Chapter 15

Itô's Formula

15.1 Itô's formula for one Brownian motion

We want a rule to "differentiate" expressions of the form $f(B(t))$, where $f(x)$ is a differentiable function. If $B(t)$ were also differentiable, then the ordinary *chain rule* would give

$$\frac{d}{dt} f(B(t)) = f'(B(t))B'(t),$$

which could be written in differential notation as

$$\begin{aligned} df(B(t)) &= f'(B(t))B'(t)\ dt \\ &= f'(B(t))dB(t) \end{aligned}$$

However, $B(t)$ is not differentiable, and in particular has nonzero quadratic variation, so the correct formula has an extra term, namely,

$$df(B(t)) = f'(B(t))\ dB(t) + \tfrac{1}{2} f''(B(t)) \underbrace{dt}_{dB(t)\ dB(t)}.$$

This is *Itô's formula in differential form.* Integrating this, we obtain *Itô's formula in integral form:*

$$f(B(t)) - \underbrace{f(B(0))}_{f(0)} = \int_0^t f'(B(u))\ dB(u) + \tfrac{1}{2} \int_0^t f''(B(u))\ du.$$

Remark 15.1 (Differential vs. Integral Forms) The mathematically meaningful form of Itô's formula is Itô's formula in integral form:

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(u))\ dB(u) + \tfrac{1}{2} \int_0^t f''(B(u))\ du.$$

This is because we have solid definitions for both integrals appearing on the right-hand side. The first,

$$\int_0^t f'(B(u))\ dB(u)$$

is an *Itô integral*, defined in the previous chapter. The second,

$$\int_0^t f''(B(u))\ du,$$

is a *Riemann integral*, the type used in freshman calculus.

For paper and pencil computations, the more convenient form of Itô's rule is *Itô's formula in differential form:*

$$df(B(t)) = f'(B(t))\ dB(t) + \tfrac{1}{2} f''(B(t))\ dt.$$

There is an intuitive meaning but no solid definition for the terms $df(B(t))$, $dB(t)$ and dt appearing in this formula. This formula becomes mathematically respectable only after we integrate it.

15.2 Derivation of Itô's formula

Consider $f(x) = \frac{1}{2}x^2$, so that

$$f'(x) = x, \quad f''(x) = 1.$$

Let x_k, x_{k+1} be numbers. Taylor's formula implies

$$f(x_{k+1}) - f(x_k) = (x_{k+1} - x_k) f'(x_k) + \tfrac{1}{2}(x_{k+1} - x_k)^2 f''(x_k).$$

In this case, Taylor's formula to second order is *exact* because f is a *quadratic function*.

In the general case, the above equation is only approximate, and the error is of the order of $(x_{k+1} - x_k)^3$. The total error will have limit zero in the last step of the following argument.

Fix $T > 0$ and let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$. Using Taylor's formula, we write:

$$\begin{aligned}
&f(B(T)) - f(B(0)) \\
&= \tfrac{1}{2}B^2(T) - \tfrac{1}{2}B^2(0) \\
&= \sum_{k=0}^{n-1} \left[f(B(t_{k+1})) - f(B(t_k)) \right] \\
&= \sum_{k=0}^{n-1} \left[B(t_{k+1}) - B(t_k) \right] f'(B(t_k)) + \tfrac{1}{2} \sum_{k=0}^{n-1} \left[B(t_{k+1}) - B(t_k) \right]^2 f''(B(t_k)) \\
&= \sum_{k=0}^{n-1} B(t_k) \left[B(t_{k+1}) - B(t_k) \right] + \tfrac{1}{2} \sum_{k=0}^{n-1} \left[B(t_{k+1}) - B(t_k) \right]^2 .
\end{aligned}$$

We let $||\Pi||\to 0$ to obtain

$$f(B(T)) - f(B(0)) = \int_0^T B(u)\ dB(u) + \tfrac{1}{2}\underbrace{\langle B\rangle(T)}_{T}$$
$$= \int_0^T f'(B(u))\ dB(u) + \tfrac{1}{2}\int_0^T \underbrace{f''(B(u))}_{1}\, du.$$

This is Itô's formula in integral form for the special case

$$f(x) = \tfrac{1}{2}x^2.$$

15.3 Geometric Brownian motion

Definition 15.1 (Geometric Brownian Motion) Geometric Brownian motion is

$$S(t) = S(0)\exp\left\{\sigma B(t) + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right\},$$

where μ and $\sigma > 0$ are constant.

