Chapter 12

Semi-Continuous Models

12.1 Discrete-time Brownian Motion

Let $\{Y_j\}_{j=1}^n$ be a collection of independent, standard normal random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, where P is the *market measure*. As before we denote the column vector $(Y_1, \dots, Y_n)^T$ by $\mathbf{Y}$. We therefore have for any real colum vector $\mathbf{u} = (u_1, \dots, u_n)^T$,

$$I\!\!E e^{\mathbf{u}^T \mathbf{Y}} = I\!\!E \exp\left\{\sum_{j=1}^n u_j Y_j\right\} = \exp\left\{\sum_{j=1}^n \frac{1}{2} u_j^2\right\}.$$

Define the *discrete-time Brownian motion* (See Fig. 12.1):

$$\begin{aligned} B_0 &= 0, \\ B_k &= \sum_{j=1}^k Y_j,\ k = 1, \dots, n. \end{aligned}$$

If we know $Y_1, Y_2, \dots, Y_k$, then we know $B_1, B_2, \dots, B_k$. Conversely, if we know $B_1, B_2, \dots, B_k$, then we know $Y_1 = B_1, Y_2 = B_2 - B_1, \dots, Y_k = B_k - B_{k-1}$. Define the filtration

$$\begin{aligned} \mathcal{F}_0 &= \{\phi, \Omega\}, \\ \mathcal{F}_k &= \sigma(Y_1, Y_2, \dots, Y_k) = \sigma(B_1, B_2, \dots, B_k),\ k = 1, \dots, n. \end{aligned}$$

Theorem 1.34 *$\{B_k\}_{k=0}^n$ is a martingale (under P).*

Proof:

$$\begin{aligned} I\!\!E[B_{k+1} | \mathcal{F}_k] &= I\!\!E[Y_{k+1} + B_k | \mathcal{F}_k] \\ &= I\!\!E Y_{k+1} + B_k \\ &= B_k. \end{aligned}$$

■

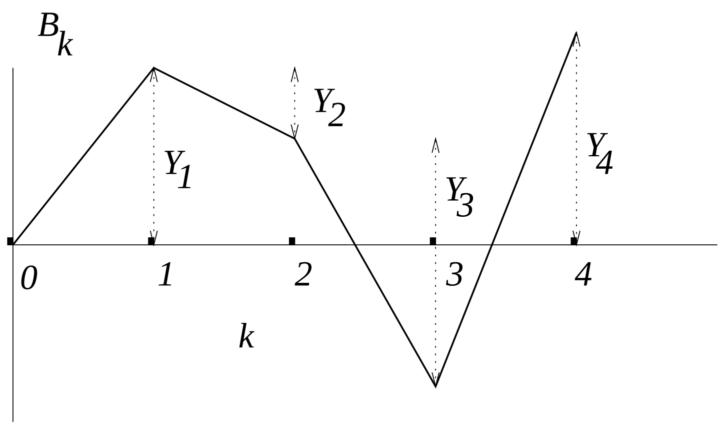

Figure 12.1: *Discrete-time Brownian motion.*

Theorem 1.35 $\{B_k\}_{k=0}^n$ *is a Markov process.*

Proof: Note that

$$\mathbb{E}[h(B_{k+1})|\mathcal{F}_k] = \mathbb{E}[h(Y_{k+1} + B_k)|\mathcal{F}_k].$$

Use the Independence Lemma. Define

$$g(b) = \mathbb{E}h(Y_{k+1} + b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(y+b) e^{-\frac{1}{2}y^2}\ dy.$$

Then

$$\mathbb{E}[h(Y_{k+1} + B_k)|\mathcal{F}_k] = g(B_k),$$

which is a function of B_k alone.

12.2 The Stock Price Process

Given parameters:

- $\mu \in \mathbb{R}$, the *mean rate of return.*
- $\sigma > 0$, the *volatility.*
- $S_0 > 0$, the initial stock price.

The *stock price process* is then given by

$$S_k = S_0 \exp\left\{\sigma B_k + (\mu - \tfrac{1}{2}\sigma^2)k\right\},\ k = 0, \ldots, n.$$

Note that

$$S_{k+1} = S_k \exp\left\{\sigma Y_{k+1} + (\mu - \tfrac{1}{2}\sigma^2)\right\},$$

$$
\begin{aligned}
\mathbb{E}[S_{k+1}|\mathcal{F}_k] &= S_k \mathbb{E}[e^{\sigma Y_{k+1}}|\mathcal{F}_k].e^{\mu-\frac{1}{2}\sigma^2} \\
&= S_k e^{\frac{1}{2}\sigma^2} e^{\mu-\frac{1}{2}\sigma^2} \\
&= e^{\mu} S_k.
\end{aligned}
$$

Thus

$$
\mu = \log \frac{\mathbb{E}[S_{k+1}|\mathcal{F}_k]}{S_k} = \log \mathbb{E}\left[\frac{S_{k+1}}{S_k}\middle|\mathcal{F}_k\right],
$$

and

$$
\operatorname{var}\left(\log \frac{S_{k+1}}{S_k}\right) = \operatorname{var}\left(\sigma Y_{k+1} + (\mu - \tfrac{1}{2}\sigma^2)\right) = \sigma^2.
$$

12.3 Remainder of the Market

The other processes in the market are defined as follows.

