Chapter 13

Brownian Motion

13.1 Symmetric Random Walk

Toss a fair coin infinitely many times. Define

$$X_j(\omega) = \begin{cases} 1 & \text{if} \quad \omega_j = H, \\ -1 & \text{if} \quad \omega_j = T. \end{cases}$$

Set

$$M_0 = 0$$
$$M_k = \sum_{j=1}^{k} X_j, \qquad k \geq 1.$$

13.2 The Law of Large Numbers

We will use the method of moment generating functions to derive the Law of Large Numbers:

Theorem 2.38 (Law of Large Numbers:)

$$\frac{1}{k} M_k \to 0 \quad \textit{almost surely, as} \quad k \to \infty.$$

Proof:

$$
\begin{aligned}
\varphi_k(u) &= I\!\!E \exp\left\{\frac{u}{k}M_k\right\} \\
&= I\!\!E \exp\left\{\sum_{j=1}^{k}\frac{u}{k}X_j\right\} && \text{(Def. of } M_k.\text{)} \\
&= \prod_{j=1}^{k} I\!\!E \exp\left\{\frac{u}{k}X_j\right\} && \text{(Independence of the } X_j\text{'s)} \\
&= \left(\tfrac{1}{2}e^{\frac{u}{k}} + \tfrac{1}{2}e^{-\frac{u}{k}}\right)^k,
\end{aligned}
$$

which implies,

$$
\log \varphi_k(u) = k \log\left(\tfrac{1}{2}e^{\frac{u}{k}} + \tfrac{1}{2}e^{-\frac{u}{k}}\right)
$$

Let $x = \frac{1}{k}$. Then

$$
\begin{aligned}
\lim_{k\longrightarrow\infty} \log \varphi_k(u) &= \lim_{x\longrightarrow 0} \frac{\log\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}{x} \\
&= \lim_{x\longrightarrow 0} \frac{\frac{u}{2}e^{ux} - \frac{u}{2}e^{-ux}}{\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}} && \text{(L'Hôpital's Rule)} \\
&= 0.
\end{aligned}
$$

Therefore,

$$
\lim_{k\longrightarrow\infty} \varphi_k(u) = e^0 = 1,
$$

which is the m.g.f. for the constant 0. ■

13.3 Central Limit Theorem

We use the method of moment generating functions to prove the Central Limit Theorem.

Theorem 3.39 (Central Limit Theorem)

$$
\frac{1}{\sqrt{k}}M_k \rightarrow \quad \textit{Standard normal, as} \quad k\rightarrow\infty.
$$

Proof:

$$
\begin{aligned}
\varphi_k(u) &= I\!\!E \exp\left\{\frac{u}{\sqrt{k}}M_k\right\} \\
&= \left(\tfrac{1}{2}e^{\frac{u}{\sqrt{k}}} + \tfrac{1}{2}e^{-\frac{u}{\sqrt{k}}}\right)^k,
\end{aligned}
$$

so that,

$$\log \varphi_k(u) = k \log \left(\tfrac{1}{2} e^{\frac{u}{\sqrt{k}}} + \tfrac{1}{2} e^{-\frac{u}{\sqrt{k}}} \right).$$

Let $x = \frac{1}{\sqrt{k}}$. Then

$$\begin{aligned}
\lim_{k \to \infty} \log \varphi_k(u) &= \lim_{x \to 0} \frac{\log \left(\frac{1}{2} e^{ux} + \frac{1}{2} e^{-ux} \right)}{x^2} \\
&= \lim_{x \to 0} \frac{\frac{u}{2} e^{ux} - \frac{u}{2} e^{-ux}}{2x \left(\frac{1}{2} e^{ux} + \frac{1}{2} e^{-ux} \right)} && \text{(L'Hôpital's Rule)} \\
&= \lim_{x \to 0} \frac{1}{\frac{1}{2} e^{ux} + \frac{1}{2} e^{-ux}} . \lim_{x \to 0} \frac{\frac{u}{2} e^{ux} - \frac{u}{2} e^{-ux}}{2x} \\
&= \lim_{x \to 0} \frac{\frac{u}{2} e^{ux} - \frac{u}{2} e^{-ux}}{2x} \\
&= \lim_{x \to 0} \frac{\frac{u^2}{2} e^{ux} - \frac{u^2}{2} e^{-ux}}{2} && \text{(L'Hôpital's Rule)} \\
&= \tfrac{1}{2} u^2 .
\end{aligned}$$