Define

$$f(t,x) = S(0)\exp\left\{\sigma x + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right\},$$

so

$$S(t) = f(t, B(t)).$$

Then

$$f_t = \left(\mu - \tfrac{1}{2}\sigma^2\right)f,\ \ f_x = \sigma f,\ \ f_{xx} = \sigma^2 f.$$

According to Itô's formula,

$$\begin{aligned} dS(t) &= df(t, B(t)) \\ &= f_t dt + f_x dB + \tfrac{1}{2} f_{xx}\underbrace{dBdB}_{dt} \\ &= (\mu - \tfrac{1}{2}\sigma^2)f\ dt + \sigma f\ dB + \tfrac{1}{2}\sigma^2 f\ dt \\ &= \mu S(t)dt + \sigma S(t)\ dB(t) \end{aligned}$$

Thus, *Geometric Brownian motion in differential form* is

$$dS(t) = \mu S(t)dt + \sigma S(t)\ dB(t),$$

and *Geometric Brownian motion in integral form* is

$$S(t) = S(0) + \int_0^t \mu S(u)\ du + \int_0^t \sigma S(u)\ dB(u).$$

15.4 Quadratic variation of geometric Brownian motion

In the integral form of Geometric Brownian motion,

$$S(t) = S(0) + \int_0^t \mu S(u)\, du + \int_0^t \sigma S(u)\, dB(u),$$

the Riemann integral

$$F(t) = \int_0^t \mu S(u)\, du$$

is differentiable with $F'(t) = \mu S(t)$. This term has zero quadratic variation. The Itô integral

$$G(t) = \int_0^t \sigma S(u)\, dB(u)$$

is not differentiable. It has quadratic variation

$$\langle G \rangle(t) = \int_0^t \sigma^2 S^2(u)\, du.$$

Thus the quadratic variation of S is given by the quadratic variation of G. In differential notation, we write

$$dS(t)\, dS(t) = (\mu S(t)dt + \sigma S(t)dB(t))^2 = \sigma^2 S^2(t)\, dt$$

15.5 Volatility of Geometric Brownian motion

Fix $0 \le T_1 \le T_2$. Let $\Pi = \{t_0, \dots, t_n\}$ be a partition of $[T_1, T_2]$. The *squared absolute sample volatility* of S on $[T_1, T_2]$ is

$$\begin{aligned} \frac{1}{T_2 - T_1} \sum_{k=0}^{n-1} [S(t_{k+1}) - S(t_k)]^2 &\simeq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \sigma^2 S^2(u)\, du \\ &\simeq \sigma^2 S^2(T_1) \end{aligned}$$

As $T_2 \downarrow T_1$, the above approximation becomes exact. In other words, the *instantaneous relative volatility* of S is σ^2. This is usually called simply the *volatility* of S.

15.6 First derivation of the Black-Scholes formula

Wealth of an investor. An investor begins with nonrandom initial wealth X_0 and at each time t, holds $\Delta(t)$ shares of stock. Stock is modelled by a geometric Brownian motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t).$$

$\Delta(t)$ can be random, but must be adapted. The investor finances his investing by borrowing or lending at interest rate r.

Let $X(t)$ denote the wealth of the investor at time t. Then

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r\left[X(t) - \Delta(t)S(t)\right]dt \\ &= \Delta(t)\left[\mu S(t)dt + \sigma S(t)dB(t)\right] + r\left[X(t) - \Delta(t)S(t)\right]dt \\ &= rX(t)dt \ + \ \Delta(t)S(t) \underbrace{(\mu - r)}_{\text{Risk premium}} \ dt \ + \ \Delta(t)S(t)\sigma dB(t). \end{aligned}$$

Value of an option. Consider an European option which pays $g(S(T))$ at time T. Let $v(t,x)$ denote the value of this option at time t if the stock price is $S(t) = x$. In other words, the value of the option at each time $t \in [0,T]$ is

$$v(t, S(t)).$$

The differential of this value is

$$\begin{aligned} dv(t, S(t)) &= v_t dt + v_x dS + \tfrac{1}{2} v_{xx} dS\ dS \\ &= v_t dt + v_x\left[\mu S\ dt + \sigma S\ dB\right] + \tfrac{1}{2} v_{xx}\sigma^2 S^2\ dt \\ &= \left[v_t + \mu S v_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx}\right]\ dt + \sigma S v_x dB \end{aligned}$$