Money market process:

$$
M_k = e^{rk},\ k = 0, 1, \ldots, n.
$$

Portfolio process:

- $\Delta_0, \Delta_1, \ldots, \Delta_{n-1}$,
- Each Δ_k is $\mathcal{F}_k$-measurable.

Wealth process:

- X_0 given, nonrandom.
- $$
\begin{aligned}
X_{k+1} &= \Delta_k S_{k+1} + e^r(X_k - \Delta_k S_k) \\
&= \Delta_k(S_{k+1} - e^r S_k) + e^r X_k
\end{aligned}
$$
- Each X_k is $\mathcal{F}_k$-measurable.

Discounted wealth process:

$$
\frac{X_{k+1}}{M_{k+1}} = \Delta_k\left(\frac{S_{k+1}}{M_{k+1}} - \frac{S_k}{M_k}\right) + \frac{X_k}{M_k}.
$$

12.4 Risk-Neutral Measure

Definition 12.1 Let $\widetilde{\mathbb{P}}$ be a probability measure on $(\Omega, \mathcal{F})$, equivalent to the market measure P. If $\left\{\frac{S_k}{M_k}\right\}_{k=0}^{n}$ is a martingale under $\widetilde{\mathbb{P}}$, we say that $\widetilde{\mathbb{P}}$ is a *risk-neutral measure.*

Theorem 4.36 *If $\widetilde{I\!P}$ is a risk-neutral measure, then every discounted wealth process $\left\{\frac{X_k}{M_k}\right\}_{k=0}^{n}$ is a martingale under $\widetilde{I\!P}$, regardless of the portfolio process used to generate it.*

Proof:

$$\begin{aligned} \widetilde{I\!E}\left[\frac{X_{k+1}}{M_{k+1}}\middle|\mathcal{F}_k\right] &= \widetilde{I\!E}\left[\Delta_k\left(\frac{S_{k+1}}{M_{k+1}} - \frac{S_k}{M_k}\right) + \frac{X_k}{M_k}\middle|\mathcal{F}_k\right] \\ &= \Delta_k\left(\widetilde{I\!E}\left[\frac{S_{k+1}}{M_{k+1}}\middle|\mathcal{F}_k\right] - \frac{S_k}{M_k}\right) + \frac{X_k}{M_k} \\ &= \frac{X_k}{M_k}. \end{aligned}$$

■

12.5 Risk-Neutral Pricing

Let V_n be the payoff at time n, and say it is $\mathcal{F}_n$-measurable. Note that V_n may be path-dependent. Hedging a short position:

- Sell the simple European derivative security V_n.
- Receive X_0 at time 0.
- Construct a portfolio process $\Delta_0, \ldots, \Delta_{n-1}$ which starts with X_0 and ends with $X_n = V_n$.
- If there is a risk-neutral measure $\widetilde{I\!P}$, then

$$X_0 = \widetilde{I\!E}\frac{X_n}{M_n} = \widetilde{I\!E}\frac{V_n}{M_n}.$$

Remark 12.1 Hedging in this "semi-continuous" model is usually not possible because there are not enough trading dates. This difficulty will disappear when we go to the fully continuous model.

12.6 Arbitrage

Definition 12.2 An *arbitrage* is a portfolio which starts with $X_0 = 0$ and ends with X_n satisfying

$$I\!P(X_n \geq 0) = 1,\ I\!P(X_n > 0) > 0.$$

(P here is the market measure).