Therefore,

$$\lim_{k \to \infty} \varphi_k(u) = e^{\frac{1}{2} u^2},$$

which is the m.g.f. for a standard normal random variable. ■

13.4 Brownian Motion as a Limit of Random Walks

Let n be a positive integer. If $t \geq 0$ is of the form $\frac{k}{n}$, then set

$$B^{(n)}(t) = \frac{1}{\sqrt{n}} M_{tn} = \frac{1}{\sqrt{n}} M_k .$$

If $t \geq 0$ is not of the form $\frac{k}{n}$, then define $B^{(n)}(t)$ by linear interpolation (See Fig. 13.1).

Here are some properties of $B^{(100)}(t)$:

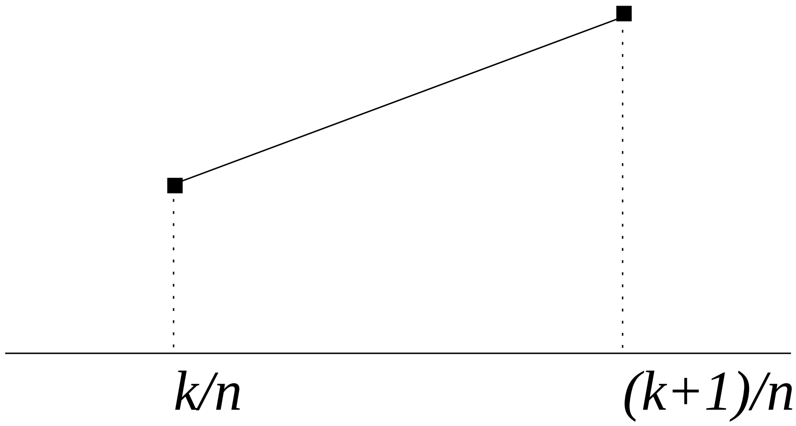

Figure 13.1: *Linear Interpolation to define* $B^{(n)}(t)$.

Properties of $B^{(100)}(1)$:

$$B^{(100)}(1) = \frac{1}{10}\sum_{j=1}^{100} X_j \qquad \text{(Approximately normal)}$$

$$\mathbb{E}B^{(100)}(1) = \frac{1}{10}\sum_{j=1}^{100} \mathbb{E}X_j = 0.$$

$$\text{var}(B^{(100)}(1)) = \frac{1}{100}\sum_{j=1}^{100} \text{var}(X_j) = 1$$

Properties of $B^{(100)}(2)$:

$$B^{(100)}(2) = \frac{1}{10}\sum_{j=1}^{200} X_j \qquad \text{(Approximately normal)}$$

$$\mathbb{E}B^{(100)}(2) = 0.$$

$$\text{var}(B^{(100)}(2)) = 2.$$

Also note that:

- $B^{(100)}(1)$ and $B^{(100)}(2) - B^{(100)}(1)$ are independent.
- $B^{(100)}(t)$ is a continuous function of t.

To get Brownian motion, let $n \to \infty$ in $B^{(n)}(t), \quad t \geq 0$.

13.5 Brownian Motion

(Please refer to Oksendal, Chapter 2.)