A hedging portfolio starts with some initial wealth X_0 and invests so that the wealth $X(t)$ at each time tracks $v(t, S(t))$. We saw above that

$$dX(t) = \left[rX + \Delta(\mu - r)S\right]\ dt + \sigma S \Delta dB.$$

To ensure that $X(t) = v(t, S(t))$ for all t, we equate coefficients in their differentials. Equating the dB coefficients, we obtain the Δ-*hedging rule*:

$$\Delta(t) = v_x(t, S(t)).$$

Equating the dt coefficients, we obtain:

$$v_t + \mu S v_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx} = rX + \Delta(\mu - r)S.$$

But we have set $\Delta = v_x$, and we are seeking to cause X to agree with v. Making these substitutions, we obtain

$$v_t + \mu S v_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx} = rv + v_x(\mu - r)S,$$

(where $v = v(t, S(t))$ and $S = S(t)$) which simplifies to

$$v_t + rSv_x + \tfrac{1}{2}\sigma^2 S^2 v_{xx} = rv.$$

In conclusion, we should let v be the solution to the *Black-Scholes partial differential equation*

$$v_t(t,x) + rxv_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = rv(t,x)$$

satisfying the terminal condition

$$v(T,x) = g(x).$$

If an investor starts with $X_0 = v(0, S(0))$ and uses the hedge $\Delta(t) = v_x(t, S(t))$, then he will have $X(t) = v(t, S(t))$ for all t, and in particular, $X(T) = g(S(T))$.

15.7 Mean and variance of the Cox-Ingersoll-Ross process

The *Cox-Ingersoll-Ross* model for interest rates is

$$dr(t) = a(b - cr(t))dt + \sigma\sqrt{r(t)}\, dB(t),$$

where a, b, c, σ and $r(0)$ are positive constants. In integral form, this equation is

$$r(t) = r(0) + a\int_0^t (b - cr(u))\, du + \sigma\int_0^t \sqrt{r(u)}\, dB(u).$$

We apply Itô's formula to compute $dr^2(t)$. This is $df(r(t))$, where $f(x) = x^2$. We obtain

$$\begin{aligned}
dr^2(t) &= df(r(t)) \\
&= f'(r(t))\, dr(t) + \tfrac{1}{2}f''(r(t))\, dr(t)\, dr(t) \\
&= 2r(t)\left[a(b - cr(t))\, dt + \sigma\sqrt{r(t)}\, dB(t)\right] + \left[a(b - cr(t))\, dt + \sigma\sqrt{r(t)}\, dB(t)\right]^2 \\
&= 2abr(t)\, dt - 2acr^2(t)\, dt + 2\sigma r^{\frac{3}{2}}(t)\, dB(t) + \sigma^2 r(t)\, dt \\
&= (2ab + \sigma^2)r(t)\, dt - 2acr^2(t)\, dt + 2\sigma r^{\frac{3}{2}}(t)\, dB(t)
\end{aligned}$$

The mean of $r(t)$. The integral form of the CIR equation is

$$r(t) = r(0) + a\int_0^t (b - cr(u))\, du + \sigma\int_0^t \sqrt{r(u)}\, dB(u).$$

Taking expectations and remembering that the expectation of an Itô integral is zero, we obtain

$$\mathbb{E}r(t) = r(0) + a\int_0^t (b - c\mathbb{E}r(u))\, du.$$

Differentiation yields

$$\frac{d}{dt}\mathbb{E}r(t) = a(b - c\mathbb{E}r(t)) = ab - ac\mathbb{E}r(t),$$

which implies that

$$\frac{d}{dt}\left[e^{act}\mathbb{E}r(t)\right] = e^{act}\left[ac\mathbb{E}r(t) + \frac{d}{dt}\mathbb{E}r(t)\right] = e^{act}ab.$$

Integration yields

$$e^{act}\mathbb{E}r(t) - r(0) = ab\int_0^t e^{acu}\, du = \frac{b}{c}(e^{act} - 1).$$

We solve for $\mathbb{E}r(t)$:

$$\mathbb{E}r(t) = \frac{b}{c} + e^{-act}\left(r(0) - \frac{b}{c}\right).$$

If $r(0) = \frac{b}{c}$, then $\mathbb{E}r(t) = \frac{b}{c}$ for every t. If $r(0) \neq \frac{b}{c}$, then $r(t)$ exhibits *mean reversion*:

$$\lim_{t\to\infty} \mathbb{E}r(t) = \frac{b}{c}.$$

Variance of $r(t)$. The integral form of the equation derived earlier for $dr^2(t)$ is

$$r^2(t) = r^2(0) + (2ab + \sigma^2)\int_0^t r(u)\,du - 2ac\int_0^t r^2(u)\,du + 2\sigma\int_0^t r^{\frac{3}{2}}(u)\,dB(u).$$