Theorem 6.37 (Fundamental Theorem of Asset Pricing: Easy part) *If there is a risk-neutral measure, then there is no arbitrage.*

Proof: Let $\widetilde{I\!P}$ be a risk-neutral measure, let $X_0 = 0$, and let X_n be the final wealth corresponding to any portfolio process. Since $\left\{\frac{X_k}{M_k}\right\}_{k=0}^{n}$ is a martingale under $\widetilde{I\!P}$,

$$\widetilde{I\!E}\frac{X_n}{M_n} = \widetilde{I\!E}\frac{X_0}{M_0} = 0. \tag{6.1}$$

Suppose $I\!P(X_n \geq 0) = 1$. We have

$$I\!P(X_n \geq 0) = 1 \Longrightarrow I\!P(X_n < 0) = 0 \Longrightarrow \widetilde{I\!P}(X_n < 0) = 0 \Longrightarrow \widetilde{I\!P}(X_n \geq 0) = 1. \tag{6.2}$$

(6.1) and (6.2) imply $\widetilde{I\!P}(X_n = 0) = 1$. We have

$$\widetilde{I\!P}(X_n = 0) = 1 \Longrightarrow \widetilde{I\!P}(X_n > 0) = 0 \Longrightarrow I\!P(X_n > 0) = 0.$$

This is not an arbitrage. ■

12.7 Stalking the Risk-Neutral Measure

Recall that

- $Y_1, Y_2, \ldots, Y_n$ are independent, standard normal random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- $S_k = S_0 \exp\left\{\sigma B_k + (\mu - \frac{1}{2}\sigma^2)k\right\}$.
-

$$\begin{aligned} S_{k+1} &= S_0 \exp\left\{\sigma(B_k + Y_{k+1}) + (\mu - \tfrac{1}{2}\sigma^2)(k+1)\right\} \\ &= S_k \exp\left\{\sigma Y_{k+1} + (\mu - \tfrac{1}{2}\sigma^2)\right\}. \end{aligned}$$

Therefore,

$$\frac{S_{k+1}}{M_{k+1}} = \frac{S_k}{M_k}.\exp\left\{\sigma Y_{k+1} + (\mu - r - \tfrac{1}{2}\sigma^2)\right\},$$

$$\begin{aligned} I\!E\left[\frac{S_{k+1}}{M_{k+1}}\middle|\mathcal{F}_k\right] &= \frac{S_k}{M_k}.I\!E\left[\exp\left\{\sigma Y_{k+1}\right\}|\mathcal{F}_k\right].\exp\{\mu - r - \tfrac{1}{2}\sigma^2\} \\ &= \frac{S_k}{M_k}.\exp\{\tfrac{1}{2}\sigma^2\}.\exp\{\mu - r - \tfrac{1}{2}\sigma^2\} \\ &= e^{\mu - r}.\frac{S_k}{M_k}. \end{aligned}$$

If $\mu = r$, the market measure is risk neutral. If $\mu \neq r$, we must seek further.

$$\begin{aligned}\frac{S_{k+1}}{M_{k+1}} &= \frac{S_k}{M_k}.\exp\left\{\sigma Y_{k+1} + (\mu - r - \tfrac{1}{2}\sigma^2)\right\}\\ &= \frac{S_k}{M_k}.\exp\left\{\sigma(Y_{k+1} + \tfrac{\mu-r}{\sigma}) - \tfrac{1}{2}\sigma^2\right\}\\ &= \frac{S_k}{M_k}.\exp\left\{\sigma\tilde{Y}_{k+1} - \tfrac{1}{2}\sigma^2\right\},\end{aligned}$$

where

$$\tilde{Y}_{k+1} = Y_{k+1} + \tfrac{\mu-r}{\sigma}.$$

The quantity $\frac{\mu-r}{\sigma}$ is denoted θ and is called the *market price of risk*.

We want a probability measure $\widetilde{I\!P}$ under which $\tilde{Y}_1, \ldots, \tilde{Y}_n$ are independent, standard normal random variables. Then we would have

$$\begin{aligned}\widetilde{I\!E}\left[\frac{S_{k+1}}{M_{k+1}}\middle|\mathcal{F}_k\right] &= \frac{S_k}{M_k}.\widetilde{I\!E}\left[\exp\{\sigma\tilde{Y}_{k+1}\}|\mathcal{F}_k\right].\exp\{-\tfrac{1}{2}\sigma^2\}\\ &= \frac{S_k}{M_k}.\exp\{\tfrac{1}{2}\sigma^2\}.\exp\{-\tfrac{1}{2}\sigma^2\}\\ &= \frac{S_k}{M_k}.\end{aligned}$$

Cameron-Martin-Girsanov's Idea: Define the random variable

$$Z = \exp\left[\sum_{j=1}^{n}(-\theta Y_j - \tfrac{1}{2}\theta^2)\right].$$

Properties of Z:

- $Z \geq 0$.
-

$$\begin{aligned}I\!EZ &= I\!E\exp\left\{\sum_{j=1}^{n}(-\theta Y_j)\right\}.\exp\left\{-\frac{n}{2}\theta^2\right\}\\ &= \exp\left\{\frac{n}{2}\theta^2\right\}.\exp\left\{-\frac{n}{2}\theta^2\right\} = 1.\end{aligned}$$

Define

$$\widetilde{I\!P}(A) = \int_A Z\, dI\!P \quad \forall A \in \mathcal{F}.$$

Then $\widetilde{I\!P}(A) \geq 0$ for all $A \in \mathcal{F}$ and

$$\widetilde{I\!P}(\Omega) = I\!EZ = 1.$$

In other words, $\widetilde{I\!P}$ is a probability measure.