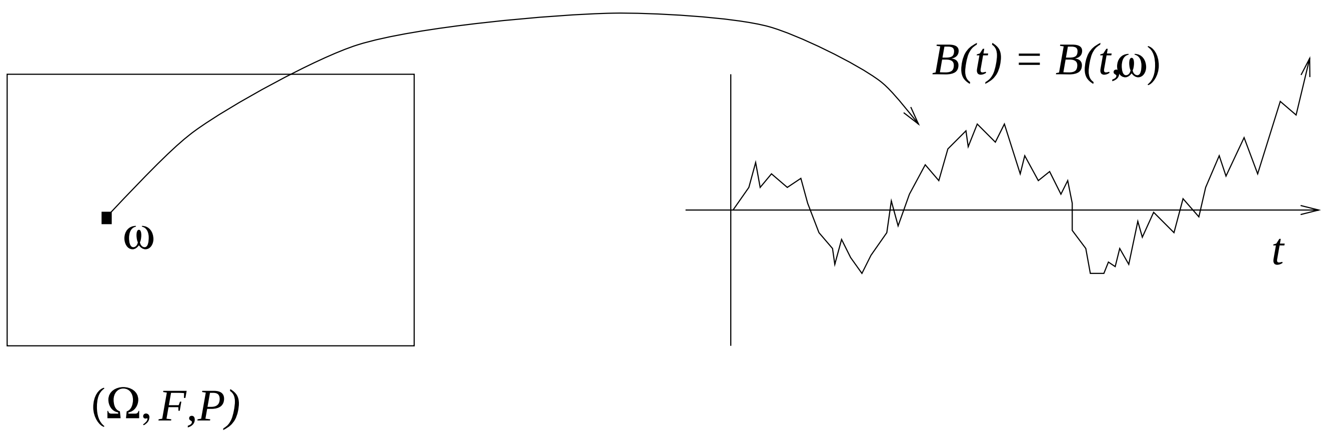

Figure 13.2: *Continuous-time Brownian Motion.*

A random variable $B(t)$ (see Fig. 13.2) is called a Brownian Motion if it satisfies the following properties:

1. $B(0) = 0$,
2. $B(t)$ is a continuous function of t;
3. B has independent, normally distributed increments: If

$$0 = t_0 < t_1 < t_2 < \ldots < t_n$$

and

$$Y_1 = B(t_1) - B(t_0), \quad Y_2 = B(t_2) - B(t_1), \quad \ldots \quad Y_n = B(t_n) - B(t_{n-1}),$$

then

- $Y_1, Y_2, \ldots, Y_n$ are independent,
- $\mathbb{E}Y_j = 0 \qquad \forall j$,
- $\text{var}(Y_j) = t_j - t_{j-1} \qquad \forall j$.

13.6 Covariance of Brownian Motion

Let $0 \le s \le t$ be given. Then $B(s)$ and $B(t) - B(s)$ are independent, so $B(s)$ and $B(t) = (B(t) - B(s)) + B(s)$ are jointly normal. Moreover,

$$\begin{aligned}
\mathbb{E}B(s) = 0, &\qquad \text{var}(B(s)) = s, \\
\mathbb{E}B(t) = 0, &\qquad \text{var}(B(t)) = t, \\
\mathbb{E}B(s)B(t) &= \mathbb{E}B(s)[(B(t) - B(s)) + B(s)] \\
&= \underbrace{\mathbb{E}B(s)(B(t) - B(s))}_{0} + \underbrace{\mathbb{E}B^2(s)}_{s} \\
&= s.
\end{aligned}$$

