Chapter 10

Capital Asset Pricing

10.1 An Optimization Problem

Consider an agent who has initial wealth X_0 and wants to invest in the stock and money markets so as to maximize

$$I\!E \log X_n.$$

Remark 10.1 Regardless of the portfolio used by the agent, $\{\zeta_k X_k\}_{k=0}^{\infty}$ is a martingale under P, so

$$I\!E \zeta_n X_n = X_0 \qquad (BC)$$

Here, (BC) stands for "Budget Constraint".

Remark 10.2 If ξ is any random variable satisfying (BC), i.e.,

$$I\!E \zeta_n \xi = X_0,$$

then there is a portfolio which starts with initial wealth X_0 and produces $X_n = \xi$ at time n. To see this, just regard ξ as a simple European derivative security paying off at time n. Then X_0 is its value at time 0, and starting from this value, there is a hedging portfolio which produces $X_n = \xi$.

Remarks 10.1 and 10.2 show that the optimal X_n for the capital asset pricing problem can be obtained by solving the following

Constrained Optimization Problem:
Find a random variable ξ which solves:

$$\text{Maximize} \quad I\!E \log \xi$$

$$\text{Subject to} \quad I\!E \zeta_n \xi = X_0.$$

Equivalently, we wish to

$$\text{Maximize} \quad \sum_{\omega \in \Omega} (\log \xi(\omega))\, I\!P(\omega)$$

$$\text{Subject to} \quad \sum_{\omega\in\Omega} \zeta_n(\omega)\xi(\omega)\mathbb{P}(\omega) \; - \; X_0 = 0.$$

There are 2^n sequences ω in Ω. Call them $\omega_1, \omega_2, \ldots, \omega_{2^n}$. Adopt the notation

$$x_1 = \xi(\omega_1),\; x_2 = \xi(\omega_2),\; \ldots,\; x_{2^n} = \xi(\omega_{2^n}).$$

We can thus restate the problem as:

$$\text{Maximize} \quad \sum_{k=1}^{2^n} (\log x_k)\mathbb{P}(\omega_k)$$

$$\text{Subject to} \quad \sum_{k=1}^{2^n} \zeta_n(\omega_k)x_k\mathbb{P}(\omega_k) \; - \; X_o = 0.$$

In order to solve this problem we use:

Theorem 1.30 (Lagrange Multiplier) *If $(x_1^*, \ldots, x_m^*)$ solve the problem*

$$\textit{Maxmize} \quad f(x_1, \ldots, x_m)$$

$$\textit{Subject to} \quad g(x_1, \ldots, x_m) = 0,$$

then there is a number λ such that

$$\frac{\partial}{\partial x_k} f(x_1^*, \ldots, x_m^*) = \lambda \frac{\partial}{\partial x_k} g(x_1^*, \ldots, x_m^*), \quad k = 1, \ldots, m, \tag{1.1}$$

and

$$g(x_1^*, \ldots, x_m^*) = 0. \tag{1.2}$$

For our problem, (1.1) and (1.2) become

$$\frac{1}{x_k^*}\mathbb{P}(\omega_k) = \lambda\zeta_n(\omega_k)\mathbb{P}(\omega_k),\; k = 1, \ldots, 2^n, \tag{1.1'}$$

$$\sum_{k=1}^{2^n} \zeta_n(\omega_k)x_k^*\mathbb{P}(\omega_k) \; = \; X_0. \tag{1.2'}$$

Equation (1.1') implies

$$x_k^* = \frac{1}{\lambda\zeta_n(\omega_k)}.$$

Plugging this into (1.2') we get

$$\frac{1}{\lambda}\sum_{k=1}^{2^n} \mathbb{P}(\omega_k) = X_0 \implies \frac{1}{\lambda} = X_0.$$

Therefore,

$$x_k^* = \frac{X_0}{\zeta_n(\omega_k)},\ k = 1, \ldots, 2^n.$$

Thus we have shown that if ξ^* solves the problem

$$\begin{array}{ll} \text{Maximize} & \mathbb{E}\log\xi \\ \text{Subject to} & \mathbb{E}(\zeta_n\xi) = X_0, \end{array} \tag{1.3}$$

then

$$\xi^* = \frac{X_0}{\zeta_n}. \tag{1.4}$$

Theorem 1.31 *If ξ^* is given by (1.4), then ξ^* solves the problem (1.3).*

Proof: Fix $Z > 0$ and define

$$f(x) = \log x - xZ.$$

We maximize f over $x > 0$:

$$f'(x) = \frac{1}{x} - Z = 0 \iff x = \frac{1}{Z},$$

$$f''(x) = -\frac{1}{x^2} < 0,\ \forall x \in \mathbb{R}.$$

The function f is maximized at $x^* = \frac{1}{Z}$, i.e.,

$$\log x - xZ \le f(x^*) = \log\frac{1}{Z} - 1,\ \forall x > 0,\ \forall Z > 0. \tag{1.5}$$

Let ξ be any random variable satisfying

$$\mathbb{E}(\zeta_n\xi) = X_0$$

and let

$$\xi^* = \frac{X_0}{\zeta_n}.$$

From (1.5) we have

$$\log\xi - \xi\left(\frac{\zeta_n}{X_0}\right) \le \log\left(\frac{X_0}{\zeta_n}\right) - 1.$$

Taking expectations, we have

$$\mathbb{E}\log\xi - \frac{1}{X_0}\mathbb{E}(\zeta_n\xi) \le \mathbb{E}\log\xi^* - 1,$$

and so

$$\mathbb{E}\log\xi \le \mathbb{E}\log\xi^*.$$

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In summary, capital asset pricing works as follows: Consider an agent who has initial wealth X_0 and wants to invest in the stock and money market so as to maximize

$$\mathbb{E} \log X_n.$$

The optimal X_n is $X_n = \frac{X_0}{\zeta_n}$, i.e.,

$$\zeta_n X_n = X_0.$$

Since $\{\zeta_k X_k\}_{k=0}^n$ is a martingale under P, we have

$$\zeta_k X_k = \mathbb{E}[\zeta_n X_n | \mathcal{F}_k] = X_0, \; k = 0, \ldots, n,$$

so

$$X_k = \frac{X_0}{\zeta_k},$$

and the optimal portfolio is given by

$$\Delta_k(\omega_1, \ldots, \omega_k) = \frac{\frac{X_0}{\zeta_{k+1}(\omega_1, \ldots, \omega_k, H)} - \frac{X_0}{\zeta_{k+1}(\omega_1, \ldots, \omega_k, T)}}{S_{k+1}(\omega_1, \ldots, \omega_k, H) - S_{k+1}(\omega_1, \ldots, \omega_k, T)}.$$