Chapter 14

The Itô Integral

The following chapters deal with *Stochastic Differential Equations in Finance*. References:

1. B. Oksendal, *Stochastic Differential Equations*, Springer-Verlag,1995
2. J. Hull, *Options, Futures and other Derivative Securities,* Prentice Hall, 1993.

14.1 Brownian Motion

(See Fig. 13.3.) $(\Omega, \mathcal{F}, \mathbb{P})$ is given, always in the background, even when not explicitly mentioned. **Brownian motion**, $B(t, \omega) : [0, \infty) \times \Omega \rightarrow I\!R$, has the following properties:

1. $B(0) = 0$; Technically, $I\!P\{\omega; B(0, \omega) = 0\} = 1$,
2. $B(t)$ is a continuous function of t,
3. If $0 = t_0 \leq t_1 \leq \ldots \leq t_n$, then the increments

$$B(t_1) - B(t_0),\ \ldots,\ B(t_n) - B(t_{n-1})$$

   are *independent,normal,* and

$$I\!E[B(t_{k+1}) - B(t_k)] = 0,$$
$$I\!E[B(t_{k+1}) - B(t_k)]^2 = t_{k+1} - t_k.$$

14.2 First Variation

Quadratic variation is a measure of volatility. First we will consider *first variation,* $FV(f)$, of a function $f(t)$.

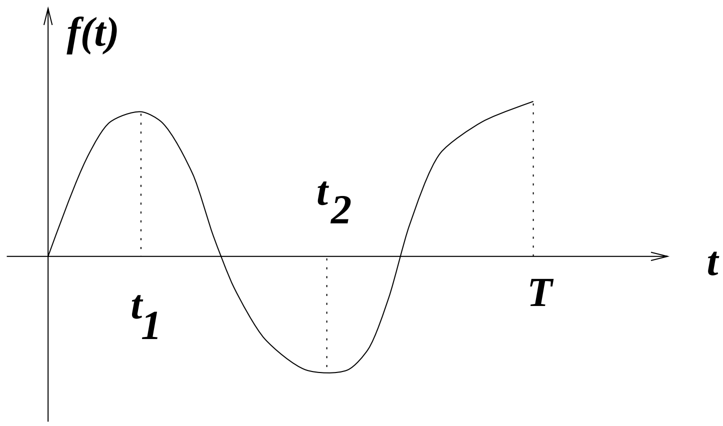

Figure 14.1: *Example function* $f(t)$.

For the function pictured in Fig. 14.1, the first variation over the interval $[0,T]$ is given by:

$$\begin{aligned} FV_{[0,T]}(f) &= [f(t_1)-f(0)] - [f(t_2)-f(t_1)] + [f(T)-f(t_2)] \\ &= \int_0^{t_1} f'(t)\,dt + \int_{t_1}^{t_2} (-f'(t))\,dt + \int_{t_2}^{T} f'(t)\,dt. \\ &= \int_0^T |f'(t)|\,dt. \end{aligned}$$

Thus, first variation measures the total amount of up and down motion of the path.

The general definition of first variation is as follows:

Definition 14.1 (First Variation) Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a *partition* of $[0,T]$, i.e.,

$$0 = t_0 \le t_1 \le \ldots \le t_n = T.$$

The *mesh* of the partition is defined to be

$$||\Pi|| = \max_{k=0,\ldots,n-1} (t_{k+1} - t_k).$$

We then define

$$FV_{[0,T]}(f) = \lim_{||\Pi||\to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|.$$

Suppose f is differentiable. Then the Mean Value Theorem implies that in each subinterval $[t_k, t_{k+1}]$, there is a point t_k^* such that

$$f(t_{k+1}) - f(t_k) = f'(t_k^*)(t_{k+1} - t_k).$$

Then

$$\sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)| = \sum_{k=0}^{n-1} |f'(t_k^*)|(t_{k+1} - t_k),$$

and

$$\begin{aligned} FV_{[0,T]}(f) &= \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|(t_{k+1} - t_k) \\ &= \int_0^T |f'(t)|\ dt. \end{aligned}$$

14.3 Quadratic Variation

Definition 14.2 (Quadratic Variation) The *quadratic variation* of a function f on an interval $[0, T]$ is

$$\langle f \rangle (T) = \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^2.$$