Thus for any $s \geq 0, t \geq 0$ (not necessarily $s \leq t$), we have

$$\mathbb{E}B(s)B(t) = s \wedge t.$$

13.7 Finite-Dimensional Distributions of Brownian Motion

Let

$$0 < t_1 < t_2 < \ldots < t_n$$

be given. Then

$$(B(t_1), B(t_2), \ldots, B(t_n))$$

is jointly normal with covariance matrix

$$C = \begin{bmatrix} \mathbb{E}B^2(t_1) & \mathbb{E}B(t_1)B(t_2) & \ldots & \mathbb{E}B(t_1)B(t_n) \\ \mathbb{E}B(t_2)B(t_1) & \mathbb{E}B^2(t_2) & \ldots & \mathbb{E}B(t_2)B(t_n) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{E}B(t_n)B(t_1) & \mathbb{E}B(t_n)B(t_2) & \ldots & \mathbb{E}B^2(t_n) \end{bmatrix}$$

$$= \begin{bmatrix} t_1 & t_1 & \ldots & t_1 \\ t_1 & t_2 & \ldots & t_2 \\ \ldots & \ldots & \ldots & \ldots \\ t_1 & t_2 & \ldots & t_n \end{bmatrix}$$

13.8 Filtration generated by a Brownian Motion

$$\{\mathcal{F}(t)\}_{t \geq 0}$$

Required properties:

- For each t, $B(t)$ is $\mathcal{F}(t)$-measurable,
- For each t and for $t < t_1 < t_2 < \cdots < t_n$, the Brownian motion increments

  $$B(t_1) - B(t), \quad B(t_2) - B(t_1), \quad \ldots, \quad B(t_n) - B(t_{n-1})$$

  are *independent of* $\mathcal{F}(t)$.

Here is one way to construct $\mathcal{F}(t)$. First fix t. Let $s \in [0, t]$ and $C \in \mathcal{B}(\mathbb{R})$ be given. Put the set

$$\{B(s) \in C\} = \{\omega : B(s, \omega) \in C\}$$

in $\mathcal{F}(t)$. Do this for all possible numbers $s \in [0, t]$ and $C \in \mathcal{B}(\mathbb{R})$. Then put in every other set required by the σ-algebra properties.

This $\mathcal{F}(t)$ contains exactly the information learned by observing the Brownian motion upto time t. $\{\mathcal{F}(t)\}_{t \geq 0}$ is called the *filtration generated by the Brownian motion.*

13.9 Martingale Property

Theorem 9.40 *Brownian motion is a martingale.*

Proof: Let $0 \le s \le t$ be given. Then

$$\begin{aligned} I\!E[B(t)|\mathcal{F}(s)] &= I\!E[(B(t)-B(s)) + B(s)|\mathcal{F}(s)] \\ &= I\!E[B(t)-B(s)] + B(s) \\ &= B(s). \end{aligned}$$

■

Theorem 9.41 *Let $\theta \in I\!R$ be given. Then*

$$Z(t) = \exp\left\{-\theta B(t) - \tfrac{1}{2}\theta^2 t\right\}$$

is a martingale.

Proof: Let $0 \le s \le t$ be given. Then

$$\begin{aligned} I\!E[Z(t)|\mathcal{F}(s)] &= I\!E\left[\exp\{-\theta(B(t)-B(s)+B(s)) - \tfrac{1}{2}\theta^2((t-s)+s)\}\Big|\mathcal{F}(s)\right] \\ &= I\!E\left[Z(s)\exp\{-\theta(B(t)-B(s)) - \tfrac{1}{2}\theta^2(t-s)\}\Big|\mathcal{F}(s)\right] \\ &= Z(s) I\!E\left[\exp\{-\theta(B(t)-B(s)) - \tfrac{1}{2}\theta^2(t-s)\}\right] \\ &= Z(s)\exp\left\{\tfrac{1}{2}(-\theta)^2 \operatorname{var}(B(t)-B(s)) - \tfrac{1}{2}\theta^2(t-s)\right\} \\ &= Z(s). \end{aligned}$$

■

13.10 The Limit of a Binomial Model

Consider the n'th Binomial model with the following parameters:

- $u_n = 1 + \frac{\sigma}{\sqrt{n}}$. "Up" factor. ($\sigma > 0$).
- $d_n = 1 - \frac{\sigma}{\sqrt{n}}$. "Down" factor.
- $r = 0$.
- $\tilde{p}_n = \frac{1-d_n}{u_n-d_n} = \frac{\sigma/\sqrt{n}}{2\sigma/\sqrt{n}} = \frac{1}{2}$.
- $\tilde{q}_n = \frac{1}{2}$.

