\right] = I\!\!E^{t,X(t)}h(X(T)) = v(t,X(t)),$$

so

$$\begin{aligned}
I\!\!E\left[v(t,X(t))|\mathcal{F}(s)\right] &= I\!\!E\left[\ I\!\!E\left[h(X(T))\Big|\mathcal{F}(t)\right]\ \Big|\mathcal{F}(s)\right] \\
&= I\!\!E\left[h(X(T))\Big|\mathcal{F}(s)\right] \\
&= I\!\!E^{s,X(s)}h(X(T)) \qquad \text{(Markov property)} \\
&= v(s,X(s)).
\end{aligned}$$

■

Itô's formula implies

$$\begin{aligned}
dv(t,X(t)) &= v_t dt + v_x dX + \tfrac{1}{2}v_{xx} dX\ dX \\
&= v_t dt + a v_x dt + \sigma v_x dB + \tfrac{1}{2}\sigma^2 v_{xx} dt.
\end{aligned}$$

In integral form, we have

$$\begin{aligned}
v(t, X(t)) = v(0, X(0)) \\
&+ \int_0^t \left[v_t(u, X(u)) + a(X(u)) v_x(u, X(u)) + \tfrac{1}{2}\sigma^2(X(u)) v_{xx}(u, X(u)) \right] du \\
&+ \int_0^t \sigma(X(u)) v_x(u, X(u)) \, dB(u).
\end{aligned}$$

We know that $v(t, X(t))$ is a martingale, so the integral $\int_0^t \left[v_t + a v_x + \frac{1}{2}\sigma^2 v_{xx} \right] du$ must be zero for all t. This implies that the integrand is zero; hence

$$v_t + a v_x + \tfrac{1}{2}\sigma^2 v_{xx} = 0.$$

Thus by two different arguments, one based on the Kolmogorov backward equation, and the other based on Itô's formula, we have come to the same conclusion.

Theorem 5.51 (Feynman-Kac) *Define*

$$v(t, x) = \mathbb{E}^{t,x} h(X(T)), \qquad 0 \le t \le T,$$

where

$$dX(t) = a(X(t)) \, dt + \sigma(X(t)) \, dB(t).$$

Then

$$v_t(t, x) + a(x) v_x(t, x) + \tfrac{1}{2}\sigma^2(x) v_{xx}(t, x) = 0 \tag{FK}$$

and

$$v(T, x) = h(x).$$

The Black-Scholes equation is a special case of this theorem, as we show in the next section.

Remark 16.1 (Derivation of KBE) We plunked down the Kolmogorov backward equation without any justification. In fact, one can use Itô's formula to prove the Feynman-Kac Theorem, and use the Feynman-Kac Theorem to derive the Kolmogorov backward equation.

16.6 Black-Scholes

Consider the SDE

$$dS(t) = rS(t) \, dt + \sigma S(t) \, dB(t).$$

With initial condition

$$S(t) = x,$$

the solution is

$$S(u) = x \exp\left\{ \sigma(B(u) - B(t)) + (r - \tfrac{1}{2}\sigma^2)(u - t) \right\}, \qquad u \ge t.$$

Define

$$\begin{aligned} v(t,x) &= I\!\!E^{t,x} h(S(T)) \\ &= I\!\!E h\left(x \exp\left\{\sigma(B(T)-B(t)) + (r - \tfrac{1}{2}\sigma^2)(T-t)\right\}\right), \end{aligned}$$

where h is a function to be specified later.

Recall the *Independence Lemma*: If $\mathcal{G}$ is a σ-field, X is $\mathcal{G}$-measurable, and Y is independent of $\mathcal{G}$, then

$$I\!\!E\left[h(X,Y)\middle|\mathcal{G}\right] = \gamma(X),$$

where

$$\gamma(x) = I\!\!E h(x,Y).$$

With geometric Brownian motion, for $0 \le t \le T$, we have

$$\begin{aligned} S(t) &= S(0)\exp\left\{\sigma B(t) + (r - \tfrac{1}{2}\sigma^2)t\right\}, \\ S(T) &= S(0)\exp\left\{\sigma B(T) + (r - \tfrac{1}{2}\sigma^2)T\right\} \\ &= \underbrace{S(t)}_{\mathcal{F}(t)\text{-measurable}} \quad \underbrace{\exp\left\{\sigma(B(T)-B(t)) + (r - \tfrac{1}{2}\sigma^2)(T-t)\right\}}_{\text{independent of } \mathcal{F}(t)} \end{aligned}$$

We thus have

$$S(T) = XY,$$

where

$$\begin{aligned} X &= S(t) \\ Y &= \exp\left\{\sigma(B(T)-B(t)) + (r - \tfrac{1}{2}\sigma^2)(T-t)\right\}. \end{aligned}$$

Now

$$I\!\!E h(xY) = v(t,x).$$

The independence lemma implies

$$\begin{aligned} I\!\!E\left[h(S(T))\middle|\mathcal{F}(t)\right] &= I\!\!E\left[h(XY)|\mathcal{F}(t)\right] \\ &= v(t,X) \\ &= v(t,S(t)). \end{aligned}$$

We have shown that

$$v(t, S(t)) = \mathbb{E}\left[h(S(T))\middle|\mathcal{F}(t)\right], \qquad 0 \le t \le T.$$

Note that the random variable $h(S(T))$ whose conditional expectation is being computed does not depend on t. Because of this, the tower property implies that $v(t, S(t)), 0 \le t \le T$, is a martingale: For $0 \le s \le t \le T$,

$$\begin{aligned}
\mathbb{E}\left[v(t, S(t))\middle|\mathcal{F}(s)\right] &= \mathbb{E}\left[\ \mathbb{E}\left[h(S(T))\middle|\mathcal{F}(t)\right]\ \middle|\mathcal{F}(s)\right] \\
&= \mathbb{E}\left[h(S(T))\middle|\mathcal{F}(s)\right] \\
&= v(s, S(s)).
\end{aligned}$$

This is a special case of Theorem 5.51.