Remark 14.1 (Quadratic Variation of Differentiable Functions) If f is differentiable, then $\langle f \rangle(T) = 0$, because

$$\begin{aligned} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^2 &= \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k)^2 \\ &\leq ||\Pi||. \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k) \end{aligned}$$

and

$$\begin{aligned} \langle f \rangle(T) &\leq \lim_{||\Pi|| \to 0} ||\Pi||. \lim_{||\Pi|| \to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|^2 (t_{k+1} - t_k) \\ &= \lim_{||\Pi|| \to 0} ||\Pi|| \int_0^T |f'(t)|^2\ dt \\ &= 0. \end{aligned}$$

Theorem 3.44

$$\langle B \rangle(T) = T,$$

or more precisely,

$$\mathbb{P}\{\omega \in \Omega; \langle B(.,\omega)\rangle(T) = T\} = 1.$$

In particular, the paths of Brownian motion are not differentiable.

Proof: (Outline) Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$. To simplify notation, set $D_k = B(t_{k+1}) - B(t_k)$. Define the *sample quadratic variation*

$$Q_\Pi = \sum_{k=0}^{n-1} D_k^2.$$

Then

$$Q_\Pi - T = \sum_{k=0}^{n-1} [D_k^2 - (t_{k+1} - t_k)].$$

We want to show that

$$\lim_{||\Pi|| \to 0} (Q_\Pi - T) = 0.$$

Consider an individual summand

$$D_k^2 - (t_{k+1} - t_k) = [B(t_{k+1}) - B(t_k)]^2 - (t_{k+1} - t_k).$$

This has expectation 0, so

$$I\!\!E(Q_\Pi - T) = I\!\!E \sum_{k=0}^{n-1} [D_k^2 - (t_{k+1} - t_k)] = 0.$$

For $j \neq k$, the terms

$$D_j^2 - (t_{j+1} - t_j) \quad \text{and} \quad D_k^2 - (t_{k+1} - t_k)$$

are independent, so

$$\begin{aligned}
\operatorname{var}(Q_\Pi - T) &= \sum_{k=0}^{n-1} \operatorname{var}[D_k^2 - (t_{k+1} - t_k)] \\
&= \sum_{k=0}^{n-1} I\!\!E[D_k^4 - 2(t_{k+1} - t_k) D_k^2 + (t_{k+1} - t_k)^2] \\
&= \sum_{k=0}^{n-1} [3(t_{k+1} - t_k)^2 - 2(t_{k+1} - t_k)^2 + (t_{k+1} - t_k)^2] \\
&\qquad \text{(if } X \text{ is normal with mean 0 and variance } \sigma^2\text{, then } I\!\!E(X^4) = 3\sigma^4\text{)} \\
&= 2 \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \\
&\leq 2||\Pi|| \sum_{k=0}^{n-1} (t_{k+1} - t_k) \\
&= 2||\Pi||\, T.
\end{aligned}$$

Thus we have

$$\begin{aligned}
I\!\!E(Q_\Pi - T) &= 0, \\
\operatorname{var}(Q_\Pi - T) &\leq 2||\Pi||.T.
\end{aligned}$$

As $||\Pi||\to 0$, $\mathrm{var}(Q_\Pi - T)\to 0$, so

$$\lim_{||\Pi||\to 0}(Q_\Pi - T) = 0.$$

■

Remark 14.2 (Differential Representation) We know that

$$I\!\!E[(B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)] = 0.$$

We showed above that

$$\mathrm{var}[(B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)] = 2(t_{k+1} - t_k)^2.$$

When $(t_{k+1} - t_k)$ is small, $(t_{k+1} - t_k)^2$ is *very* small, and we have the approximate equation

$$(B(t_{k+1}) - B(t_k))^2 \simeq t_{k+1} - t_k,$$

which we can write informally as

$$dB(t)\ dB(t) = dt.$$

14.4 Quadratic Variation as Absolute Volatility

On any time interval $[T_1, T_2]$, we can sample the Brownian motion at times

$$T_1 = t_0 \le t_1 \le \ldots \le t_n = T_2$$

and compute the *squared sample absolute volatility*

$$\frac{1}{T_2 - T_1}\sum_{k=0}^{n-1}(B(t_{k+1}) - B(t_k))^2.$$

This is approximately equal to

$$\frac{1}{T_2 - T_1}[\langle B\rangle(T_2) - \langle B\rangle(T_1)] = \frac{T_2 - T_1}{T_2 - T_1} = 1.$$

As we increase the number of sample points, this approximation becomes exact. In other words, Brownian motion has *absolute volatility 1*.