Let $\sharp_k(H)$ denote the number of H in the first k tosses, and let $\sharp_k(T)$ denote the number of T in the first k tosses. Then

$$\begin{aligned}\sharp_k(H) + \sharp_k(T) &= k,\\ \sharp_k(H) - \sharp_k(T) &= M_k,\end{aligned}$$

which implies,

$$\begin{aligned}\sharp_k(H) &= \tfrac{1}{2}(k + M_k)\\ \sharp_k(T) &= \tfrac{1}{2}(k - M_k).\end{aligned}$$

In the n'th model, take n steps per unit time. Set $S_0^{(n)} = 1$. Let $t = \frac{k}{n}$ for some k, and let

$$S^{(n)}(t) = \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt + M_{nt})} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt - M_{nt})}.$$

Under $\widetilde{I\!P}$, the price process $S^{(n)}$ is a martingale.

Theorem 10.42 *As $n\to\infty$, the distribution of $S^{(n)}(t)$ converges to the distribution of*

$$\exp\{\sigma B(t) - \tfrac{1}{2}\sigma^2 t\},$$

where B is a Brownian motion. Note that the correction $-\frac{1}{2}\sigma^2 t$ is necessary in order to have a martingale.

Proof: Recall that from the Taylor series we have

$$\log(1 + x) = x - \tfrac{1}{2}x^2 + O(x^3),$$

so

$$\begin{aligned}\log S^{(n)}(t) &= \tfrac{1}{2}(nt + M_{nt}) \log(1 + \frac{\sigma}{\sqrt{n}}) + \tfrac{1}{2}(nt - M_{nt}) \log(1 - \frac{\sigma}{\sqrt{n}})\\ &= nt \left(\tfrac{1}{2}\log(1 + \frac{\sigma}{\sqrt{n}}) + \tfrac{1}{2}\log(1 - \frac{\sigma}{\sqrt{n}})\right)\\ &\quad + M_{nt}\left(\tfrac{1}{2}\log(1 + \frac{\sigma}{\sqrt{n}}) - \tfrac{1}{2}\log(1 - \frac{\sigma}{\sqrt{n}})\right)\\ &= nt\left(\tfrac{1}{2}\frac{\sigma}{\sqrt{n}} - \frac{1}{4}\frac{\sigma^2}{n} - \tfrac{1}{2}\frac{\sigma}{\sqrt{n}} - \frac{1}{4}\frac{\sigma^2}{n} + O(n^{-3/2})\right)\\ &\quad + M_{nt}\left(\tfrac{1}{2}\frac{\sigma}{\sqrt{n}} - \frac{1}{4}\frac{\sigma^2}{n} + \tfrac{1}{2}\frac{\sigma}{\sqrt{n}} + \frac{1}{4}\frac{\sigma^2}{n} + O(n^{-3/2})\right)\\ &= -\tfrac{1}{2}\sigma^2 t + O(n^{-\frac{1}{2}})\\ &\quad + \sigma\underbrace{\left(\frac{1}{\sqrt{n}}M_{nt}\right)}_{\to B_t} + \underbrace{\left(\frac{1}{n}M_{nt}\right)}_{\to 0} O(n^{-\frac{1}{2}})\end{aligned}$$

As $n\to\infty$, the distribution of $\log S^{(n)}(t)$ approaches the distribution of $\sigma B(t) - \frac{1}{2}\sigma^2 t$. ■