Because $v(t, S(t))$ is a martingale, the sum of the dt terms in $dv(t, S(t))$ must be 0. By Itô's formula,

$$\begin{aligned}
dv(t, S(t)) = &\left[v_t(t, S(t))\ dt + rS(t)v_x(t, S(t)) + \tfrac{1}{2}\sigma^2 S^2(t) v_{xx}(t, S(t))\right]\ dt \\
&+ \sigma S(t) v_x(t, S(t))\ dB(t).
\end{aligned}$$

This leads us to the equation

$$v_t(t, x) + rxv_x(t, x) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = 0, \qquad 0 \le t < T, \quad x \ge 0.$$

This is a special case of Theorem 5.51 (Feynman-Kac).

Along with the above partial differential equation, we have the *terminal condition*

$$v(T, x) = h(x), \qquad x \ge 0.$$

Furthermore, if $S(t) = 0$ for some $t \in [0, T]$, then also $S(T) = 0$. This gives us the *boundary condition*

$$v(t, 0) = h(0), \qquad 0 \le t \le T.$$

Finally, we shall eventually see that the value at time t of a contingent claim paying $h(S(T))$ is

$$\begin{aligned}
u(t, x) &= e^{-r(T-t)} \mathbb{E}^{t,x} h(S(T)) \\
&= e^{-r(T-t)} v(t, x)
\end{aligned}$$

at time t if $S(t) = x$. Therefore,

$$\begin{aligned}
v(t, x) &= e^{r(T-t)} u(t, x), \\
v_t(t, x) &= -re^{r(T-t)} u(t, x) + e^{r(T-t)} u_t(t, x), \\
v_x(t, x) &= e^{r(T-t)} u_x(t, x), \\
v_{xx}(t, x) &= e^{r(T-t)} u_{xx}(t, x).
\end{aligned}$$

Plugging these formulas into the partial differential equation for v and cancelling the $e^{r(T-t)}$ appearing in every term, we obtain the *Black-Scholes partial differential equation*:

$$-ru(t,x) + u_t(t,x) + rxu_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 u_{xx}(t,x) = 0, \qquad 0 \le t < T, \;\; x \ge 0. \tag{BS}$$

Compare this with the earlier derivation of the Black-Scholes PDE in Section 15.6.

In terms of the transition density

$$p(t,T;\; x,y) = \frac{1}{\sigma y\sqrt{2\pi(T-t)}} \exp\left\{-\frac{1}{2(T-t)\sigma^2}\left[\log\frac{y}{x} - (r - \tfrac{1}{2}\sigma^2)(T-t)\right]^2\right\}$$

for geometric Brownian motion (See Example 16.4), we have the "stochastic representation"

$$\begin{aligned} u(t,x) &= e^{-r(T-t)} I\!E^{t,x} h(S(T)) \\ &= e^{-r(T-t)} \int_0^\infty h(y)p(t,T;\; x,y)\, dy. \end{aligned} \tag{SR}$$

In the case of a call,

$$h(y) = (y-K)^+$$

and

$$\begin{aligned} u(t,x) = x\; N\left(\frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + r(T-t) + \tfrac{1}{2}\sigma^2(T-t)\right]\right) \\ - e^{-r(T-t)} K\; N\left(\frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + r(T-t) - \tfrac{1}{2}\sigma^2(T-t)\right]\right) \end{aligned}$$

Even if $h(y)$ is some other function (e.g., $h(y) = (K-y)^+$, a put), $u(t,x)$ is still given by and satisfies the Black-Scholes PDE (BS) derived above.

16.7 Black-Scholes with price-dependent volatility

$$\begin{aligned} dS(t) &= rS(t)\; dt + \beta(S(t))\; dB(t), \\ v(t,x) &= e^{-r(T-t)} I\!E^{t,x}(S(T)-K)^+. \end{aligned}$$

The Feynman-Kac Theorem now implies that

$$-rv(t,x) + v_t(t,x) + rxv_x(t,x) + \tfrac{1}{2}\beta^2(x)v_{xx}(t,x) = 0, \qquad 0 \le t < T, \;\; x > 0.$$

v also satisfies the *terminal condition*

$$v(T,x) = (x-K)^+, \qquad x \ge 0,$$

and the *boundary condition*

$$v(t,0) = 0, \qquad 0 \le t \le T.$$

An example of such a process is the following from J.C. Cox, *Notes on options pricing I: Constant elasticity of variance diffusions,* Working Paper, Stanford University, 1975:

$$dS(t) = rS(t)\ dt + \sigma S^{\delta}(t)\ dB(t),$$

where $0 \le \delta < 1$. The "volatility" $\sigma S^{\delta-1}(t)$ decreases with increasing stock price. The corresponding Black-Scholes equation is

$$\begin{aligned} -rv + v_t + rxv_x + \tfrac{1}{2}\sigma^2 x^{2\delta} v_{xx} &= 0, \qquad 0 \le t < T \quad x > 0; \\ v(t,0) &= 0, \qquad 0 \le t \le T \\ v(T,x) &= (x-K)^+, \qquad x \ge 0. \end{aligned}$$