Furthermore, consider the equation

$$\langle B\rangle(T) = T = \int_0^T 1\ dt, \qquad \forall T \ge 0.$$

This says that quadratic variation for Brownian motion accumulates at rate 1 *at all times along almost every path*.

14.5 Construction of the Itô Integral

The **integrator** is Brownian motion $B(t), t \geq 0$, with associated filtration $\mathcal{F}(t), t \geq 0$, and the following properties:

1. $s \leq t \Longrightarrow$ every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$,
2. $B(t)$ is $\mathcal{F}(t)$-measurable, $\forall t$,
3. For $t \leq t_1 \leq \ldots \leq t_n$, the increments $B(t_1) - B(t), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent of $\mathcal{F}(t)$.

The **integrand** is $\delta(t), t \geq 0$, where

1. $\delta(t)$ is $\mathcal{F}(t)$-measurable $\forall t$ (i.e., δ is adapted)
2. δ is square-integrable:

$$\mathbb{E} \int_0^T \delta^2(t)\, dt < \infty, \qquad \forall T.$$

We want to define the **Itô Integral:**

$$I(t) = \int_0^t \delta(u)\, dB(u), \qquad t \geq 0.$$

Remark 14.3 (Integral w.r.t. a differentiable function) If $f(t)$ is a differentiable function, then we can define

$$\int_0^t \delta(u)\, df(u) = \int_0^t \delta(u) f'(u)\, du.$$

This won't work when the integrator is Brownian motion, because the paths of Brownian motion are not differentiable.

14.6 Itô integral of an elementary integrand

Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$, i.e.,

$$0 = t_0 \leq t_1 \leq \ldots \leq t_n = T.$$

Assume that $\delta(t)$ is constant on each subinterval $[t_k, t_{k+1}]$ (see Fig. 14.2). We call such a δ an *elementary process*.

The functions $B(t)$ and $\delta(t_k)$ can be interpreted as follows:

- Think of $B(t)$ as the *price per unit share* of an asset at time t.

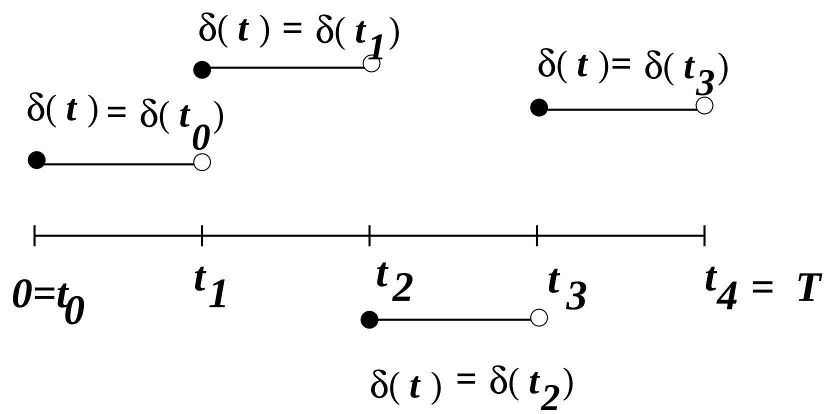

Figure 14.2: *An elementary function δ.*

- Think of $t_0, t_1, \ldots, t_n$ as the *trading dates* for the asset.
- Think of $\delta(t_k)$ as the *number of shares of the asset acquired* at trading date t_k and held until trading date t_{k+1}.

Then the Itô integral $I(t)$ can be interpreted as the *gain from trading* at time t; this gain is given by:

$$I(t) = \begin{cases} \delta(t_0)[B(t) - \underbrace{B(t_0)}_{=B(0)=0}], & 0 \le t \le t_1 \\ \delta(t_0)[B(t_1) - B(t_0)] + \delta(t_1)[B(t) - B(t_1)], & t_1 \le t \le t_2 \\ \delta(t_0)[B(t_1) - B(t_0)] + \delta(t_1)[B(t_2) - B(t_1)] + \delta(t_2)[B(t) - B(t_2)], & t_2 \le t \le t_3. \end{cases}$$

In general, if $t_k \le t \le t_{k+1}$,

$$I(t) = \sum_{j=0}^{k-1} \delta(t_j)[B(t_{j+1}) - B(t_j)] + \delta(t_k)[B(t) - B(t_k)].$$

14.7 Properties of the Itô integral of an elementary process

Adaptedness For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.