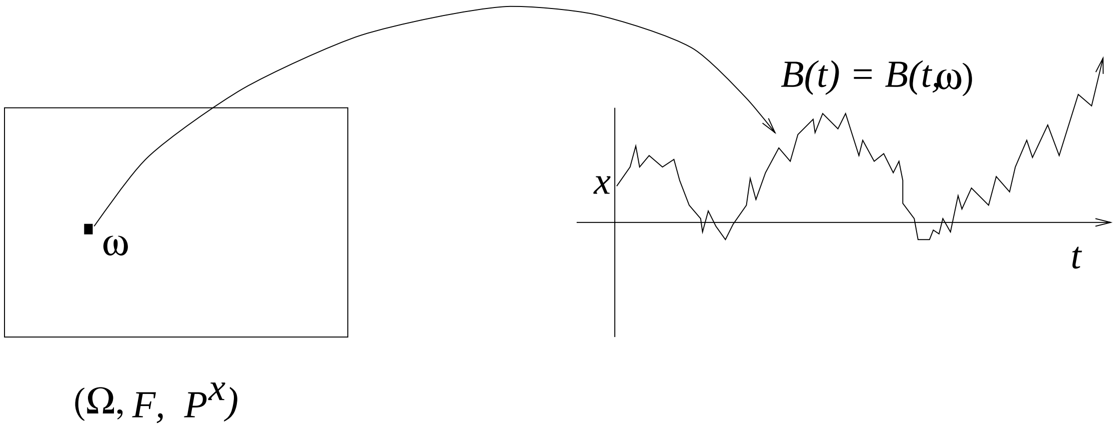

Figure 13.3: *Continuous-time Brownian Motion, starting at* $x \neq 0$.

13.11 Starting at Points Other Than 0

(The remaining sections in this chapter were taught Dec 7.)

For a Brownian motion $B(t)$ that starts at 0, we have:

$$\mathbb{P}(B(0) = 0) = 1.$$

For a Brownian motion $B(t)$ that starts at x, denote the corresponding probability measure by $\mathbb{P}^x$ (See Fig. 13.3), and for such a Brownian motion we have:

$$\mathbb{P}^x(B(0) = x) = 1.$$

Note that:

- If $x \neq 0$, then $\mathbb{P}^x$ puts all its probability on a completely different set from P.
- The distribution of $B(t)$ under $\mathbb{P}^x$ is the same as the distribution of $x + B(t)$ under P.

13.12 Markov Property for Brownian Motion

We prove that

Theorem 12.43 *Brownian motion has the Markov property.*

Proof:

Let $s \geq 0, \quad t \geq 0$ be given (See Fig. 13.4).

$$\mathbb{E}\left[h(B(s+t))\Big|\mathcal{F}(s)\right] = \mathbb{E}\left[h(\underbrace{B(s+t) - B(s)}_{\text{Independent of } \mathcal{F}(s)} + \underbrace{B(s)}_{\mathcal{F}(s)\text{-measurable}})\Big|\mathcal{F}(s)\right]$$

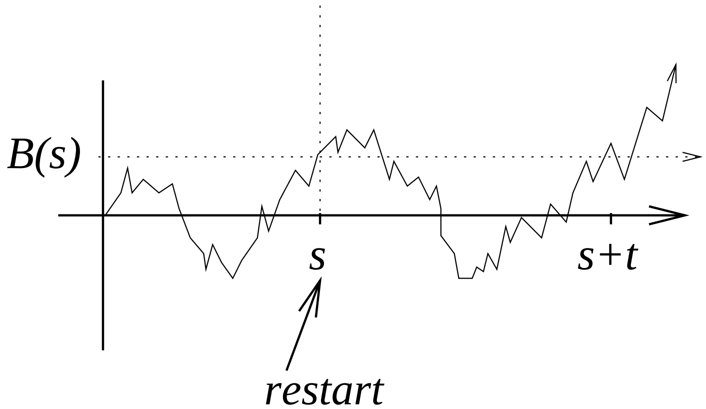

Figure 13.4: *Markov Property of Brownian Motion.*

Use the Independence Lemma. Define

$$\begin{aligned} g(x) &= \mathbb{E}\left[h(\,B(s+t) - B(s)\, + x\,)\right] \\ &= \mathbb{E}\left[h(\,x + \underbrace{B(t)}_{\text{same distribution as } B(s+t)-B(s)}\,)\right] \\ &= \mathbb{E}^x h(B(t)). \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}\left[h\left(B(s+t)\,\right)\Big|\mathcal{F}(s)\right] &= g(B(s)) \\ &= E^{B(s)} h(B(t)). \end{aligned}$$