We show that $\widetilde{I\!\!P}$ is a risk-neutral measure. For this, it suffices to show that

$$\tilde{Y}_1 = Y_1 + \theta,\ \ldots,\ \tilde{Y}_n = Y_n + \theta$$

are independent, standard normal under $\widetilde{I\!\!P}$.

Verification:

- $Y_1, Y_2, \ldots, Y_n$: Independent, standard normal under P, and

$$I\!\!E \exp\left[\sum_{j=1}^{n} u_j Y_j\right] = \exp\left[\sum_{j=1}^{n} \tfrac{1}{2} u_j^2\right].$$

- $\tilde{Y} = Y_1 + \theta,\ \ldots,\ \tilde{Y}_n = Y_n + \theta$.
- $Z > 0$ almost surely.
- $Z = \exp\left[\sum_{j=1}^{n}(-\theta Y_j - \frac{1}{2}\theta^2)\right]$,

$$\widetilde{I\!\!P}(A) = \int_A Z\, dI\!\!P \quad \forall A \in \mathcal{F},$$

$\widetilde{I\!\!E} X = I\!\!E(XZ)$ for every random variable X.

- Compute the moment generating function of $(\tilde{Y}_1, \ldots, \tilde{Y}_n)$ under $\widetilde{I\!\!P}$:

$$\begin{aligned}
\widetilde{I\!\!E} \exp\left[\sum_{j=1}^{n} u_j \tilde{Y}_j\right] &= I\!\!E \exp\left[\sum_{j=1}^{n} u_j (Y_j + \theta) + \sum_{j=1}^{n}(-\theta Y_j - \tfrac{1}{2}\theta^2)\right] \\
&= I\!\!E \exp\left[\sum_{j=1}^{n} (u_j - \theta) Y_j\right] . \exp\left[\sum_{j=1}^{n}(u_j\theta - \tfrac{1}{2}\theta^2)\right] \\
&= \exp\left[\sum_{j=1}^{n} \tfrac{1}{2}(u_j - \theta)^2\right] . \exp\left[\sum_{j=1}^{n}(u_j\theta - \tfrac{1}{2}\theta^2)\right] \\
&= \exp\left[\sum_{j=1}^{n} \left((\tfrac{1}{2}u_j^2 - u_j\theta + \tfrac{1}{2}\theta^2) + (u_j\theta - \tfrac{1}{2}\theta^2)\right)\right] \\
&= \exp\left[\sum_{j=1}^{n} \tfrac{1}{2} u_j^2\right].
\end{aligned}$$

12.8 Pricing a European Call

Stock price at time n is

$$\begin{aligned} S_n &= S_0 \exp\left\{\sigma B_n + (\mu - \tfrac{1}{2}\sigma^2)n\right\} \\ &= S_0 \exp\left\{\sigma \sum_{j=1}^{n} Y_j + (\mu - \tfrac{1}{2}\sigma^2)n\right\} \\ &= S_0 \exp\left\{\sigma \sum_{j=1}^{n} (Y_j + \tfrac{\mu - r}{\sigma}) - (\mu - r)n + (\mu - \tfrac{1}{2}\sigma^2)n\right\} \\ &= S_0 \exp\left\{\sigma \sum_{j=1}^{n} \tilde{Y}_j \; + \; (r - \tfrac{1}{2}\sigma^2)n\right\}. \end{aligned}$$

Payoff at time n is $(S_n - K)^+$. Price at time zero is

$$\begin{aligned} \widetilde{\mathbb{E}} \frac{(S_n - K)^+}{M_n} &= \widetilde{\mathbb{E}} \left[e^{-rn} \left(S_0 \exp\left\{\sigma \sum_{j=1}^{n} \tilde{Y}_j \; + \; (r - \tfrac{1}{2}\sigma^2)n\right\} - K \right)^+ \right] \\ &= \int_{-\infty}^{\infty} e^{-rn} \left(S_0 \exp\left\{\sigma b + (r - \tfrac{1}{2}\sigma^2)n\right\} - K \right)^+ . \frac{1}{\sqrt{2\pi n}} e^{-\frac{b^2}{2n^2}} \; db \end{aligned}$$

since $\sum_{j=1}^{n} \tilde{Y}_j$ is normal with mean 0, variance n, under $\widetilde{\mathbb{P}}$.

This is the *Black-Scholes* price. It does not depend on μ.