Linearity If

$$I(t) = \int_0^t \delta(u)\, dB(u), \qquad J(t) = \int_0^t \gamma(u)\, dB(u)$$

then

$$I(t) \pm J(t) = \int_0^t (\delta(u) \pm \gamma(u))\, dB(u)$$

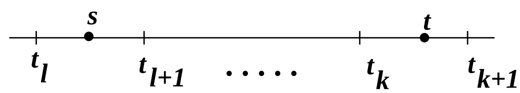

Figure 14.3: *Showing s and t in different partitions.*

and

$$cI(t) = \int_0^t c\delta(u)dB(u).$$

Martingale $I(t)$ is a martingale.

We prove the martingale property for the elementary process case.

Theorem 7.45 (Martingale Property)

$$I(t) = \sum_{j=0}^{k-1} \delta(t_j)[B(t_{j+1}) - B(t_j)] \ + \ \delta(t_k)[B(t) - B(t_k)], \qquad t_k \le t \le t_{k+1}$$

is a martingale.

Proof: Let $0 \le s \le t$ be given. We treat the more difficult case that s and t are in different subintervals, i.e., there are partition points t_ℓ and t_k such that $s \in [t_\ell, t_{\ell+1}]$ and $t \in [t_k, t_{k+1}]$ (See Fig. 14.3).

Write

$$\begin{aligned} I(t) = \sum_{j=0}^{\ell-1} \delta(t_j)[B(t_{j+1}) - B(t_j)] \ + \ \delta(t_\ell)[B(t_{\ell+1}) - B(t_\ell)] \\ + \sum_{j=\ell+1}^{k-1} \delta(t_j)[B(t_{j+1}) - B(t_j)] \ + \ \delta(t_k)[B(t) - B(t_k)] \end{aligned}$$

We compute conditional expectations:

$$\begin{aligned} I\!E\left[\sum_{j=0}^{\ell-1} \delta(t_j)(B(t_{j+1}) - B(t_j)) \middle| \mathcal{F}(s)\right] &= \sum_{j=0}^{\ell-1} \delta(t_j)(B(t_{j+1}) - B(t_j)). \\ I\!E\left[\delta(t_\ell)(B(t_{\ell+1}) - B(t_\ell)) \middle| \mathcal{F}(s)\right] &= \delta(t_\ell)\left(I\!E[B(t_{\ell+1})|\mathcal{F}(s)] - B(t_\ell)\right) \\ &= \delta(t_\ell)[B(s) - B(t_\ell)] \end{aligned}$$

These first two terms add up to $I(s)$. We show that the third and fourth terms are zero.

$$\mathbb{E}\left[\sum_{j=\ell+1}^{k-1} \delta(t_j)(B(t_{j+1}) - B(t_j))\Big|\mathcal{F}(s)\right] = \sum_{j=\ell+1}^{k-1} \mathbb{E}\left[\mathbb{E}\left[\delta(t_j)(B(t_{j+1}) - B(t_j))\Big|\mathcal{F}(t_j)\right]\Big|\mathcal{F}(s)\right]$$

$$= \sum_{j=\ell+1}^{k-1} \mathbb{E}\left[\delta(t_j)\underbrace{(\mathbb{E}[B(t_{j+1})|\mathcal{F}(t_j)] - B(t_j))}_{=0}\Big|\mathcal{F}(s)\right]$$

$$\mathbb{E}\left[\delta(t_k)(B(t) - B(t_k))\Big|\mathcal{F}(s)\right] = \mathbb{E}\left[\delta(t_k)\underbrace{(\mathbb{E}[B(t)|\mathcal{F}(t_k)] - B(t_k))}_{=0}\Big|\mathcal{F}(s)\right]$$