■

In fact Brownian motion has the *strong Markov property.*

Example 13.1 (Strong Markov Property) See Fig. 13.5. Fix $x > 0$ and define

$$\tau = \min\left\{t \geq 0; \quad B(t) = x\right\}.$$

Then we have:

$$\mathbb{E}\left[h(\,B(\tau+t)\,)\Big|\mathcal{F}(\tau)\right] = g(B(\tau)) = \mathbb{E}^x h(B(t)).$$

■

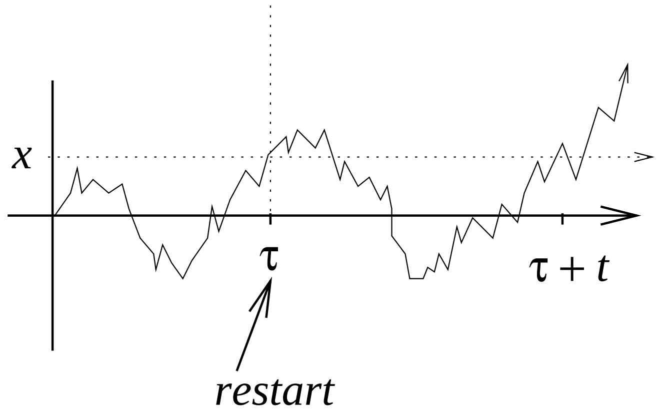

Figure 13.5: *Strong Markov Property of Brownian Motion.*

13.13 Transition Density

Let $p(t,x,y)$ be the probability that the Brownian motion changes value from x to y in time t, and let τ be defined as in the previous section.

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$$

$$g(x) = \mathbb{E}^x h(B(t)) = \int_{-\infty}^{\infty} h(y) p(t,x,y)\, dy.$$

$$\mathbb{E}\left[h(B(s+t)) \middle| \mathcal{F}(s) \right] = g(B(s)) = \int_{-\infty}^{\infty} h(y) p(t, B(s), y)\, dy.$$

$$\mathbb{E}\left[h(B(\tau+t)) \middle| \mathcal{F}(\tau) \right] = \int_{-\infty}^{\infty} h(y) p(t,x,y)\, dy.$$

13.14 First Passage Time

Fix $x > 0$. Define

$$\tau = \min\{t \geq 0; \quad B(t) = x\}.$$

Fix $\theta > 0$. Then

$$\exp\left\{ \theta B(t\wedge\tau) - \tfrac{1}{2}\theta^2 (t\wedge\tau) \right\}$$

is a martingale, and

$$\mathbb{E} \exp\left\{ \theta B(t\wedge\tau) - \tfrac{1}{2}\theta^2 (t\wedge\tau) \right\} = 1.$$

We have

$$\lim_{t\to\infty} \exp\left\{-\tfrac{1}{2}\theta^2(t\wedge\tau)\right\} = \begin{cases} e^{-\frac{1}{2}\theta^2\tau} & \text{if } \tau<\infty, \\ 0 & \text{if } \tau=\infty, \end{cases} \tag{14.1}$$

$$0 \le \exp\{\theta B(t\wedge\tau) - \tfrac{1}{2}\theta^2(t\wedge\tau)\} \le e^{\theta x}.$$

Let $t\to\infty$ in (14.1), using the Bounded Convergence Theorem, to get

$$I\!\!E\left[\exp\{\theta x - \tfrac{1}{2}\theta^2\tau\}\mathbf{1}_{\{\tau<\infty\}}\right] = 1.$$