Taking expectations, we obtain

$$\mathbb{E}r^2(t) = r^2(0) + (2ab + \sigma^2)\int_0^t \mathbb{E}r(u)\,du - 2ac\int_0^t \mathbb{E}r^2(u)\,du.$$

Differentiation yields

$$\frac{d}{dt}\mathbb{E}r^2(t) = (2ab + \sigma^2)\mathbb{E}r(t) - 2ac\mathbb{E}r^2(t),$$

which implies that

$$\begin{aligned}\frac{d}{dt}e^{2act}\mathbb{E}r^2(t) &= e^{2act}\left[2ac\mathbb{E}r^2(t) + \frac{d}{dt}\mathbb{E}r^2(t)\right]\\ &= e^{2act}(2ab + \sigma^2)\mathbb{E}r(t).\end{aligned}$$

Using the formula already derived for $\mathbb{E}r(t)$ and integrating the last equation, after considerable algebra we obtain

$$\begin{aligned}\mathbb{E}r^2(t) &= \frac{b\sigma^2}{2ac^2} + \frac{b^2}{c^2} + \left(r(0) - \frac{b}{c}\right)\left(\frac{\sigma^2}{ac} + \frac{2b}{c}\right)e^{-act}\\ &\qquad + \left(r(0) - \frac{b}{c}\right)^2\frac{\sigma^2}{ac}e^{-2act} + \frac{\sigma^2}{ac}\left(\frac{b}{2c} - r(0)\right)e^{-2act}.\\ \operatorname{var} r(t) &= \mathbb{E}r^2(t) - (\mathbb{E}r(t))^2\\ &= \frac{b\sigma^2}{2ac^2} + \left(r(0) - \frac{b}{c}\right)\frac{\sigma^2}{ac}e^{-act} + \frac{\sigma^2}{ac}\left(\frac{b}{2c} - r(0)\right)e^{-2act}.\end{aligned}$$

15.8 Multidimensional Brownian Motion

Definition 15.2 (d-dimensional Brownian Motion) A *d-dimensional Brownian Motion* is a process

$$B(t) = (B_1(t), \ldots, B_d(t))$$

with the following properties:

- Each $B_k(t)$ is a one-dimensional Brownian motion;
- If $i \neq j$, then the processes $B_i(t)$ and $B_j(t)$ are independent.

Associated with a d-dimensional Brownian motion, we have a filtration $\{\mathcal{F}(t)\}$ such that

- For each t, the random vector $B(t)$ is $\mathcal{F}(t)$-measurable;
- For each $t \leq t_1 \leq \ldots \leq t_n$, the vector increments

  $$B(t_1) - B(t), \ldots, B(t_n) - B(t_{n-1})$$

  are independent of $\mathcal{F}(t)$.

15.9 Cross-variations of Brownian motions

Because each component B_i is a one-dimensional Brownian motion, we have the informal equation

$$dB_i(t)\ dB_i(t) = dt.$$

However, we have:

Theorem 9.49 *If* $i \neq j$,

$$dB_i(t)\ dB_j(t) = 0$$

Proof: Let $\Pi = \{t_0, \dots, t_n\}$ be a partition of $[0, T]$. For $i \neq j$, define the *sample cross variation* of B_i and B_j on $[0, T]$ to be

$$C_\Pi = \sum_{k=0}^{n-1} [B_i(t_{k+1}) - B_i(t_k)]\, [B_j(t_{k+1}) - B_j(t_k)]\, .$$

The increments appearing on the right-hand side of the above equation are all independent of one another and all have mean zero. Therefore,

$$I\!\!E C_\Pi = 0.$$

We compute $\text{var}(C_\Pi)$. First note that

$$\begin{aligned} C_\Pi^2 = \sum_{k=0}^{n-1} \Big[B_i(t_{k+1}) - B_i(t_k)\Big]^2 \Big[B_j(t_{k+1}) - B_j(t_k)\Big]^2 \\ + 2 \sum_{\ell<k}^{n-1} [B_i(t_{\ell+1}) - B_i(t_\ell)]\, [B_j(t_{\ell+1}) - B_j(t_\ell)]\, . [B_i(t_{k+1}) - B_i(t_k)]\, [B_j(t_{k+1}) - B_j(t_k)] \end{aligned}$$

All the increments appearing in the sum of cross terms are independent of one another and have mean zero. Therefore,

$$\begin{aligned} \text{var}(C_\Pi) &= I\!\!E C_\Pi^2 \\ &= I\!\!E \sum_{k=0}^{n-1} [B_i(t_{k+1}) - B_i(t_k)]^2\, [B_j(t_{k+1}) - B_j(t_k)]^2\, . \end{aligned}$$