■

Theorem 7.46 (Itô Isometry)

$$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \delta^2(u)\,du.$$

Proof: To simplify notation, assume $t = t_k$, so

$$I(t) = \sum_{j=0}^{k} \delta(t_j)[\underbrace{B(t_{j+1}) - B(t_j)}_{D_j}]$$

Each D_j has expectation 0, and different D_j are independent.

$$\begin{aligned} I^2(t) &= \left(\sum_{j=0}^{k} \delta(t_j)D_j\right)^2 \\ &= \sum_{j=0}^{k} \delta^2(t_j)D_j^2 + 2\sum_{i<j} \delta(t_i)\delta(t_j)D_iD_j. \end{aligned}$$

Since the cross terms have expectation zero,

$$\begin{aligned} \mathbb{E}I^2(t) &= \sum_{j=0}^{k} \mathbb{E}[\delta^2(t_j)D_j^2] \\ &= \sum_{j=0}^{k} \mathbb{E}\left[\delta^2(t_j)\mathbb{E}\left[(B(t_{j+1}) - B(t_j))^2\Big|\mathcal{F}(t_j)\right]\right] \\ &= \sum_{j=0}^{k} \mathbb{E}\delta^2(t_j)(t_{j+1} - t_j) \\ &= \mathbb{E}\sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \delta^2(u)\,du \\ &= \mathbb{E}\int_0^t \delta^2(u)\,du \end{aligned}$$

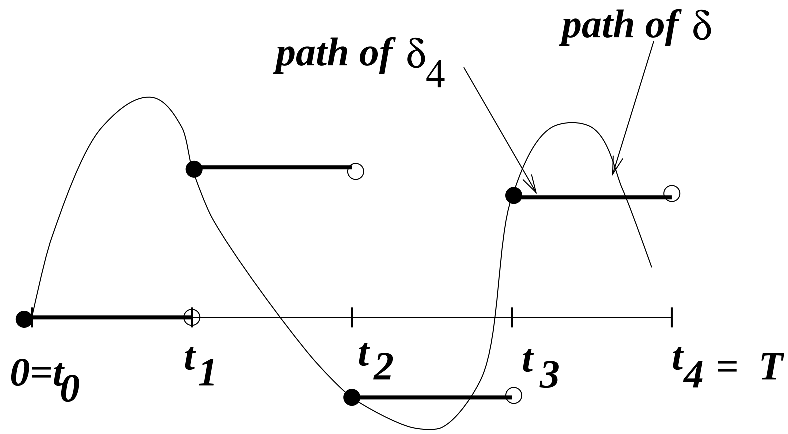

Figure 14.4: *Approximating a general process by an elementary process δ_4, over* $[0, T]$.

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14.8 Itô integral of a general integrand

Fix $T > 0$. Let δ be a process (not necessarily an elementary process) such that

- $\delta(t)$ is $\mathcal{F}(t)$-measurable, $\forall t \in [0, T]$,
- $I\!\!E \int_0^T \delta^2(t)\ dt < \infty$.

Theorem 8.47 *There is a sequence of elementary processes* $\{\delta_n\}_{n=1}^{\infty}$ *such that*

$$\lim_{n \to \infty} I\!\!E \int_0^T |\delta_n(t) - \delta(t)|^2\ dt = 0.$$

Proof: Fig. 14.4 shows the main idea.

■

In the last section we have defined

$$I_n(T) = \int_0^T \delta_n(t)\ dB(t)$$

for every n. We now define

$$\int_0^T \delta(t)\ dB(t) = \lim_{n \to \infty} \int_0^T \delta_n(t)\ dB(t).$$

The only difficulty with this approach is that we need to make sure the above limit exists. Suppose n and m are large positive integers. Then

$$\begin{aligned}\mathrm{var}(I_n(T) - I_m(T)) &= \mathbb{E}\left(\int_0^T [\delta_n(t) - \delta_m(t)]\, dB(t)\right)^2 \\ \text{(Itô Isometry:)} &= \mathbb{E}\int_0^T [\delta_n(t) - \delta_m(t)]^2\, dt \\ &= \mathbb{E}\int_0^T [\, |\delta_n(t) - \delta(t)| + |\delta(t) - \delta_m(t)|\,]^2\, dt \\ ((a+b)^2 \le 2a^2 + 2b^2 :) &\le 2\mathbb{E}\int_0^T |\delta_n(t) - \delta(t)|^2\, dt + 2\mathbb{E}\int_0^T |\delta_m(t) - \delta(t)|^2\, dt,\end{aligned}$$

which is small. This guarantees that the sequence $\{I_n(T)\}_{n=1}^{\infty}$ has a limit.