Let $\theta\downarrow 0$ to get $I\!\!E\mathbf{1}_{\{\tau<\infty\}} = 1$, so

$$I\!\!P\{\tau<\infty\} = 1,$$

$$I\!\!E\exp\{-\tfrac{1}{2}\theta^2\tau\} = e^{-\theta x}. \tag{14.2}$$

Let $\alpha = \frac{1}{2}\theta^2$. We have the m.g.f.:

$$I\!\!E e^{-\alpha\tau} = e^{-x\sqrt{2\alpha}}, \qquad \alpha>0. \tag{14.3}$$

Differentiation of (14.3) w.r.t. α yields

$$-I\!\!E\left[\tau e^{-\alpha\tau}\right] = -\frac{x}{\sqrt{2\alpha}}e^{-x\sqrt{2\alpha}}.$$

Letting $\alpha\downarrow 0$, we obtain

$$I\!\!E\tau = \infty. \tag{14.4}$$

Conclusion. Brownian motion reaches level x with probability 1. The expected time to reach level x is infinite.

We use the Reflection Principle below (see Fig. 13.6).

$$I\!\!P\{\tau\le t, \quad B(t)<x\} = I\!\!P\{B(t)>x\}$$

$$\begin{aligned} I\!\!P\{\tau\le t\} &= I\!\!P\{\tau\le t, B(t)<x\} + I\!\!P\{\tau\le t, B(t)>x\} \\ &= I\!\!P\{B(t)>x\} + I\!\!P\{B(t)>x\} \\ &= 2I\!\!P\{B(t)>x\} \\ &= \frac{2}{\sqrt{2\pi t}}\int_x^\infty e^{-\frac{y^2}{2t}}\,dy \end{aligned}$$

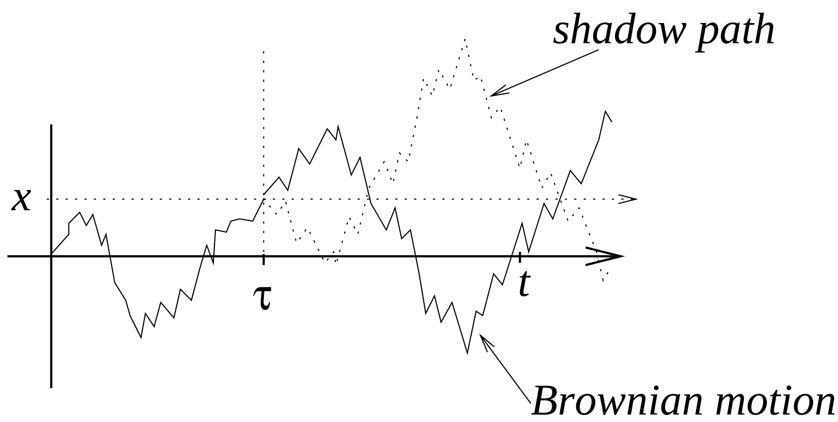

Figure 13.6: *Reflection Principle in Brownian Motion.*

Using the substitution $z = \frac{y}{\sqrt{t}}, \quad dz = \frac{dy}{\sqrt{t}}$ we get

$$I\!P\{\tau \le t\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{-\frac{z^2}{2}} dz.$$

Density:

$$f_\tau(t) = \frac{\partial}{\partial t} I\!P\{\tau \le t\} = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}},$$

which follows from the fact that if

$$F(t) = \int_{a(t)}^{b} g(z)\, dz,$$

then

$$\frac{\partial F}{\partial t} = -\frac{\partial a}{\partial t} g(a(t)).$$

Laplace transform formula:

$$I\!E e^{-\alpha\tau} = \int_0^\infty e^{-\alpha t} f_\tau(t) dt = e^{-x\sqrt{2\alpha}}.$$