But $[B_i(t_{k+1}) - B_i(t_k)]^2$ and $[B_j(t_{k+1}) - B_j(t_k)]^2$ are independent of one another, and each has expectation $(t_{k+1} - t_k)$. It follows that

$$\text{var}(C_\Pi) = \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \leq ||\Pi|| \sum_{k=0}^{n-1} (t_{k+1} - t_k) = ||\Pi||.T.$$

As $||\Pi|| \to 0$, we have $\text{var}(C_\Pi) \to 0$, so C_Π converges to the constant $I\!\!E C_\Pi = 0$. ∎

15.10 Multi-dimensional Itô formula

To keep the notation as simple as possible, we write the Itô formula for *two* processes driven by a *two*-dimensional Brownian motion. The formula generalizes to *any number* of processes driven by a Brownian motion of *any number* (not necessarily the same number) of dimensions.

Let X and Y be processes of the form

$$X(t) = X(0) + \int_0^t \alpha(u)\ du + \int_0^t \delta_{11}(u)\ dB_1(u) + \int_0^t \delta_{12}(u)\ dB_2(u),$$
$$Y(t) = Y(0) + \int_0^t \beta(u)\ du + \int_0^t \delta_{21}(u)\ dB_1(u) + \int_0^t \delta_{22}(u)\ dB_2(u).$$

Such processes, consisting of a nonrandom initial condition, plus a Riemann integral, plus one or more Itô integrals, are called *semimartingales*. The integrands $\alpha(u), \beta(u)$, and $\delta_{ij}(u)$ can be any adapted processes. The adaptedness of the integrands guarantees that X and Y are also adapted. In differential notation, we write

$$dX = \alpha\ dt + \delta_{11}\ dB_1 + \delta_{12}\ dB_2,$$
$$dY = \beta\ dt + \delta_{21}\ dB_1 + \delta_{22}\ dB_2.$$

Given these two semimartingales X and Y, the quadratic and cross variations are:

$$\begin{aligned}
dX\ dX &= (\alpha\ dt + \delta_{11}\ dB_1 + \delta_{12}\ dB_2)^2, \\
&= \delta_{11}^2 \underbrace{dB_1\ dB_1}_{dt} + 2\delta_{11}\delta_{12} \underbrace{dB_1\ dB_2}_{0} + \delta_{12}^2 \underbrace{dB_2\ dB_2}_{dt} \\
&= (\delta_{11}^2 + \delta_{12}^2)^2\ dt, \\
dY\ dY &= (\beta\ dt + \delta_{21}\ dB_1 + \delta_{22}\ dB_2)^2 \\
&= (\delta_{21}^2 + \delta_{22}^2)^2\ dt, \\
dX\ dY &= (\alpha\ dt + \delta_{11}\ dB_1 + \delta_{12}\ dB_2)(\beta\ dt + \delta_{21}\ dB_1 + \delta_{22}\ dB_2) \\
&= (\delta_{11}\delta_{21} + \delta_{12}\delta_{22})\ dt
\end{aligned}$$

Let $f(t, x, y)$ be a function of three variables, and let $X(t)$ and $Y(t)$ be semimartingales. Then we have the corresponding Itô formula:

$$df(t, x, y) \ = \ f_t\, dt \ + \ f_x\, dX \ + \ f_y\, dY \ + \ \tfrac{1}{2}\left[f_{xx}\ dX\ dX + 2f_{xy}\ dX\ dY + f_{yy}\ dY\ dY\right].$$

In integral form, with X and Y as decribed earlier and with all the variables filled in, this equation is

$$\begin{aligned}
&f(t, X(t), Y(t)) - f(0, X(0), Y(0)) \\
&= \int_0^t [f_t + \alpha f_x + \beta f_y + \tfrac{1}{2}(\delta_{11}^2 + \delta_{12}^2) f_{xx} + (\delta_{11}\delta_{21} + \delta_{12}\delta_{22}) f_{xy} + \tfrac{1}{2}(\delta_{21}^2 + \delta_{22}^2) f_{yy}]\ du \\
&\quad + \int_0^t [\delta_{11} f_x + \delta_{21} f_y]\ dB_1 \ + \ \int_0^t [\delta_{12} f_x + \delta_{22} f_y]\ dB_2,
\end{aligned}$$

where $f = f(u, X(u), Y(u))$, for $i, j \in \{1, 2\}$, $\delta_{ij} = \delta_{ij}(u)$, and $B_i = B_i(u)$.