14.9 Properties of the (general) Itô integral

$$I(t) = \int_0^t \delta(u)\, dB(u).$$

Here δ is any adapted, square-integrable process.

Adaptedness. For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.

Linearity. If

$$I(t) = \int_0^t \delta(u)\, dB(u), \qquad J(t) = \int_0^t \gamma(u)\, dB(u)$$

then

$$I(t) \pm J(t) = \int_0^t (\delta(u) \pm \gamma(u))\, dB(u)$$

and

$$cI(t) = \int_0^t c\delta(u) dB(u).$$

Martingale. $I(t)$ is a martingale.

Continuity. $I(t)$ is a continuous function of the upper limit of integration t.

Itô Isometry. $\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \delta^2(u)\, du$.

Example 14.1 () Consider the Itô integral

$$\int_0^T B(u)\, dB(u).$$

We approximate the integrand as shown in Fig. 14.5

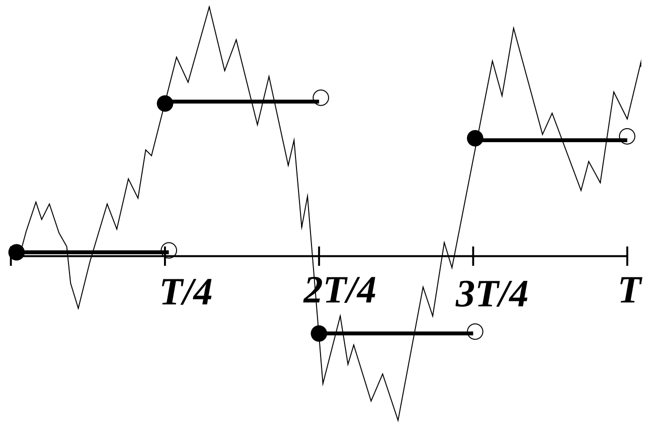

Figure 14.5: *Approximating the integrand $B(u)$ with δ_4, over $[0, T]$.*

$$\delta_n(u) = \begin{cases} B(0) = 0 & \text{if} \quad 0 \le u < T/n; \\ B(T/n) & \text{if} \quad T/n \le u < 2T/n; \\ \dots & \\ B\left(\frac{(n-1)T}{T}\right) & \text{if} \quad \frac{(n-1)T}{n} \le u < T. \end{cases}$$

By definition,

$$\int_0^T B(u)\ dB(u) = \lim_{n \to \infty} \sum_{k=0}^{n-1} B\left(\frac{kT}{n}\right)\left[B\left(\frac{(k+1)T}{n}\right) - B\left(\frac{kT}{n}\right)\right].$$

To simplify notation, we denote

$$B_k \triangleq B\left(\frac{kT}{n}\right),$$

so

$$\int_0^T B(u)\ dB(u) = \lim_{n \to \infty} \sum_{k=0}^{n-1} B_k(B_{k+1} - B_k).$$

We compute

$$\begin{aligned} \tfrac{1}{2}\sum_{k=0}^{n-1}(B_{k+1} - B_k)^2 &= \tfrac{1}{2}\sum_{k=0}^{n-1} B_{k+1}^2 - \sum_{k=0}^{n-1} B_k B_{k+1} + \tfrac{1}{2}\sum_{k=0}^{n-1} B_k^2 \\ &= \tfrac{1}{2}B_n^2 + \tfrac{1}{2}\sum_{j=0}^{n-1} B_j^2 - \sum_{k=0}^{n-1} B_k B_{k+1} + \tfrac{1}{2}\sum_{k=0}^{n-1} B_k^2 \\ &= \tfrac{1}{2}B_n^2 + \sum_{k=0}^{n-1} B_k^2 - \sum_{k=0}^{n-1} B_k B_{k+1} \\ &= \tfrac{1}{2}B_n^2 - \sum_{k=0}^{n-1} B_k(B_{k+1} - B_k). \end{aligned}$$

Therefore,

$$\sum_{k=0}^{n-1} B_k(B_{k+1} - B_k) = \tfrac{1}{2}B_n^2 - \tfrac{1}{2}\sum_{k=0}^{n-1}(B_{k+1} - B_k)^2,$$

or equivalently

$$\sum_{k=0}^{n-1} B\left(\frac{kT}{n}\right)\left[B\left(\frac{(k+1)T}{n}\right) - B\left(\frac{kT}{n}\right)\right] = \tfrac{1}{2}B^2(T) - \tfrac{1}{2}\sum_{k=0}^{n-1}\left[B\left(\frac{(k+1)T}{n}\right)\left(\frac{k}{T}\right)\right]^2.$$

Let $n \to \infty$ and use the definition of quadratic variation to get

$$\int_0^T B(u)\ dB(u) = \tfrac{1}{2}B^2(T) - \tfrac{1}{2}T.$$

■

Remark 14.4 (Reason for the $\frac{1}{2}T$ term) If f is differentiable with $f(0) = 0$, then

$$\begin{aligned}\int_0^T f(u)\ df(u) &= \int_0^T f(u)f'(u)\ du \\ &= \tfrac{1}{2}f^2(u)\Big|_0^T \\ &= \tfrac{1}{2}f^2(T).\end{aligned}$$

In contrast, for Brownian motion, we have

$$\int_0^T B(u)dB(u) = \tfrac{1}{2}B^2(T) - \tfrac{1}{2}T.$$

The extra term $\frac{1}{2}T$ comes from the nonzero quadratic variation of Brownian motion. It has to be there, because

$$I\!E\int_0^T B(u)\ dB(u) = 0 \qquad \text{(Itô integral is a martingale)}$$

but

$$I\!E\tfrac{1}{2}B^2(T) = \tfrac{1}{2}T.$$

14.10 Quadratic variation of an Itô integral

Theorem 10.48 (Quadratic variation of Itô integral) *Let*

$$I(t) = \int_0^t \delta(u)\ dB(u).$$

Then

$$\langle I\rangle(t) = \int_0^t \delta^2(u)\ du.$$

This holds even if δ is not an elementary process. The quadratic variation formula says that at each time u, the *instantaneous absolute volatility* of I is $\delta^2(u)$. This is the absolute volatility of the Brownian motion scaled by the size of the position (i.e. $\delta(t)$) in the Brownian motion. Informally, we can write the quadratic variation formula in differential form as follows:

$$dI(t)\ dI(t) = \delta^2(t)\ dt.$$

Compare this with

$$dB(t)\ dB(t) = dt.$$

Proof: (For an elementary process δ). Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be the partition for δ, i.e., $\delta(t) = \delta(t_k)$ for $t_k \le t \le t_{k+1}$. To simplify notation, assume $t = t_n$. We have

$$\langle I\rangle(t) = \sum_{k=0}^{n-1} \left[\langle I\rangle(t_{k+1}) - \langle I\rangle(t_k)\right].$$

Let us compute $\langle I\rangle(t_{k+1}) - \langle I\rangle(t_k)$. Let $\Xi = \{s_0, s_1, \ldots, s_m\}$ be a partition

$$t_k = s_0 \le s_1 \le \ldots \le s_m = t_{k+1}.$$

Then

$$\begin{aligned} I(s_{j+1}) - I(s_j) &= \int_{s_j}^{s_{j+1}} \delta(t_k)\ dB(u) \\ &= \delta(t_k)\left[B(s_{j+1}) - B(s_j)\right], \end{aligned}$$

so

$$\begin{aligned} \langle I\rangle(t_{k+1}) - \langle I\rangle(t_k) &= \sum_{j=0}^{m-1} \left[I(s_{j+1}) - I(s_j)\right]^2 \\ &= \delta^2(t_k) \sum_{j=0}^{m-1} \left[B(s_{j+1}) - B(s_j)\right]^2 \\ &\xrightarrow{||\Xi||\to 0} \delta^2(t_k)(t_{k+1} - t_k). \end{aligned}$$

It follows that

$$\begin{aligned} \langle I\rangle(t) &= \sum_{k=0}^{n-1} \delta^2(t_k)(t_{k+1} - t_k) \\ &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \delta^2(u)\ du \\ &\xrightarrow{||\Pi||\to 0} \int_0^t \delta^2(u)\ du. \end{aligned}$$

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